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# HEEGAARD FLOER HOMOLOGY AND RATIONAL CUSPIDAL CURVES

MACIEJ BORODZIK[1] and CHARLES LIVINGSTON[2]

[1] Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland;
email: mcboro@mimuw.edu.pl

[2] Department of Mathematics, Indiana University, Bloomington, IN 47405, USA;
email: livingst@indiana.edu

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**Abstract**

We apply the methods of Heegaard Floer homology to identify topological properties of complex curves in $\mathbb{C}P^2$. As one application, we resolve an open conjecture that constrains the Alexander polynomial of the link of the singular point of the curve in the case that there is exactly one singular point, having connected link, and the curve is of genus zero. Generalizations apply in the case of multiple singular points.

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## 1. Introduction

We consider irreducible algebraic curves $C \subset \mathbb{C}P^2$. Such a curve has a finite set of singular points, $\{z_i\}_{i=1}^n$; a neighborhood of each intersects $C$ in a cone on a link $L_i \subset S^3$. A fundamental question asks what possible configurations of links $\{L_i\}$ arise in this way. In this generality the problem is fairly intractable, and research has focused on a restricted case, in which each $L_i$ is connected (and is thus denoted $K_i$), and $C$ is a rational curve, meaning that there is a rational surjective map $\mathbb{C}P^1 \to C$. Such a curve is called *rational cuspidal*. Being rational cuspidal is equivalent to $C$ being homeomorphic to $S^2$.

Our results apply in the case of multiple singular points, but the following statement gives an indication of the nature of the results and their consequences.

THEOREM 1.1. *Suppose that $C$ is a rational cuspidal curve of degree $d$ with one singular point, a cone on the knot $K$, and the Alexander polynomial of $K$ is expanded at $t = 1$ to be $\Delta_K(t) = 1+((d-1)(d-2)/2)(t-1)+(t-1)^2\sum_l k_l t^l$. Then, for all $j$, $0 \leqslant j \leqslant d-3$, $k_{d(d-j-3)} = (j+1)(j+2)/2$.*

There are three facets to the work here.

(1) We begin with a basic observation that a neighborhood $Y$ of $C$ is built from the 4-ball by attaching a 2-handle along the knot $K = \#K_i$ with framing $d^2$, where $d$ is the degree of the curve. Thus, its boundary, $S^3_{d^2}(K)$, bounds the rational homology ball $\mathbb{C}P^2\backslash Y$. From this, it follows that the Heegaard Floer correction term satisfies $d(S^3_{d^2}(K), \mathfrak{s}_m) = 0$ if $d|m$, for properly enumerated Spin$^c$ structures $\mathfrak{s}_m$.

(2) Because each $K_i$ is an algebraic knot (in particular an $L$-space knot), the Heegaard Floer complex $CFK^\infty(S^3, K_i)$ is determined by the Alexander polynomial of $K_i$, and thus the complex $CFK^\infty(S^3, K)$ and the $d$-invariants are also determined by the Alexander polynomials of the $K_i$.

(3) The constraints that arise on the Alexander polynomials, although initially appearing quite intricate, can be reinterpreted in compact form using semigroups of singular points. In this way, we can relate these constraints to well-known conjectures.

### 1.1. The conjecture of Fernández de Bobadilla, Luengo, Melle-Hernandez, and Némethi.

In [7], the following conjecture was proposed. It was also verified for most of the known examples of rational cuspidal curves.

CONJECTURE 1.2 [7]. Suppose that the rational cuspidal curve $C$ of degree $d$ has critical points $z_1, \ldots, z_n$. Let $K_1, \ldots, K_n$ be the corresponding links of singular points, and let $\Delta_1, \ldots, \Delta_n$ be their Alexander polynomials. Let $\Delta = \Delta_1 \cdot \ldots \cdot \Delta_n$, expanded as

$$\Delta(t) = 1 + \frac{(d-1)(d-2)}{2}(t-1) + (t-1)^2 \sum_{j=0}^{2g-2} k_l t^l.$$

Then, for any $j = 0, \ldots, d-3$, $k_{d(d-j-3)} \leqslant (j+1)(j+2)/2$, with equality for $n = 1$.

We remark that the case $n = 1$ of the conjecture is Theorem 1.1. We will prove this result in Section 4.4. Later, we will also prove an alternative generalization of Theorem 1.1 for the case $n > 1$, stated as Theorem 5.4, which is the main

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result of the present article. The advantage of this formulation over the original conjecture lies in the fact that it gives precise values of the coefficients $k_{d(d-j-3)}$. Theorem 6.5 provides an equivalent statement of Theorem 5.4.

REMARK 1.3. After a preliminary version of this paper appeared, Bodnár and Némethi [2] found a counterexample to Conjecture 1.2 with three singular points; see Remark 5.5 for more details.

## 2. Background: Algebraic geometry and rational cuspidal curves

In this section, we will present some of the general theory of rational cuspidal curves. Section 2.1 includes basic information about singular points of plane curves. In Section 2.2, we discuss the semigroup of a singular point and its connections to the Alexander polynomial of the link. We shall use results from this section later in the article to simplify the equalities that we obtain. In Section 2.3, we describe results from [7] to give some flavor of the theory. In Section 2.4, we provide a rough sketch of some methods used to study rational cuspidal curves. We refer to [15] for an excellent and fairly up-to-date survey of results on rational cuspidal curves.

**2.1. Singular points and algebraic curves.** For a general introduction and references to this subsection, we refer to [4, 8], or to [14, Section 10] for a more topological approach. In this article, we will be considering algebraic curves embedded in $\mathbb{C}P^2$. Thus we will use the word *curve* to refer to a zero set of an irreducible homogeneous polynomial $F$ of degree $d$. The *degree* of the curve is the degree of the corresponding polynomial.

Let $C$ be a curve. A point $z \in C$ is called *singular* if the gradient of $F$ vanishes at $z$. Singular points of irreducible curves in $\mathbb{C}P^2$ are always isolated. Given a singular point and a sufficiently small ball $B \subset \mathbb{C}P^2$ around $z$, we call $K = C \cap \partial B$ the *link* of the singular point. The singular point is called *cuspidal* or *unibranched* if $K$ is a knot, that is, a link with one component, or, equivalently, if there is an analytic map $\psi$ from a disk in $\mathbb{C}$ onto $C \cap B$.

Unless specified otherwise, all singular points are assumed to be cuspidal.

Two unibranched singular points are called *topologically equivalent* if the links of these singular points are isotopic; see for instance [8, Definition I.3.30] for more details. A unibranched singular point is topologically equivalent to one for which the local parameterization $\psi$ is given in local coordinates $(x, y)$ on $B$ by $t \mapsto (x(t), y(t))$, where $x(t) = t^p$, $y(t) = t^{q_1} + \cdots + t^{q_n}$ for some positive integers $p, q_1, \ldots, q_n$ satisfying $p < q_1 < q_2 < \cdots < q_n$. Furthermore, if we set $D_i = \gcd(p, q_1, \ldots, q_i)$, then $D_i$ does not divide $q_{i+1}$, and $D_n = 1$. The sequence

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$(p; q_1, \ldots, q_n)$ is called the *characteristic sequence* of the singular point, and $p$ is called the *multiplicity*. Sometimes $n$ is referred to as the *number of Puiseux pairs*, a notion which comes from an alternative way of encoding the sequence $(p; q_1, \ldots, q_n)$. We will say that a singular point is of *type* $(p; q_1, \ldots, q_n)$ if it has a presentation of this sort in local coordinates.

The link of a singular point with a characteristic sequence $(p; q_1, \ldots, q_n)$ is an $(n-1)$-fold iterate of a torus knot $T(p', q')$, where $p' = p/D_1$ and $q' = q_1/D_1$; see for example [4, Sections 8.3 and 8.5] or [28, Ch. 5.2]. In particular, if $n = 1$, the link is a torus knot $T(p, q_1)$. In all cases, the genus of the link is equal to $\mu/2 = \delta$, where $\mu$ is the Milnor number and $\delta$ is the so-called $\delta$-invariant of the singular point; see [8, page 205] or [14, Section 10]. The genus is also equal to half the degree of the Alexander polynomial of the link of the singular point. The Milnor number can be computed from the following formula; see [14, Remark 10.10]:

$$\mu = (p-1)(q_1-1) + \sum_{i=2}^{n}(D_i - 1)(q_i - q_{i-1}).$$

Suppose that $C$ is a degree-$d$ curve with singular points $z_1, \ldots, z_n$ (and $L_1, \ldots, L_n$ are their links). The genus formula, due to Serre (see [14, Property 10.4]), states that the genus of $C$ is equal to

$$g(C) = \frac{1}{2}(d-1)(d-2) - \sum_{i=1}^{n} \delta_i.$$

If all the critical points are cuspidal, we have $\delta_i = g(L_i)$, so the above formula can be written as

$$g(C) = \frac{1}{2}(d-1)(d-2) - \sum_{i=1}^{n} g(L_i). \tag{2.1}$$

In particular, $C$ is rational cuspidal (that is, it is a homeomorphic image of a sphere) if and only $\sum g(L_i) = \frac{1}{2}(d-1)(d-2)$.

**2.2. Semigroup of a singular point.** The notion of the semigroup associated to a singular point is a central notion in the subject, although in the present work we use only the language of semigroups, not the algebraic aspects. We refer to [28, Ch. 4] or [8, page 214] for details and proofs. Suppose that $z$ is a cuspidal singular point of a curve $C$, and that $B$ is a sufficiently small ball around $z$. Let $\psi(t) = (x(t), y(t))$ be a local parameterization of $C \cap B$ near $z$; see Section 2.1. For any polynomial $G(x, y)$, we look at the order at zero of an analytic map $t \mapsto G(x(t), y(t)) \in \mathbb{C}$. Let $S$ be the set of integers that can be realized as the order for some $G$. Then $S$ is clearly a semigroup of $\mathbb{Z}_{\geqslant 0}$. We call it the *semigroup of the singular point*. The semigroup can be computed from the characteristic sequence: for example, for a sequence $(p; q_1)$, $S$ is generated by $p$ and $q_1$. The *gap sequence*, $G := \mathbb{Z}_{\geqslant 0} \backslash S$, has precisely $\mu/2$ elements, and the largest one is $\mu - 1$, where $\mu$ is the Milnor number.

We now assume that $K$ is the link of the singular point $z$. Explicit computations of the Alexander polynomial of $K$ show that it is of the form

$$\Delta_K(t) = \sum_{i=0}^{2m} (-1)^i t^{n_i}, \tag{2.2}$$

where $n_i$ form an increasing sequence with $n_0 = 0$ and $n_{2m} = 2g$, twice the genus of $K$.

Expanding $t^{n_{2i}} - t^{n_{2i-1}}$ as $(t-1)(t^{n_{2i}-1} + t^{n_{2i}-2} + \cdots + t^{n_{2i-1}})$ yields

$$\Delta_K(t) = 1 + (t-1) \sum_{j=1}^{k} t^{g_j}, \tag{2.3}$$

for some finite sequence $0 < g_1 < \cdots < g_k$. We have the following result (see [28, Exercise 5.7.7]).

LEMMA 2.4. *The sequence $g_1, \ldots, g_k$ is the gap sequence of the semigroup of the singular point. In particular $k = \#G = \mu/2$, where $\mu$ is the Milnor number, so $\#G$ is the genus.*

Writing $t^{g_j}$ as $(t-1)(t^{g_j-1} + t^{g_j-2} + \cdots + t + 1) + 1$ in (2.3) yields the following formula:

$$\Delta_K(t) = 1 + (t-1)g(K) + (t-1)^2 \sum_{j=0}^{\mu-2} k_j t^j, \tag{2.5}$$

where $k_j = \#\{m > j : m \notin S\}$.

We shall use the following definition.

DEFINITION 2.6. For any finite increasing sequence of positive integers $G$, we define

$$I_G(m) = \#\{k \in G \cup \mathbb{Z}_{<0} : k \geqslant m\}, \tag{2.7}$$

where $\mathbb{Z}_{<0}$ is the set of the negative integers. We shall call $I_G$ the *gap function*, because in most applications $G$ will be a gap sequence of some semigroup.

REMARK 2.8. We point out that, for $j = 0, \ldots, \mu - 2$, we have $I_G(j+1) = k_j$, where the $k_j$ are as in (2.5).

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EXAMPLE 2.9. Consider the knot $T(3,7)$. Its Alexander polynomial is

$$\begin{aligned}\frac{(t^{21}-1)(t-1)}{(t^3-1)(t^7-1)} &= 1-t+t^3-t^4+t^6-t^8+t^9-t^{11}+t^{12}\\ &= 1+(t-1)(t+t^2+t^4+t^5+t^8+t^{11})\\ &= 1+6(t-1)+(t-1)^2(6+5t+4t^2\\ &\quad +4t^3+3t^4+2t^5+2t^6+2t^7+t^8+t^9+t^{10}).\end{aligned}$$

The semigroup is $(0,3,6,7,9,10,12,13,14,\ldots)$. The gap sequence is 1, 2, 4, 5, 8, 11.

REMARK 2.10. The passage from (2.2) through (2.3) to (2.5) is just an algebraic manipulation, and thus it applies to any knot whose Alexander polynomial has form (2.2). In particular, according to [24, Theorem 1.2], it applies to any $L$-space knot. In this setting, we will also call the sequence $g_1,\ldots,g_k$ the *gap sequence* of the knot, and denote it by $G_K$; we will write $I_K(m)$ for the gap function relative to $G_K$. Even though the complement $\mathbb{Z}_{\geqslant 0}\setminus G_K$ is not always a semigroup, we still have $\#G_K=\frac{1}{2}\deg\Delta_K$. This property follows immediately from the symmetry of the Alexander polynomial.

**2.3. Rational cuspidal curves with one cusp.** The classification of rational cuspidal curves is a challenging old problem, with some conjectures (like the Coolidge–Nagata conjecture [5, 16]) remaining open for many decades. The classification of curves with a unique critical point is far from being accomplished; the special case when the unique singular point has only one Puiseux term (its link is a torus knot) is complete [7], but even in this basic case the proof is quite difficult.

To give some indication of the situation, consider two families of rational cuspidal curves. The first one is written in projective coordinates on $\mathbb{C}P^2$ as $x^d+y^{d-1}z=0$ for $d>1$; the other one is $(zy-x^2)^{d/2}-xy^{d-1}=0$ for $d$ even and $d>1$. These are of degree $d$. Both families have a unique singular point: in the first case, it is of type $(d-1;d)$; in the second case, it is of type $(d/2;2d-1)$. In both cases, the Milnor number is $(d-1)(d-2)$, so the curves are rational. An explicit parameterization can be easily given as well.

There also exist more complicated examples. For instance, Orevkov [21] constructed rational cuspidal curves of degree $\phi_j$ having a single singular point of type $(\phi_{j-2};\phi_{j+2})$, where $j$ is odd and $j>5$. Here, the $\phi_j$ are the Fibonacci numbers, $\phi_0=0$, $\phi_1=1$, $\phi_{j+2}=\phi_{j+1}+\phi_j$. As an example, there exists a rational cuspidal curve of degree 13 with a single singular point of type (5; 34). Orevkov's construction is inductive and by no means trivial. Another family found by Orevkov is rational cuspidal curves of degree $\phi_{j-1}^2 - 1$ having a single singular point of type $(\phi_{j-2}^2; \phi_j^2)$, for $j > 5$, odd.

The main result of [7] is that, apart from these four families of rational cuspidal curves, there are only two sporadic curves with a unique singular point having one Puiseux pair, one of degree eight, and the other of degree 16.

**2.4. Constraints on rational cuspidal curves.** Here, we review some constraints for rational cuspidal curves. We refer to [15] for more details and references. The article [7] shows how these constraints can be used in practice. The fundamental constraint is given by (2.1). Then Matsuoka and Sakai [13] proved that, if $(p_1; q_{11}, \ldots, q_{1k_1}), \ldots, (p_n; q_{n1}, \ldots, q_{nk_n})$ are the only singular points occurring on a rational cuspidal curve of degree $d$ with $p_1 \geqslant \cdots \geqslant p_n$, then $p_1 > d/3$. Later, Orevkov [21] improved this to $\alpha(p_1 + 1) + 1/\sqrt{5} > d$, where $\alpha = (3 + \sqrt{5})/2 \sim 2.61$, and showed that this inequality is asymptotically optimal (it is related to the curves described in Section 2.1). Both proofs use very deep algebro-geometric tools. We re-prove the result of [13] in Proposition 6.7 below.

Another obstruction comes from the semicontinuity of the spectrum, a concept that arises from Hodge theory. Even a rough definition of the spectrum of a singular point is beyond the scope of this article. We refer to [1, Ch. 14] for a definition of the spectrum, and to [7] for illustrations of its use. We point out that recently (see [3]) a tight relation has been drawn between the spectrum of a singular point and the Tristram–Levine signatures of its link. In general, semicontinuity of the spectrum is a very strong tool, but it is also very difficult to apply.

Using tools from algebraic geometry, such as the Hodge index theorem, Tono in [27] proved that any rational cuspidal curve can have at most eight singular points. An old conjecture is that a rational cuspidal curve can have at most four singular points; see [25] for a precise statement.

In [6], a completely new approach was proposed, motivated by a conjecture on Seiberg–Witten invariants of links of surface singularities made by Némethi and Nicolaescu; see [19]. Specifically, Conjecture 1.2 in the present article arises from these considerations. Another reference for the general conjecture on Seiberg–Witten invariants is [18].

## 3. Topology, algebraic topology, and Spin$^c$ structures

Let $C \subset \mathbb{C}P^2$ be a rational cuspidal curve. Let $d$ be its degree, and let $z_1, \ldots, z_n$ be its singular points. We let $Y$ be a closed manifold regular neighborhood of $C$, let $M = \partial Y$, and let $W = \overline{\mathbb{C}P^2 - Y}$.

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**3.1. Topological descriptions of $Y$ and $M$.** The neighborhood $Y$ of $C$ can be built in three steps. First, disk neighborhoods of the $z_i$ are selected. Then neighborhoods of $N-1$ embedded arcs on $C$ are adjoined, yielding a 4-ball. Finally, the remainder of $C$ is a disk, so its neighborhood forms a 2-handle attached to the 4-ball. Thus, $Y$ is a 4-ball with a 2-handle attached. The attaching curve is easily seen to be $K = \#K_i$. Finally, since the self-intersection of $C$ is $d^2$, the framing of the attaching map is $d^2$. In particular, $M = S^3_{d^2}(K)$.

One quickly computes that $H_2(\mathbb{C}P^2, C) = \mathbb{Z}_d$, and $H_4(\mathbb{C}P^2, C) = \mathbb{Z}$, with the remaining homology groups 0. Using excision, we see that the groups $H_i(W, M)$ are the same. Via Lefschetz duality and the universal coefficient theorem, we find that $H_0(W) = \mathbb{Z}$, $H_1(W) = \mathbb{Z}_d$, and all the other groups are 0. Finally, the long exact sequence of the pair $(W, M)$ yields

$$0 \to H_2(W, M) \to H_1(M) \to H_1(W) \to 0,$$

which in this case is

$$0 \to \mathbb{Z}_d \to \mathbb{Z}_{d^2} \to \mathbb{Z}_d \to 0.$$

This is realized geometrically by letting the generator of $H_2(W, M)$ be $H \cap W$, where $H \subset \mathbb{C}P^2$ is a generic line. Its boundary is algebraically $d$ copies of the meridian of the attaching curve $K$ in the 2-handle decomposition of $Y$.

Taking duals, we see that the map $H^2(W) \to H^2(M)$, which maps $\mathbb{Z}_d \to \mathbb{Z}_{d^2}$, takes the canonical generator to $d$ times the dual to the meridian in $M = S^3_{d^2}(K)$.

**3.2. Spin$^c$ structures.** For any space $X$, there is a transitive action of $H^2(X)$ on Spin$^c$ $(X)$. Thus, $W$ has $d$ Spin$^c$ structures, and $M$ has $d^2$ such structures.

Since $\mathbb{C}P^2$ has a Spin$^c$ structure with first Chern class a dual to the class of the line, its restriction to $W$ is a structure whose restriction to $M$ has first Chern class equal to $d$ times the dual to the meridian.

For a cohomology class $z \in H^2(X)$ and a Spin$^c$ structure $\mathfrak{s}$, one has $c_1(z \cdot \mathfrak{s}) - c_1(\mathfrak{s}) = 2z$. Thus, for each $k \in \mathbb{Z}$, there is a Spin$^c$ structure on $M$ which extends to $W$ having first Chern class of the form $d + 2kd$. Notice that, for $d$ odd, all $md \in \mathbb{Z}_{d^2}$ for $m \in \mathbb{Z}$ occur as first Chern classes of Spin$^c$ structures that extend over $W$, but, for $d$ even, only elements of the form $md$ with $m$ odd occur. (Thus, for $d$ even, there are $d$ extending structures, but only $d/2$ first Chern classes that occur.)

According to [23, Section 3.4], the Spin$^c$ structures on $M$ have an enumeration $\mathfrak{s}_m$, for $m \in [-d^2/2, d^2/2]$, which can be defined via the manifold $Y$. Specifically, $\mathfrak{s}_m$ is defined to be the restriction to $M$ of the Spin$^c$ structure on $Y$, $\mathfrak{t}_m$, with the property that $\langle c_1(\mathfrak{t}_m), C\rangle + d^2 = 2m$. We point out that, if $d$ is even, $\mathfrak{s}_{d^2/2}$ and $\mathfrak{s}_{-d^2/2}$ denote the same structure; see Remark 4.5 below.

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It now follows from our previous observations that the structures $\mathfrak{s}_m$ that extend to $W$ are those with $m = kd$ for some integer $k$, $-d/2 \leqslant k \leqslant d/2$ if $d$ is odd. If $d$ is even, then those that extend have $m = kd/2$ for some odd $k$, $-d \leqslant k \leqslant d$. For future reference, we summarize this with the following lemma.

LEMMA 3.1. *If $W^4 = \overline{\mathbb{C}P^2 - Y}$, where $Y$ is a neighborhood of a rational cuspidal curve $C$ of degree $d$ (as constructed above), then the Spin$^c$ structure $\mathfrak{s}_m$ on $\partial W^4$ extends to $W^4$ if $m = kd$ for some integer $k$, $-d/2 \leqslant k \leqslant d/2$ if $d$ is odd. If $d$ is even, then those that extend have $m = kd/2$ for some odd $k$, $-d \leqslant k \leqslant d$. Here, $\mathfrak{s}_m$ is the Spin$^c$ structure on $\partial W$ that extends to a structure $\mathfrak{t}$ on $Y$ satisfying $\langle c_1(\mathfrak{t}_m), C\rangle + d^2 = 2m$.*

## 4. Heegaard Floer theory

Heegaard Floer theory [22] associates to a 3-manifold $M$, with Spin$^c$ structure $\mathfrak{s}$, a filtered graded chain complex $CF^\infty(M, \mathfrak{s})$ over the field $\mathbb{Z}_2$ defined up to a filtered chain homotopy equivalence. A fundamental invariant of the pair $(M, \mathfrak{s})$, the *correction term* or $d$-invariant, $d(M, \mathfrak{s}) \in \mathbb{Q}$, is determined by $CF^\infty(M, \mathfrak{s})$. The manifold $M$ is called an *$L$-space* if certain associated homology groups are of rank one [24].

A knot $K$ in $M$ provides a second filtration on $CF^\infty(M, \mathfrak{s})$ [22]. In particular, for $K \subset S^3$, there is a bifiltered graded chain complex $CFK^\infty(K)$ over the field $\mathbb{Z}_2$. It is known that for algebraic knots the complex is determined by the Alexander polynomial of $K$. More generally, this holds for any knot upon which some surgery yields an $L$-space; these knots are called $L$-space knots.

The Heegaard Floer invariants of surgery on $K$, in particular the $d$-invariants of $S^3_q(K)$, are determined by this complex, and for $q > 2(\text{genus}(K))$ the computation of $d(S^3_q(K), \mathfrak{s})$ from $CFK^\infty(K)$ is particularly simple. In this section, we will illustrate the general theory, leaving the details to references such as [9, 11].

### 4.1. $CFK^\infty(K)$ for $K$ an algebraic knot.

Figure 1 is a schematic illustration of a finite complex over $\mathbb{Z}_2$. Each dot represents a generator, and the arrows indicate boundary maps. Abstractly, it is of the form $0 \to \mathbb{Z}_2^4 \to \mathbb{Z}_2^5 \to 0$ with homology $\mathbb{Z}_2$. The complex is bifiltered, with the horizontal and vertical coordinates representing the filtrations levels of the generators. We will refer to the two filtrations levels as the $(i, j)$-filtrations levels. The complex has an absolute grading which is not indicated in the diagram; the generator at filtration level $(0, 6)$ has grading zero, and the boundary map lowers the grading by one. Thus, there are five generators at grading level zero and four at grading level one. We call the first set of generators type **A**, and the second type **B**.

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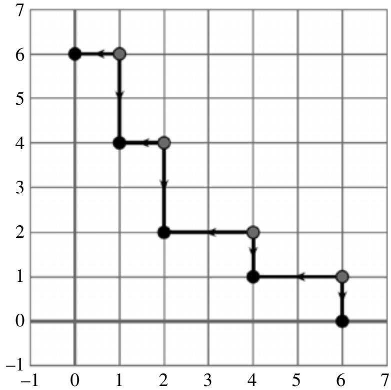


Figure 1. The staircase complex $\mathrm{St}(K)$ for the torus knot $T(3, 7)$.

We will refer to a complex such as this as a *staircase complex* of length $n$, $\mathrm{St}(v)$, where $v$ is a $(n-1)$-tuple of positive integers designating the length of the segments starting at the top left and moving to the bottom right in alternating right and downward steps. Furthermore, we require that the top left vertex lies on the vertical axis and the bottom right vertex lies on the horizontal axis. Thus, the illustration is of $\mathrm{St}(1, 2, 1, 2, 2, 1, 2, 1)$. The absolute grading of $\mathrm{St}(v)$ is defined by setting the grading of the top left generator to be equal to zero and the boundary map to lower the grading by one.

The vertices of $\mathrm{St}(K)$ will be denoted $\mathrm{Vert}(St(K))$. We shall write $\mathrm{Vert}_A(\mathrm{St}(K))$ to denote the set of type **A** vertices, and write $\mathrm{Vert}_B(\mathrm{St}(K))$ for the set of vertices of type **B**.

If $K$ is a knot admitting an $L$-space surgery, in particular an algebraic knot (see [10]), then it has Alexander polynomial of the form $\Delta_K(t) = \sum_{i=0}^{2m}(-1)t^{n_i}$. To such a knot we associate a staircase complex, $\mathrm{St}(K) = \mathrm{St}(n_{i+1} - n_i)$, where $i$ runs from zero to $2m - 1$. As an example, the torus knot $T(3, 7)$ has Alexander polynomial $1 - t + t^3 - t^4 + t^6 - t^8 + t^9 - t^{11} + t^{12}$. The corresponding staircase complex is $\mathrm{St}(1, 2, 1, 2, 2, 1, 2, 1)$.

Given any finitely generated bifiltered complex $S$, one can form a larger complex $S \otimes \mathbb{Z}_2[U, U^{-1}]$, with differentials defined by $\partial(x \otimes U^i) = (\partial x) \otimes U^i$. It is graded by $gr(x \otimes U^k) = gr(x) - 2k$. Similarly, if $x$ is at filtration level $(i, j)$, then $x \otimes U^i$ is at filtration level $(i - k, j - k)$. If $K$ admits an $L$-space surgery, then $\mathrm{St}(K) \otimes \mathbb{Z}_2[U, U^{-1}]$ is isomorphic to $CFK^\infty(K)$. Figure 2 illustrates a portion of $\mathrm{St}(T(3, 7)) \otimes \mathbb{Z}_2[U, U^{-1}]$, that is, a portion of the Heegaard Floer complex $CFK^\infty(T(3, 7))$.

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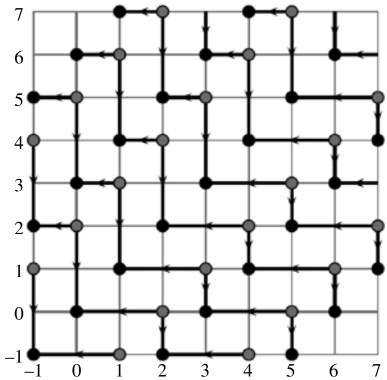


Figure 2. A portion of $CFK^\infty(T(3,7))$.

**4.2. $d$-invariants from $CFK^\infty(K)$.** We will not present the general definition of the $d$-invariant of a 3-manifold with Spin$^c$ structure; details can be found in [22]. However, in the case that a 3-manifold is of the form $S^3_q(K)$, where $q \geqslant 2(\text{genus}(K))$, there is a simple algorithm (originating from [23, Section 4]; we use the approach of [9, 11]) to determine this invariant from $CFK^\infty(K)$.

If $m$ satisfies $-d/2 \leqslant m \leqslant d/2$, one can form the quotient complex

$$CFK^\infty(K)/\,CFK^\infty(K)\{i < 0,\ j < m\}.$$

We let $d_m$ denote the least grading in which this complex has a nontrivial homology class, say $[z]$, where $[z]$ must satisfy the added constraint that, for all $i > 0$, $[z] = U^i[z_i]$ for some homology class $[z_i]$ of grading $d_m + 2i$.

In [23, Theorem 4.4], we find the following result.

THEOREM 4.1. *For the Spin$^c$ structure $\mathfrak{s}_m$, $d(S^3_q(K), \mathfrak{s}_m) = d_m + ((-2m+q)^2 - q)/4q$.*

**4.3. From staircase complexes to the $d$-invariants.** Let us now define a distance function for a staircase complex by the formula

$$J_K(m) = \min_{(v_1,v_2)\in \mathrm{Vert}(\mathrm{St}(K))} \max(v_1, v_2 - m),$$

where $v_1, v_2$ are coordinates of the vertex $v$. Observe that the minimum can always be taken with respect to the set of vertices of type **A**. The function $J_K(m)$ represents the greatest $r$ such that the region $\{i \leqslant 0,\ j \leqslant m\}$ intersects $\mathrm{St}(K) \otimes U^r$ nontrivially. It is immediately clear that $J_K(m)$ is a nonincreasing function. It is

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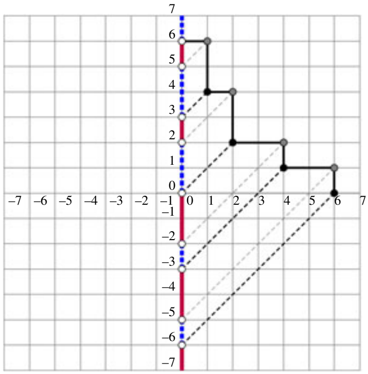


Figure 3. The function $J(m)$ for the knot $T(3,7)$. When $(0,m)$ lies on the dashed vertical intervals, the function $J(m)$ is constant; when it is on solid vertical intervals, the function $J(m)$ is decreasing. The dashed lines connecting vertices to points on the vertical axis indicate how the ends of dashed and solid intervals are constructed.

also immediate that for $m \geqslant g$ we have $J_K(m) = 0$. Figure 3 illustrates properties of the function $J_K(m)$ for the $K = T(3.7)$.

For the sake of the next lemma, we define $n_{-1} = -\infty$.

LEMMA 4.2. *Suppose that $m \leqslant g$. We have $J_K(m+1) - J_K(m) = -1$ if $n_{2i-1} - g \leqslant m < n_{2i} - g$ for some $i$, and $J_K(m+1) = J_K(m)$ otherwise.*

*Proof.* The proof is purely combinatorial. We order the type **A** vertices of $\mathrm{St}(K)$ so that the first coordinate is increasing, and we denote these vertices $v_0, \ldots, v_k$. For example, for $\mathrm{St}(T(3,7))$ as depicted in Figure 1, we have $v_0 = (0,6)$, $v_1 = (1,4)$, $v_2 = (2,2)$, $v_3 = (4,1)$, and $v_4 = (6,0)$. We denote by $(v_{i1}, v_{i2})$ the coordinates of the vertex $v_i$.

A verification of the two following facts is straightforward.

$$\begin{aligned} &\max(v_{i1}, v_{i2} - m) = v_{i1} \quad \text{if and only if} \quad m \geqslant v_{i1} - v_{i2}. \\ &\max(v_{i1}, v_{i2} - m) \geqslant \max(v_{i-1,1}, v_{i-1,2} - m) \quad \text{if and only if} \quad m \leqslant v_{i1} - v_{i-1,2}. \end{aligned} \tag{4.3}$$

By the definition of the staircase complex, we also have $v_{i1} - v_{i2} = n_{2i} - g$ and $v_{i1} - v_{i-1,2} = n_{2i-1} - g$. The second equation of (4.3) yields

$$J_K(m) = \max(v_{i1}, v_{i2} - m) \quad \text{if and only if} \quad m \in [n_{2i-1}, n_{2i+1}].$$

Then the first equation of (4.3) allows to compute the difference $J_K(m+1) - J_K(m)$. $\square$

The relationship between $J_K$ and the $d$-invariant is given by the next result.

PROPOSITION 4.4. *Let $K$ be an algebraic knot, let $q > 2g(K)$, and let $m \in [-q/2, q/2]$ be an integer. Then*

$$d(S^3_q(K), \mathfrak{s}_m) = \frac{(-2m+q)^2 - q}{4q} - 2J(m).$$

*Proof.* Denote by $S_i$ the subcomplex $\mathrm{St}(K) \otimes U^i$ in $CFK^\infty(K)$. The result depends on understanding the homology of the image of $S_i$ in $CFK^\infty(K)/CFK^\infty(K)\{i < 0, j < m\}$. Because of the added constraint (see the paragraph before Theorem 4.1), we only have to look at the homology classes supported on images of the type **A** vertices. Notice that, if $i > J_K(m)$, then at least one of the type **A** vertices is in $CFK^\infty(K)\{i < 0, j < m\}$. But all the type **A** vertices are homologous in $S_i$, and since these generate $H_0(S_i)$, the homology of the image in the quotient is zero. On the other hand, if $i \leqslant J_K(m)$, then none of the vertices of $S_i$ are in $CFK^\infty(K)\{i < 0, j < m\}$, and thus the homology of $S_i$ survives in the quotient.

It follows that the least grading of a nontrivial class in the quotient arises from the $U^{J_K(m)}$ translate of one of type **A** vertices of $S_0 = \mathrm{St}(K)$. Since $U$ lowers grading by two, the grading is $-2J_K(m)$. The result follows by applying the shift described in Theorem 4.1. $\square$

REMARK 4.5. Notice that, in the case that $q$ is even, the integer values $m = -q/2$ and $m = q/2$ are both in the given interval. One easily checks that Proposition 4.4 yields the same value at these two endpoints.

We now relate the $J$ function to the semigroup of the singular point. Let $I_K$ be the gap function as in Definition 2.6 and Remark 2.10.

PROPOSITION 4.6. *If $K$ is the link of an algebraic singular point, then, for $-g \leqslant m \leqslant g$, $J_K(m) = I_K(m+g)$.*

*Proof.* In Section 2.2, we described the gap sequence in terms of the exponents $n_i$. It follows immediately that the growth properties of $I_K(m+g)$ are identical to those of $J_K(m)$, as described in Lemma 4.2. Furthermore, since the largest element in the gap sequence is $2g-1$, we have $I_K(2g) = J_K(g) = 0$. $\square$

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**4.4. Proof of Theorem 1.1.** According to Lemma 3.1, the Spin$^c$ structures on $S^3_{d^2}(K)$ that extend to the complement $W$ of a neighborhood of $C$ are precisely those $\mathfrak{s}_m$ where $m = kd$ for some $k$, where $-d/2 \leqslant k \leqslant d/2$; here, $k \in \mathbb{Z}$ if $d$ odd, and $k \in \mathbb{Z} + \frac{1}{2}$ if $d$ is even. Since $W$ is a rational homology sphere, by [22, Proposition 9.9], the associated $d$-invariants are 0, so, by Proposition 4.4, letting $q = d^2$ and $m = kd$, we have

$$2J_K(kd) = \frac{(-2kd + d^2)^2 - d^2}{4d^2}.$$

By Proposition 4.6, we can replace this with

$$8I_{G_K}(kd + g) = (d - 2k - 1)(d - 2k + 1).$$

Now $g = d((d-3)/2) + 1$, so, by substituting $j = k + (d-3)/2$, we obtain

$$8I_K(jd + 1) = 4(d - j + 1)(d - j + 2),$$

and $j \in [-3/2, \ldots, d - 3/2]$ is an integer regardless of the parity of $d$. The proof is accomplished by recalling that $k_{jd} = I_K(jd + 1)$; see Remark 2.8.

## 5. Constraints on general rational cuspidal curves

**5.1. Products of staircase complexes and the $d$-invariants.** In the case that there is more than one cusp, the previous approach continues to apply, except the knot $K$ is now a connected sum of algebraic knots.

For the connected sum $K = \# K_i$, the complex $CFK^\infty(K)$ is the tensor product of the $CFK^\infty(K_i)$. To analyze this, we consider the tensor product of the staircase complexes $\mathrm{St}(K_i)$. Although this is not a staircase complex, the homology is still $\mathbb{Z}_2$, supported at grading level zero. For the tensor product, we shall denote by $\mathrm{Vert}(\mathrm{St}(K_1) \otimes \cdots \otimes \mathrm{St}(K_n))$ the set of vertices of the corresponding complex. These are of the form $v_1 + \cdots + v_n$, where $v_j \in \mathrm{Vert}(K_j)$, $j = 1, \ldots, n$.

Any element of the form $a_{1q_1} \otimes a_{2q_2} \otimes \cdots \otimes a_{nq_n}$ represents a generator of the homology of the tensor product, where the $a_{iq_i}$ are vertices of type **A** taken from each $\mathrm{St}(K_i)$. Furthermore, if the translated subcomplex $\mathrm{St}(K) \otimes U^i \subset \mathrm{St}(K) \otimes \mathbb{Z}_2[U, U^{-1}]$ intersects $CFK^\infty(K)\{i < 0, j < m\}$ nontrivially, then the intersection contains one of these generators. Thus, the previous argument applies to prove the following.

Proposition 5.1. *Let $q > 2g - 1$, where $g = g(K)$ and $m \in [-q/2, q/2]$. Then we have*

$$d(S^3_q(K), \mathfrak{s}_m) = -2J_K(m) + \frac{(-2m + q)^2 - q}{4q},$$

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*where* $J_K(m)$ *is the minimum of* $\max(\alpha, \beta - m)$ *over all elements of form* $a_{1q_1} \otimes a_{2q_2} \otimes \cdots \otimes a_{nq_n}$, *where* $(\alpha, \beta)$ *is the filtration level of the corresponding element.*

Since the $d$-invariants vanish for all Spin$^c$ structures that extend to $W$, we have the following.

THEOREM 5.2. *If $C$ is a rational cuspidal curve of degree $d$ with singular points $K_i$ and $K = \#K_i$, then, for all $k$ in the range* $[-d/2, d/2]$, *with $k \in \mathbb{Z}$ for $d$ odd and $k \in \mathbb{Z} + \frac{1}{2}$ for $d$ even,*

$$J_K(kd) = \frac{(d-2k-1)(d-2k+1)}{8}.$$

*Proof.* We have, from the vanishing of the $d$-invariants, $d(S^3_{d^2}(K), \mathfrak{s}_m)$ (for $m = kd$), the condition

$$J_K(m) = \frac{(-2m+d^2)^2 - d^2}{8d^2}.$$

The result then follows by substituting $m = kd$ and performing algebraic simplifications. □

## 5.2. Restatement in terms of $I_{K_i}(m)$.

We now wish to restate Theorem 5.2 in terms of the coefficients of the Alexander polynomial, properly expanded. As before, for the gap sequence for the knot $K_i$, denoted $G_{K_i}$, let

$$I_i(s) = \#\{k \geqslant s : k \in G_{K_i} \cup \mathbb{Z}_{<0}\}.$$

For two functions $I, I' : \mathbb{Z} \to \mathbb{Z}$ bounded below, we define the following operation:

$$I \diamond I'(s) = \min_{m \in \mathbb{Z}} I(m) + I'(s-m). \tag{5.3}$$

As pointed out to us by Krzysztof Oleszkiewicz, in real analysis this operation is sometimes called the *infimum convolution*.

The following is the main result of this article.

THEOREM 5.4. *Let $C$ be a rational cuspidal curve of degree $d$. Let $I_1, \ldots, I_n$ be the gap functions associated to each singular point on $C$. Then, for any $j \in \{-1, 0, \ldots, d-2\}$, we have*

$$I_1 \diamond I_2 \diamond \cdots \diamond I_n(jd+1) = \frac{(j-d+1)(j-d+2)}{2}.$$

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REMARK 5.5.

- For $j = -1$, the left-hand side is $d(d-1)/2 = d - 1 + (d-1)(d-2)/2$. The meaning of the equality is that $\sum \# G_j = (d-1)(d-2)/2$, which follows from (2.1) and Lemma 2.4. Thus, the case $j = -1$ does not provide any new constraints. Likewise, for $j = d - 2$, both sides are equal to zero.

- We refer to Section 6.2 for a reformulation of Theorem 5.4.

- The relation of Theorem 5.4 to Conjecture 1.2 for $n > 1$ is more complicated. It is shown in [**17**] and [**2**] (see also [**20**, Proposition 7.1.3]) that Theorem 5.4 implies Conjecture 1.2 in the case when $n = 2$. For $n > 2$, Conjecture 1.2 is false in general. An explicit counterexample with $n = 3$ is given in [**2**, Example 2.2.4]. We refer to that paper also for a detailed discussion of the relation of Theorem 5.4 to Conjecture 1.2, as well as for a formulation of a new variant of Conjecture 1.2, [**2**, Conjecture 2.1.3]. This new conjecture does not follow from Theorem 5.4.

Theorem 5.4 is an immediate consequence of the arguments in Section 4.4 together with the following proposition.

PROPOSITION 5.6. *As in (5.3), let $I_K$ be given by $I_1 \diamond \cdots \diamond I_n$, for the gap functions $I_1, \ldots, I_n$. Then $J_K(m) = I_K(m + g)$.*

*Proof.* The proof follows by induction over $n$. For $n = 1$, the statement is equivalent to Proposition 4.6. Suppose we have proved it for $n - 1$. Let $K' = K_1 \# \cdots \# K_{n-1}$, and let $J_{K'}(m)$ be the corresponding $J$ function. Let us consider a vertex $v \in \mathrm{Vert}(\mathrm{St}_1(K) \otimes \cdots \otimes \mathrm{St}_n(K))$. We can write this as $v' + v_n$, where $v' \in \mathrm{Vert}(\mathrm{St}(K_1) \otimes \cdots \otimes \mathrm{St}(K_{n-1}))$ and $v_n \in \mathrm{Vert}(\mathrm{St}(K_n))$. We write the coordinates of the vertices as $(v_1, v_2)$, $(v_1', v_2')$ and $(v_{n1}, v_{n2})$, respectively. We have $v_1 = v_1' + v_{n1}$, $v_2 = v_2' + v_{n2}$. We shall need the following lemma.

LEMMA 5.7. *For any four integers $x, y, z, w$, we have*

$$\max(x + y, z + w) = \min_{k \in \mathbb{Z}}(\max(x, z - k) + \max(y, w + k)).$$

*Proof of Lemma 5.7.* The direction '$\leqslant$' is trivial. The equality is attained at $k = z - x$. $\square$

*Continuation of the proof of Proposition 5.6.*

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Applying Lemma 5.7 to $v_1', v_2', v_{n1}, v_{n2} - m$, and taking the minimum over all vertices $v$, we obtain

$$\begin{aligned} J_K(m) &= \min_{v \in \mathrm{Vert}(\mathrm{St}(K_1)\otimes\cdots\otimes\mathrm{St}(K_n))} \max(v_1, v_2 - m) \\ &= \min_{v' \in \mathrm{Vert}'} \min_{v_n \in \mathrm{Vert}_n} \min_{k\in\mathbb{Z}} (\max(v_1', v_2' - k) + \max(v_{n1}, v_{n2} + k - m)), \end{aligned}$$

where we denote $\mathrm{Vert}' = \mathrm{Vert}(\mathrm{St}(K_1)\otimes\cdots\otimes\mathrm{St}(K_{n-1}))$ and $\mathrm{Vert}_n = \mathrm{Vert}(\mathrm{St}(K_n))$. The last expression is clearly $\min_{k\in\mathbb{Z}} J_{K'}(k) + J_{K_n}(m-k)$. By the induction assumption, this is equal to

$$\min_{k\in\mathbb{Z}} I_{K'}(k + g') + I_{K_n}(m - k + g_n) = I_K(m + g),$$

where $g' = g(K')$ and $g_n = g(K_n)$ are the genera, and we use the fact that $g = g' + g_n$. □

## 6. Examples and applications

**6.1. A certain curve of degree six.** As described, for instance, in [7, Section 2.3, Table 1], there exists an algebraic curve of degree six with two singular points, the links of which are $K = T(4,5)$ and $K' = T(2,9)$. The values of $I_K(m)$ for $m \in \{0, \dots, 11\}$ are $\{6, 6, 5, 4, 3, 3, 3, 2, 1, 1, 1, 1\}$. The values of $I_{K'}(m)$ for $m \in \{0, \dots, 7\}$ are $\{4, 4, 3, 3, 2, 2, 1, 1\}$. We readily get

$$I \diamond I'(1) = 10, \quad I \diamond I'(7) = 6, \quad I \diamond I'(13) = 3, \quad I \diamond I'(19) = 1,$$

exactly as predicted by Theorem 5.4.

On the other hand, the computations in [7] confirm Conjecture 1.2, but we sometimes have an inequality. For example, $k_6 = 5$, whereas Conjecture 1.2 states that $k_6 \leqslant 6$. This shows that Theorem 5.4 is indeed more precise.

**6.2. Reformulations of Theorem 5.4.** Theorem 5.4 was formulated in a way that fits best with its theoretical underpinnings. In some applications, it is advantageous to reformulate the result in terms of the function counting semigroup elements in the interval $[0, k]$. To this end, we introduce some notation.

Recall that, for a semigroup $S \subset \mathbb{Z}_{\geqslant 0}$, the gap sequence of $G$ is $\mathbb{Z}_{\geqslant 0} \setminus S$. We put $g = \#G$, and for $m \geqslant 0$ we define

$$R(m) = \#\{j \in S \colon j \in [0, m)\}. \tag{6.1}$$

LEMMA 6.2. *For $m \geqslant 0$, $R(m)$ is related to the gap function $I(m)$ (see (2.7)) by the following relation:*

$$R(m) = m - g + I(m). \tag{6.3}$$

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*Proof.* Let us consider an auxiliary function $K(m) = \#\{j \in [0, m) : j \in G\}$. Then $K(m) = g - I(m)$. Now $R(m) + K(m) = m$, which completes the proof. $\square$

We extend $R(m)$ by (6.3) for all $m \in \mathbb{Z}$. We remark that $R(m) = m - g$ for $m > \sup G$, and $R(m) = 0$ for $m < 0$. In particular, $R$ is a nonnegative, nondecreasing function.

We have the following result.

LEMMA 6.4. *Let $I_1, \ldots, I_n$ be the gap functions corresponding to the semigroups $S_1, \ldots, S_n$. Let $g_1, \ldots, g_n$ be given by $g_j = \#\mathbb{Z}_{\geqslant 0} \backslash S_j$. Let $R_1, \ldots, R_n$ be as in (6.1). Then*

$$R_1 \diamond R_2 \diamond \cdots \diamond R_n(m) = m - g + I_1 \diamond \cdots \diamond I_n(m),$$

*where $g = g_1 + \cdots + g_n$.*

*Proof.* To simplify the notation, we assume that $n = 2$; the general case follows by induction. We have

$$\begin{aligned} R_1 \diamond R_2(m) &= \min_{k \in \mathbb{Z}} R_1(k) + R_2(m - k) \\ &= \min_{k \in \mathbb{Z}} (k - g_1 + I_1(k) + m - k - g_2 + I_2(m - k)) \\ &= m - g_1 - g_2 + I_1 \diamond I_2(m). \end{aligned}$$ $\square$

Now we can reformulate Theorem 5.4.

THEOREM 6.5. *For any rational cuspidal curve of degree $d$ with singular points $z_1, \ldots, z_n$, and for $R_1, \ldots, R_n$ the functions as defined in (6.1), one has that, for any $j = \{-1, \ldots, d - 2\}$,*

$$R_1 \diamond R_2 \diamond \cdots \diamond R_n(jd + 1) = \frac{(j + 1)(j + 2)}{2}.$$

This formulation follows from Theorem 5.4 by an easy algebraic manipulation, together with the observation that, by (2.1) and Lemma 2.4, the quantity $g$ from Lemma 6.4 is given by $((d - 1)(d - 2))/2$.

The formula bears strong resemblance to [7, Proposition 2], but in that article only the '$\geqslant$' part is proved, and an equality in the case when $n = 1$ is conjectured.

REMARK 6.6. Observe that, by definition,

$$R_1 \diamond \cdots \diamond R_n(k) = \min_{\substack{k_1, \ldots, k_n \in \mathbb{Z} \\ k_1 + \cdots + k_n = k}} R_1(k_1) + \cdots + R_n(k_n).$$

Since for negative values $R_j(k) = 0$ and $R_j$ is nondecreasing on $[0, \infty)$, the minimum will always be achieved for $k_1, \ldots, k_n \geqslant -1$.

**6.3. Applications.** From Theorem 6.5, we can deduce many general estimates for rational cuspidal curves. Throughout this subsection we shall be assuming that $C$ has degree $d$, its singular points are $z_1, \ldots, z_n$, the semigroups are $S_1, \ldots, S_n$, and the corresponding $R$-functions are $R_1, \ldots, R_n$. Moreover, we assume that the characteristic sequence of the singular point $z_i$ is $(p_i; q_{i1}, \ldots, q_{ik_i})$. We order the singular points so that $p_1 \geqslant p_2 \geqslant \cdots \geqslant p_n$.

We can immediately prove the result of Matsuoka and Sakai, [13], following the ideas in [7, Section 3.5.1].

PROPOSITION 6.7. *We have $p_1 > d/3$.*

*Proof.* Suppose that $3p_1 \leqslant d$. It follows that, for any $j$, $3p_j \leqslant d$. Let us choose $k_1, \ldots, k_n \geqslant -1$ such that $\sum k_j = d + 1$. For any $j$, the elements $0, p_j, 2p_j, \ldots$ all belong to the $S_j$. The function $R_j(k_j)$ counts elements in $S_j$ strictly smaller than $k_j$; hence, for any $\varepsilon > 0$, we have

$$R_j(k_j) \geqslant 1 + \left\lfloor \frac{k_j - \varepsilon}{p_j} \right\rfloor.$$

Using $3p_j \leqslant d$, we rewrite this as $R_j(k_j) \geqslant 1 + \lfloor (3k_j - 3\varepsilon)/d \rfloor$. Since $\varepsilon > 0$ is arbitrary, setting $\delta_j = 1$ if $d | 3k_j$, and zero otherwise, we write

$$R_j(k_j) \geqslant 1 + \left\lfloor \frac{3k_j}{d} \right\rfloor - \delta_j.$$

We get

$$\sum_{j\,:\, d|3k_j} R_j(k_j) \geqslant \left\lfloor \frac{\sum 3k_j}{d} \right\rfloor. \tag{6.8}$$

Using the fact that $\lfloor a/d \rfloor + \lfloor b/d \rfloor \geqslant \lfloor (a+b)/d \rfloor - 1$ for any $a, b \in \mathbb{Z}$, we estimate the other terms:

$$\sum_{j\,:\, d\nmid 3k_j} R_j(k_j) \geqslant 1 + \left\lfloor \frac{3\sum k_j}{d} \right\rfloor. \tag{6.9}$$

Since $\sum k_j = d + 1$, there must be at least one $j$ for which $d$ does not divide $3k_j$. Hence, adding (6.8) to (6.9), we obtain

$$R_1(k_1) + \cdots + R_n(k_n) \geqslant 1 + \left\lfloor \frac{\sum_{j=1}^n 3k_j}{d} \right\rfloor = 1 + \left\lfloor \frac{3d+3}{d} \right\rfloor = 4.$$

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This contradicts Theorem 6.5 for $j = 1$, and the contradiction concludes the proof. $\square$

We also have the following simple result.

PROPOSITION 6.10. *Suppose that $p_1 > (d+n-1)/2$. Then $q_{11} < d+n-1$.*

*Proof.* Suppose that $p_1 > (d+n-1)/2$ and $q_{11} > d+n-1$. It follows that $R_1(d+n) = 2$. But then we choose $k_1 = d+n$, $k_2 = \cdots = k_n = -1$, and we get $\sum_{j=1}^{n} R_j(k_j) = 2$; hence

$$R_1 \diamond R_2 \diamond \cdots \diamond R_n(d+1) \leqslant 2,$$

contradicting Theorem 6.5. $\square$

**6.4. Some examples and statistics.** We will now present some examples and statistics, where we compare our new criterion with the semicontinuity of the spectrum as used in [**7**, Property $(SS_l)$] and the Orevkov criterion [**21**, Corollary 2.2]. It will turn out that the semigroup distribution property is quite strong and is closely related to the semicontinuity of the spectrum, but they are not the same. There are cases which pass one criterion and fail to another. Checking the semigroup property is definitely a much faster task than comparing spectra; refer to [**6**, Section 3.6] for more examples.

EXAMPLE 6.11. Among the 1920 593 cuspidal singular points with Milnor number of the form $(d-1)(d-2)$ for $d$ ranging between 8 and 64, there are only 481 that pass the semigroup distribution criterion, that is, Theorem 1.1. All of these pass the Orevkov criterion $\overline{M} < 3d-4$. Of those 481, we compute that 475 satisfy the semicontinuity of the spectrum condition and 6 of them fail the condition; these are $(8; 28, 45)$, $(12; 18, 49)$, $(16; 56, 76, 85)$, $(24; 36, 78, 91)$, $(24; 84, 112, 125)$, $(36; 54, 114, 133)$.

REMARK 6.12. The computations in Example 6.11 were made on a PC computer during one afternoon. Applying the spectrum criteria for all these cases would take much longer. The computations for degrees between 12 and 30 is approximately 15 times faster for semigroups; the difference seems to grow with the degree. The reason is that, even though the spectrum can be given explicitly from the characteristic sequence (see [**26**]), it is a set of fractional numbers and the algorithm is complicated.

EXAMPLE 6.13. There are 28 cuspidal singular points with Milnor number equal to $110 = (12-1)(12-2)$. We ask the following question: which of these singular points can possibly occur as a unique singular point on a degree 12 rational curve?

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Table 1. Semigroup property for cuspidal singular points with Milnor number 12. If a cuspidal singular point fails the semigroup criterion, we indicate the first $j$ for which $I(12j+1) \neq ((j-d+1)(j-d+2))/2$.

| | | | | | |
|---|---|---|---|---|---|
| (3; 56) | Fails at $j=1$ | (6; 9, 44) | Fails at $j=1$ | (8; 12, 14, 41) | Fails at $j=3$ |
| (4; 6, 101) | Fails at $j=1$ | (6; 10, 75) | Fails at $j=1$ | (8; 12, 18, 33) | Fails at $j=4$ |
| (4; 10, 93) | Fails at $j=1$ | (6; 14, 59) | Fails at $j=2$ | (8; 12, 22, 25) | **Passes** |
| (4; 14, 85) | Fails at $j=1$ | (6; 15, 35) | Fails at $j=2$ | (8; 12, 23) | **Passes** |
| (4; 18, 77) | Fails at $j=1$ | (6; 16, 51) | Fails at $j=2$ | (8; 14, 33) | Fails at $j=1$ |
| (4; 22, 69) | Fails at $j=1$ | (6; 20, 35) | Fails at $j=4$ | (9; 12, 23) | **Passes** |
| (4; 26, 61) | Fails at $j=1$ | (6; 21, 26) | **Passes** | (10; 12, 23) | **Passes** |
| (4; 30, 53) | Fails at $j=1$ | (6; 22, 27) | **Passes** | (11; 12) | **Passes** |
| (4; 34, 45) | Fails at $j=1$ | (6; 23) | **Passes** | | |
| (6; 8, 83) | Fails at $j=1$ | (8; 10, 57) | Fails at $j=2$ | | |

We apply the semigroup distribution criterion. Only eight singular points pass the criterion, as is seen in Table 1.

Among the curves in Table 1, all those that are obstructed by the semigroup distribution are also obstructed by the semicontinuity of the spectrum. The spectrum also obstructs the case of (8; 12, 23).

EXAMPLE 6.14. There are 2330 pairs $(a, b)$ of coprime integers, such that $(a-1)(b-1)$ is of form $(d-1)(d-2)$ for $d = 5, \ldots, 200$. Again we ask if there exists a degree $d$ rational cuspidal curve having a single singular point with characteristic sequence $(a; b)$. Among these 2330 cases, precisely 302 satisfy the semigroup distribution property. Out of these 302 cases, only one, namely (2; 13), does not appear on the list from [7]; see Section 2.3 for the list. It is therefore very likely that the semigroup distribution property alone is strong enough to obtain the classification of [7].

REMARK 6.15. After a preliminary version of this article appeared, Liu [12, Theorem 2.3] proved the classification result of [7] using the semigroup distribution property as the main tool.

EXAMPLE 6.16. In Table 2, we present all the cuspidal points with Milnor number $(30-1)(30-2)$ that satisfy the semicontinuity of the spectrum. Out of these, all but the three ((18; 42, 65), (18; 42, 64, 69), and (18; 42, 63, 48)) satisfy the semigroup property. All three fail the semigroup property for $j = 1$. In particular, for these three cases the semigroup property obstructs the cases which pass the semicontinuity of the spectrum criterion.

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Table 2. Cuspidal singular points with Milnor number 752 satisfying the semicontinuity of the spectrum criterion.

| | | | |
|---|---|---|---|
| (15; 55, 69) | (18; 42, 64, 69) | (20; 30, 59) | (25; 30, 59) |
| (15; 57, 71) | (18; 42, 63, 68) | (24; 30, 57, 62) | (27; 30, 59) |
| (15; 59) | (20; 30, 55, 64) | (24; 30, 58, 63) | (28; 30, 59) |
| (18; 42, 65) | (20; 30, 58, 67) | (24; 30, 59) | (29; 30) |

REMARK 6.17. We refer to the thesis of Liu, [12], for a thorough study of the semigroup conditions. In particular, there are listed all instances of singular points with two Puiseux pairs that satisfy the semigroup property.

EXAMPLE 6.18. The configuration of five critical points (2; 3), (2; 3), (2; 5), (5; 7), and (5; 11) passes the semigroup, the spectrum, and the Orevkov criterion for a degree 10 curve. In other words, none of the aforementioned criteria obstructs the existence of such curve. We point out that it is conjectured (see [15, 25]) that a rational cuspidal curve can have at most four singular points. In other words, these three criteria alone are insufficient to prove that conjecture.

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