EVEN CANONICAL SURFACES WITH SMALL K^2, III

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Introduction

This is a continuation of [5]. In the present part, we study irregular even surfaces of general type with $K^2 < 4\chi$, where K and χ denote respectively a canonical divisor and the holomorphic Euler-Poincaré characteristic. As the first main theorem, we show the following:

THEOREM 1. *For any irregular even surface of general type with $K^2 < 4\chi$, the image of the Albanese map is a curve.*

Therefore, Severi's conjecture is true for even surfaces. The proof uses the unramified covering trick and rests heavily on Xiao's theorem: We take a finite unramified covering of an even surface, which is again even, and show that the semi-canonical map induces a pencil. Then we apply [10, Theorem 3] and [11, Theorem 1] to see that the genus of the base of the pencil equals the irregularity.

We next turn our attention to canonical surfaces and give the genus bound on the Albanese pencil. As in [5, I, §5], we employ an argument modeled on [9] (see also [7]) to show the following:

THEOREM 2. *For any irregular even canonical surface with $K^2 < 4\chi$, the Albanese pencil is a trigonal pencil of genus $g \leq 6$. Furthermore,*

$$K^2 \geq \begin{cases} 3\chi + 10(q-1) & \text{if } g = 3, \\ \dfrac{8g(g-1)}{g^2 + 7g - 16}\left(\chi + 8\left(1 - \dfrac{2}{g}\right)(g-1)\right) & \text{if } 4 \leq g \leq 6. \end{cases}$$

As we noticed in [5, I, §8], the existence of a trigonal pencil on a canonical surface implies that the canonical image cannot be cut out by hyperquadrics. Thus Theorem 2 and [5, I, Lemma 8.2 and Theorem 8.3] give us

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THEOREM 3.   *The canonical image of an irregular even canonical surface with* $K^2 < 4\chi$ *is not cut out by hyperquadrics. If the numerical characters further satisfy* $K^2 \leq 4p_g + q - 12$, *then the quadric hull of the surface is of dimension 3. In particular, Reid's conjecture* [8, p.541] *is true for irregular even surfaces with* $q \leq 4$.

1.   Inequality

In this section, we shall show some inequalities generalizing one in [5, I, §5] by using a method modeled on [9]. See also [7].

Let S be an even surface of general type. Note that it is automatically minimal, and we have a line bundle L on S satisfying $K = 2L$. Such a line bundle L is called a semi-canonical bundle of S. Suppose that we have a fibration $f : S \to B$ over a non-singular projective curve B of genus b. Let g be the genus of a general fibre D of f. Since B is a curve, f_*L is a locally free sheaf [2, Corollary 1.7].

LEMMA 1.1. $h^0(L) = h^0(f_*L)$ *and* $\chi(L) = 2\chi(f_*L) - \text{length}(R^1f_*L)_{tor}$.

Proof.   By the relative duality theorem, the dual of R^1f_*L is isomorphic to $f_*L \otimes \omega_B^{\otimes(-1)}$ since $K = 2L$. Then, using the Leray spectral sequence $E_2^{p,q} = H^p(R^qf_*L) \Rightarrow H^{p+q}(L)$, we get the assertion.   Q.E.D.

Let $\mathscr{E}$ be the subbundle of f_*L generically generated by global sections. Then $h^0(B, \mathscr{E}) = h^0(L)$. Let

$$0 = \mathscr{E}_0 \subset \mathscr{E}_1 \subset \cdots \subset \mathscr{E}_l = \mathscr{E}$$

be the Harder–Narashimhan filtration of $\mathscr{E}$ [1], that is, the unique filtration of $\mathscr{E}$ by subbundles $\mathscr{E}_i$ which satisfies

(i) $\mathscr{E}_i/\mathscr{E}_{i-1}$ is semi-stable,
(ii) $\mu_i > \mu_{i+1}$, where $\mu_i := \mu(\mathscr{E}_i/\mathscr{E}_{i-1}) = \deg(\mathscr{E}_i/\mathscr{E}_{i-1})/\text{rk}(\mathscr{E}_i/\mathscr{E}_{i-1})$.

Let $\phi_i : \mathbf{P}(\mathscr{E}_i) \to B$ be the associated projective bundle. We denote by H_i and F a (relatively ample) tautological divisor and a fibre of ϕ_i, respectively. By [4, Lemma 4.6], the $\mathbf{Q}$-divisors $H_i - \mu_i F$ and $H_i - \mu_1 F$ are respectively nef and pseudo-effective.

The natural sheaf homomorphism $f^*\mathscr{E} \to f^*f_*L \to L$ induces a rational map $\psi_i : S \to \mathbf{P}(\mathscr{E}_i)$ such that $f = \phi_i \circ \psi_i$. Let $\rho_i : X_i \to S$ be a composite of blowing-ups which eliminates the indeterminacy of ψ_i, and let $\tilde{\psi}_i : X_i \to \mathbf{P}(\mathscr{E}_i)$ be the induced holomorphic map. We denote by $\tilde{M}_i$ the pull-back to X_i of H_i via $\tilde{\psi}_i$.

Then $\tilde{M}_i - \mu_i \rho_i^* D$ is nef, since so is $H_i - \mu_i F$. Put $M_i = (\rho_i)_* \tilde{M}_i$. Then $M_i - \mu_i D$ is also nef. Note that we have $L \equiv M_i + Z_i$ with an effective divisor Z_i, where the symbol $\equiv$ means the numerical equivalence. Put $d_i = M_i D$. When $i = l$, we drop the subscript l and write d, M, Z, instead of d_l, M_l, Z_l for the sake of simplicity.

Put $a_i = g - 1 - d_i = DZ_i$. Then the multiplicity of any irreducible component of Z_i having positive intersection with D is at most a_i. Using this and the nefness of $K_{S/B}$, we can show that $a_i K_{S/B} + M_i - \mu_i D + Z_i$ is a nef $\mathbf{Q}$-divisor as in [9, Lemma 4 and Corollary 1] (see also [7, Lemma 2.1]). Since $K = 2L$, we get the following:

LEMMA 1.2. *For each i, $(2a_i + 1)L - (\mu_i + 2a_i(b - 1))D$ is nef.*

LEMMA 1.3. *Let the notation and assumption be as above.*
(1) *If* $\mathrm{rk}(\mathscr{E}) = 1$, *then*

$$L^2 \geq \frac{2(g-1)}{2g-1}\{g \deg(\mathscr{E}) + (g-1)(b-1)\}.$$

(2) *If* $\mathrm{rk}(\mathscr{E}) = 2$, *then*

$$L^2 \geq \frac{g-1}{2g-d-1}\{g \deg(\mathscr{E}) + 2(g-d-1)(b-1)\}.$$

Proof. Since $L - \mu_1 D$ is pseudo-effective, we have $((2a_i + 1)L - (\mu_i + 2a_i(b-1))D)(L - \mu_1 D) \geq 0$ which is equivalent to

$$(1.1) \qquad (2a_i + 1)L^2 \geq (g-1)((2a_i + 1)\mu_1 + \mu_i + 2a_i(b-1)).$$

If $\mathscr{E}$ is a line bundle, then (1.1) for $i = 1$ gives (1) since $d = 0$ in this case.

Assume that $\mathscr{E}$ is of rank 2. When $\mathscr{E}$ is semi-stable, we put $\mu_1 = \mu_2 = \mu(\mathscr{E})$ in the following calculation. If $d = g - 1$, then we get (2) using (1.1) for $i = 2$. Hence we assume that $a := g - 1 - d > 0$. We have

$$\begin{aligned} L^2 &= L(M - \mu_2 D) + (g-1)\mu_2 + LZ \\ &= (M - \mu_1 D + Z_1)(M - \mu_2 D) + d\mu_1 + (g-1)\mu_2 + LZ \\ &= (g-1-a)\mu_1 + (g-1)\mu_2 + LZ. \end{aligned}$$

Since $(2a+1)L - (\mu_2 + 2a(b-1))D$ is nef, we have $(2a+1)LZ \geq a(\mu_2 + 2a(b-1))$.
Hence

$$(1.2)\ (2a+1)L^2 \geq (2a+1)(g-1-a)\mu_1 + ((2a+1)(g-1)+a)\mu_2 + 2a^2(b-1)$$

from which we get (2) if $(2g-1)\mu_2 \geq (2a+1)\mu_1 + 2d(b-1)$. Otherwise, (1.1) for $i=2$ is sufficient to imply (2), since $\deg(\mathscr{E}) = \mu_1 + \mu_2$.   Q.E.D.

In order to estimate $\deg(\mathscr{E})$, we introduce a quotient bundle $\mathscr{F}$ of $\mathscr{E}$ according to [9]. Let $\mathscr{F}'$ be the subbundle of $\mathscr{E}^* := \operatorname{Hom}(\mathscr{E}, \omega_B)$ generically generated by $H^0(\mathscr{E}^*)$, and put $\mathscr{F} = (\mathscr{F}')^*$. Then $\mathscr{F}$ and $\mathscr{F}^*$ are both nef, and we have $h^1(\mathscr{E}) = h^1(\mathscr{F}) = h^0(\mathscr{F}^*)$. Hence Clifford's theorem [9] gives us

$$h^1(\mathscr{E}) \leq \frac{1}{2}\deg(\mathscr{F}^*) + \operatorname{rk}(\mathscr{F}^*) = b\operatorname{rk}(\mathscr{F}) - \frac{1}{2}\deg(\mathscr{F}).$$

Since $h^0(\mathscr{E}) = h^0(L)$, the Riemann–Roch theorem gives us the following:

LEMMA 1.4.   If $\operatorname{rk}(\mathscr{F}) = \operatorname{rk}(\mathscr{E})$, then $\deg(\mathscr{E}) \geq 2(h^0(L) - \operatorname{rk}(\mathscr{E}))$. If $\operatorname{rk}(\mathscr{F}) < \operatorname{rk}(\mathscr{E})$, then $\deg(\mathscr{E}) \geq h^0(L) - \operatorname{rk}(\mathscr{E}) + b(\operatorname{rk}(\mathscr{E}) - \operatorname{rk}(\mathscr{F})) + \deg(\mathscr{F})/2$.

2. Proof of Theorem 1

In what follows, let S be an irregular even surface of general type satisfying $K^2 < 4\chi(\mathscr{O}_S)$, and let L be a semi-canonical bundle. In this section, we show Theorem 1.

Since S is an irregular surface, we can find an m-torsion element $\eta \in \operatorname{Pic}^0(S)$ for any positive integer m. Let $\pi_m: S_m \to S$ be the cyclic m-sheeted unramified covering associated with η. If K_m denotes the canonical bundle of S_m, then $K_m = \pi_m^* K$. Therefore, S_m is also an even surface of general type with a semi-canonical bundle $L_m = \pi_m^* L$. By the universality of the Albanese map, we have a commutative diagram

$$\begin{array}{ccc} S_m & \to & \operatorname{Alb}(S_m) \\ \pi_m \downarrow & & \downarrow \\ S & \to & \operatorname{Alb}(S), \end{array}$$

where the horizontal maps are the Albanese maps. Hence, in order to prove Theorem 1, it suffices to show that the Albanese image of S_m is a curve for some m.

By the Riemann–Roch theorem, we have

$$(2.1) \qquad\qquad \chi(L_m) = 2h^0(L_m) - h^1(L_m) = -\frac{L_m^2}{2} + \chi_m,$$

where we put $\chi_m = \chi(\mathcal{O}_{S_m})$. Since $K^2 = 4L^2$, we have $K^2 \leq 4\chi - 4$. Hence

$$(2.2) \qquad\qquad K_m^2 = mK^2 \leq 4m(\chi - 1) = 4\chi_m - 4m.$$

Put $n = n(m) = h^0(L_m) - 1$. Since S_m is an even surface of general type. L_m^2 is a positive even integer. Hence there is an integer k satisfying $L_m^2 = 4n - 2k$. By (2.2), we have $\chi_m \geq L_m^2 + m = 4n - 2k + m$. Then it follows from (2.1) that $k \geq m - 2 + h^1(L_m) \geq m - 2$. Thus

$$(2.3) \qquad\qquad L_m^2 \leq 4n + 4 - 2m.$$

Since L_m^2 is a positive even integer, we have $n > 0$ when $m \geq 2$. Assuming that m is sufficiently large, we consider the semi-canonical map, that is, the rational map $\Phi_m : S_m \to \mathbf{P}^n$ associated with $|L_m|$.

Case 1. Assume that $|L_m|$ is composed of a pencil for some $m \geq 4$. We have $L_m \equiv \nu D_m + G_m$, where $\{D_m\}$ is a pencil of irreducible curves, G_m is the fixed part of $|L_m|$ and ν is a positive integer with $\nu \geq n$.

We show that $\{D_m\}$ is a non-linear pencil. Since $L_m^2 \geq 2$, we have $n > 1$ by (2.3). We have

$$4n + 4 - 2m \geq L_m^2 = \nu L_m D_m + L_m G_m \geq n L_m D_m.$$

Hence $L_m D_m \leq 3$. Since S_m is an even surface, D_m^2 is a non-negative even integer. From

$$3 \geq L_m D_m = \nu D_m^2 + D_m G_m \geq n D_m^2 \geq 2 D_m^2$$

it follows $D_m^2 = 0$. Therefore, $\{D_m\}$ is a pencil of curves of genus $g_m \leq 4$. If it were a linear pencil, then the numerical characters must satisfy

$$(2.4) \qquad\qquad K_m^2 \geq 4\chi_m - 4(g_m - 1)$$

by a result of Xiao [11]. Since $m \geq 4$ and $g_m \leq 4$, this is impossible by (2.2).

Hence $\{D_m\}$ defines a holomorphic map $f_m : S_m \to B_m$ onto a non-singular projective curve B_m of positive genus. Then, by [11] again, we know that B_m is a curve of genus $q_m := q(S_m)$. It follows that the Albanese image of S_m is a curve, since we have a commutative diagram

$$(2.5) \qquad \begin{array}{ccc} S_m & \to & \mathrm{Alb}(S_m) \\ f_m \downarrow & & \downarrow \\ B_m & \to & \mathrm{Jac}(B_m), \end{array}$$

where $\mathrm{Jac}(B_m)$ denotes the Jacobian of B_m.

Case 2. Suppose that $|L_m|$ is not composed of a pencil and $m \geq 8$. Since we have $L_m^2 < 4n - 6$ by (2.3), it follows from [5, I, Lemma 2.1] that the semi-canonical map Φ_m is not birational onto its image. Let V_m be the image of Φ_m. Since $\deg V_m \geq n - 1$, we have $L_m^2 \geq (\deg \Phi_m)(\deg V_m) \geq (\deg \Phi_m)(n - 1)$. It follows from (2.3) that $\deg \Phi_m = 2, 3$. Furthermore, since

$$\deg V_m \leq \frac{1}{2} L_m^2 \leq 2n + 2 - m < 2n - 2,$$

V_m is birationally equivalent to a ruled surface as is well-known (see e.g., [3, Theorem 1.1]).

Assume that $\deg \Phi_m = 2$. Since V_m is a ruled surface, Φ_m induces on S_m a pencil of hyperelliptic curves. Since we have $K_m^2 < 4\chi_m - 28$ by (2.2), [10, Theorem 3] shows that the Albanese image of S_m is a curve.

Assume that $\deg \Phi_m = 3$. We claim that V_m is not a rational surface. Assume the contrary. We can show that V_m is not isomorphic to $\mathbf{P}^2$ as in [5, I, Lemma 4.3]. Since we have $\deg V_m < (4/3)(n - 2)$, it follows from [9, Lemma 1] that V_m is ruled by straight lines. Then, as in [5, I, Theorem 5.1], one can show that a ruling of V_m induces on S_m a linear pencil of curves of genus $g_m \leq 6$ without base points. On the other hand, it follows from [11] that the numerical characters of S_m must satisfy the inequality (2.4), which is impossible by (2.2). Thus we have shown that V_m is birationally equivalent to an irrational ruled surface.

We consider the unique ruling $V_m \to B_m$, and let f_m be the composite $S_m \to V_m \to B_m$. Since B_m is not $\mathbf{P}^1$, f_m is a holomorphic map. It follows from [11] that B_m is of genus q_m, and we can identify f_m with the Albanese map of S_m.

In sum, Theorem 1 has been established.

3.   Proof of Theorem 2

In this section, we shall prove Theorem 2. For this purpose, we freely use the notation in the previous sections. Recall that a minimal surface is called a canonical surface if its canonical map is a birational map onto the image. We remark the following:

LEMMA 3.1.   *Let $\tilde{X}$ be any finite unramified covering of a canonical surface X. Then $\tilde{X}$ has no pencils of hyperelliptic curves.*

Proof. If $\tilde{X}$ has a hyperelliptic pencil, then its canonical image is a ruled surface or a curve. This is impossible, since it dominates the canonical image

of X. Q.E.D.

Now, let S be an irregular even canonical surface with $K^2 < 4\chi$. As we saw in Section 2, for a sufficiently large integer m, the semi-canonical map induces a fibration $f_m : S_m \to B_m$ of genus g_m, where B_m is of genus $q_m > 0$. Lemma 3.1 in particular implies that $\deg \Phi_m = 3$ in Case 2 of Section 2.

LEMMA 3.2.   *Assume that $m \geq 4$ and the semi-canonical map of S_m is composed of a pencil. Then $g_m = 3$ or 4, and*

$$L_m^2 \geq \frac{2g_m(g_m - 1)}{2g_m - 1} h^0(L_m) + 2(g_m - 1)(q_m - 1).$$

Assume that $m \geq 7$ and the semi-canonical map of S_m is of degree 3 onto its image. Then $4 \leq g_m \leq 6$ and

$$L_m^2 \geq \begin{cases} 3(\chi(L_m)/2 + 2(q_m - 1)) & \text{if } g_m = 4, \\ \dfrac{g_m(g_m - 1)}{2(g_m - 2)} h^0(L_m) + 2(g_m - 1)(q_m - 1)) & \text{if } g_m = 5, 6. \end{cases}$$

Proof.   As in Section 1, let $\tilde{\mathscr{E}}$ be the subbundle of $(f_m)_* L_m$ generically generated by $H^0(B_m, (f_m)_* L_m)$.

Assume that the semi-canonical map of S_m is composed of a pencil. Then $\tilde{\mathscr{E}}$ is a line bundle. If it is special, then $\deg(\tilde{\mathscr{E}}) \geq 2h^0(L_m) - 2 = 2n$ by Clifford's theorem, and Lemma 1.3, (1) shows

$$L_m^2 \geq \frac{2(g_m - 1)}{2g_m - 1}(2g_m n + (g_m - 1)(q_m - 1)) > 4n,$$

which is impossible by (2.3). Hence $\tilde{\mathscr{E}}$ is non-special and $\deg(\tilde{\mathscr{E}}) = h^0(L_m) + q_m - 1$. We get the disired inequality from Lemma 1.3. Since f_m must be a non-hyperelliptic fibration, we have $g_m \geq 3$.

Assume that the semi-canonical map of S_m is degree 3 onto the image V_m. If $m \geq 7$, then $\deg V_m < (4/3)(n - 2)$. From [9. Lemma 1] it follows that V_m is ruled by straight lines. This means that Φ_m maps D_m onto a straight line, and the restriction map $H^0(L_m) \to H^0(L_m|_{D_m})$ is of rank 2. It follows that $\tilde{\mathscr{E}}$ is of rank 2 and, putting $d = 3$, Lemma 1.3 gives us

(3.1) $$L_m^2 \geq \frac{g_m - 1}{2(g_m - 2)}(g_m \deg(\tilde{\mathscr{E}}) + 2(g_m - 4)(q_m - 1)).$$

We remark that $g_m - 1 \geq d = 3$.

Assume that $g_m = 4$. Since D_m is non-hyperelliptic and $L_m D_m = 3$, Clifford's theorem shows $h^0(L_m|_{D_m}) \leq 2$. Therefore, the restriction map $H^0(L_m) \to H^0(L_m|_{D_m})$ is surjective. It follows that $\tilde{\tilde{\mathscr{E}}} \simeq (f_m)_* L_m$. Hence $\deg(\tilde{\tilde{\mathscr{E}}}) = \deg((f_m)_* L_m) \geq \chi(L_m)/2 + 2(q_m - 1)$ by Lemma 1.1. The desired inequality follows from Lemma 1.3.

Assume that $g_m > 4$. Let $\mathscr{F}$ be the quotient bundle of $\tilde{\tilde{\mathscr{E}}}$ defined as in Section 1. If $\mathrm{rk}(\mathscr{F}) = 2$, then $\deg(\tilde{\tilde{\mathscr{E}}}) \geq 2(h^0(L_m) - 2)$ by Lemma 1.4. It follows from (3.1) that $L_m^2 \geq 6(n-1)$ which is impossible by (2.3). If $\mathrm{rk}(\mathscr{F}) = 1$, then it follows from Lamma 1.4 that $\deg(\tilde{\tilde{\mathscr{E}}}) \geq h^0(L_m) - 2 + q_m + \deg(\mathscr{F})/2$. Let $\mathscr{G}$ be the kernel of $\tilde{\tilde{\mathscr{E}}} \to \mathscr{F}$. Since $\mathscr{F}$ is special, we have $\deg(\mathscr{F}) \leq 2q_m - 2$. Hence

$$\mu_1(\tilde{\tilde{\mathscr{E}}}) \geq \deg(\mathscr{G}) \geq h^0(L_m) - 2 + q_m - \deg(\mathscr{F})/2 \geq n.$$

Using (1.1) for $i = 1$, we get

$$L_m^2 \geq \frac{2(g_m - 1)}{2g_m - 1}\left(g_m n + (g_m - 1)(q_m - 1)\right)$$

which is impossible for $g_m > 4$. If $\mathscr{F} = 0$, then $\deg(\tilde{\tilde{\mathscr{E}}}) = h^0(L_m) + 2(q_m - 1)$ by the Riemann-Roch theorem. Hence we get the desired inequality from Lemma 1.3. Then, by (2.3), we get $g_m \leq 6$.   Q.E.D.

Let $f : S \to B$ be the fibration induced by the Albanese map. Since we can assume that f_m is induced by the Albanese map, we get a commutative diagram

$$\begin{array}{ccc} S_m & \to & B_m \\ \pi_m \downarrow & & \downarrow \\ S & \to & B. \end{array}$$

The Galois group $\mathrm{Gal}(S_m/S)$ of $\pi_m : S_m \to S$ acts naturally on B_m, and we can identify B with the quotient $B_m/\mathrm{Gal}(S_m/S)$. We assume that m is a prime number with $m \geq 7$. Then $\varpi : B_m \to B$ is an m-sheeted cyclic covering.

LEMMA 3.3. *The covering ϖ is unramified.*

Proof. Assume that $\mathrm{Gal}(S_m/S)$ has a fixed point $t \in B_m$. Then the induced map $f_m^{-1}(t) \to f^{-1}(\varpi(t))$ is m-sheeted. Since $g_m \leq 6$, we have

$$10 \geq K_m f_m^{-1}(t) = mK f^{-1}(\varpi(t)) \geq 4m,$$

which is impossible.   Q.E.D.

Since S_m is obtained by the base change $\varpi : B_m \to B$, we have $g_m = g$, where g denotes the genus of a general fibre D of $f : S \to B$. We remark that D is trigonal, since so is D_m. Therefore, f is a trigonal fibration of genus $g \leq 6$.

We finish the proof of Theorem 2 with the following:

LEMMA 3.4. *Let S be an irregular even canonical surface with $K^2 < 4\chi$, and let g denote the genus of the Albanese pencil. Then $3 \leq g \leq 6$ and the following inequalities hold.*

$$K^2 \geq \begin{cases} 3\chi + 10(q - 1) & \text{if } g = 3, \\ \dfrac{8g(g-1)}{g^2 + 7g - 16}\left(\chi + 8\left(1 - \dfrac{2}{g}\right)(q - 1)\right) & \text{if } 4 \leq g \leq 6. \end{cases}$$

In other words, the slope of the Albanese pencil is not less than 3, $24/7$, $40/11$ or $120/31$ according to $g = 3$, 4, 5 or 6, respectively.

Proof. Since $\varpi : B_m \to B$ is an m-sheeted unramified covering, we have $q_m - 1 = m(q - 1)$ by Riemann-Hurwitz's formula.

Assume that Φ_m is composed of a pencil. It follows from (2.1) that $2h^0(L_m) \geq - L_m^2/2 + \chi_m$. From this and Lemma 3.2, (1), we get

$$4L_m^2 \geq \frac{2g(g-1)}{2g - 1}\left(-L_m^2 + 2\chi_m\right) + 8(g - 1)(q_m - 1).$$

Since $L_m^2 = mL^2$, $X_m = m\chi$ and $q_m - 1 = m(q - 1)$, we can drop the subscript m from the above inequality. Then, since $K^2 = 4L^2$, we get

$$(3.2) \qquad K^2 \geq \frac{8g(g-1)}{g^2 + 3g - 2}\left(\chi + 2\left(2 - \frac{1}{g}\right)(q - 1)\right).$$

Assume that $\deg \Phi_m = 3$. Then, similarly as above, we get

$$(3.3) \qquad K^2 \geq \frac{8g(g-1)}{g^2 + 7g - 16}\left(\chi + 8\left(1 - \frac{2}{g}\right)(q - 1)\right)$$

by using (2.1) and Lemma 3.2. Note that (3.2) is stronger than (3.3) when $g = 4$.

Q.E.D.

Remark 3.5. When $3 \leq g \leq 5$, the assertion on the slope in Lemma 3.4 holds for any (relatively minimal, non-isotrivial) non-hyperelliptic fibrations, as we have shown in [6].

The lower bound on K^2 in Lemma 3.4 is sharp for $g = 3$, 4. Indeed, for

$g \leq 4$, even canonical surfaces which attain the lower bound can be constructed as in [5, I, §8]. The following example shows that the bound is also sharp for $g = 6$.

EXAMPLE 1. For simplicity, we denote the pull-back of objects by the same symbol. Let B be a non-singular projective curve of genus q and let L_0 be a line bundle of degree $e > 0$ on B. We consider a ruled surface $R = \mathbf{P}(\mathcal{O}_B(6L_0) \oplus \mathcal{O}_B)$ and a $\mathbf{P}^1$-bundle $P = \mathbf{P}(\mathcal{O}_R(2\Delta_0 + 14L_0) \oplus \mathcal{O}_R)$ over R, where Δ_0 is the minimal section of R. Let H be a tautological divisor on P, and take a general member $S \in |3H + \Delta_0|$. We choose sections X_0 and X_1 of $[H]$ and $[H - 2\Delta_0 - 14L_0]$ respectively so that (X_0, X_1) forms a system of homogeneous coordinates on fibres of $P \to R$. Then S is defined in P by

$$\zeta(X_0^3 + \phi_1 X_0^2 X_1 + \phi_2 X_0 X_1^2) = \phi_3 X_1^3,$$

where ζ, ϕ_1, ϕ_2 and ϕ_3 are respectively general sections of $[\Delta_0]$, $[2\Delta_0 + 14L_0]$, $[4\Delta_0 + 28L_0]$ and $7[\Delta_0 + 6L_0]$ on R. Hence we can assume that S is irreducible and non-singular. Furthermore, it has a natural trigonal fibration $f : S \to B$ of genus 6. We remark that ϕ_3 is a non-zero constant on Δ_0. From the above equation, we know that f has a section B_0 defined in P by $\zeta = X_1 = 0$, and Δ_0 is linearly equivalent to $3(X_1)$ on S. By the adjunction formula, the canonical bundle of S is induced by $H + \Delta_0 + 8L_0 + K_B$ which is equivalent to $10(X_1) + 22L_0 + K_B$ on S. If B is a hyperelliptic curve and if we choose $q - 1$ distinct Weierstrass points $p_1, \ldots, p_{q-1}$, then $K_B = 2\sum p_i$ and S is an even surface with a semi-canonical bundle $[5(X_1) + 11L_0 + \sum p_i]$. By a standard calculation, we have $p_g(S) = 62e + 6(q - 1)$, $q(S) = q$ and $K_S^2 = 240e + 40(q - 1)$. Therefore, when e is sufficiently large, S is an even canonical surface with a trigonal fibration $f : S \to B$ whose slope is $120/31$.

The next example shows that we cannot weaken the assumption $K^2 < 4\chi$ in Theorems 1 and 2.

EXAMPLE 2. Let A be a principally polarized abelian surface, and let Θ be a theta divisor on A. For $k \geq 2$, choose a general member $C \in |4k\Theta|$, and let S be the double covering of A with branch locus C. Then S is an even surface, since the canonical bundle is induced by $2k\Theta$. Furthermore, the numerical characters satisfy $K_S^2 = 16k^2$, $p_g = 4k^2 + 1$ and $q = 2$ (hence $K_S^2 = 4\chi(\mathcal{O}_S)$). The Albanese map of S is clearly surjective. It is easy to see that the canonical image of S is cut out by hyperquadrics.

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