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Artinianness of Certain Graded Local Cohomology Modules

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Abstract. We show that if $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$, irrelevant ideal R_+, and M a finitely generated graded R-module, then $H^j_{\mathfrak{m}_0R}(H^t_{R_+}(M))$ is Artinian for $j = 0, 1$ where $t = \inf\{i \in \mathbb{N}_0 : H^i_{R_+}(M)$ is not finitely generated$\}$. Also, we prove that if $\operatorname{cd}(R_+, M) = 2$, then for each $i \in \mathbb{N}_0$, $H^i_{\mathfrak{m}_0R}(H^2_{R_+}(M))$ is Artinian if and only if $H^{i+2}_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian, where $\operatorname{cd}(R_+, M)$ is the cohomological dimension of M with respect to R_+. This improves some results of R. Sazeedeh.

1 Introduction

Throughout this note, we assume that $R = \bigoplus_{n\in\mathbb{N}_0} R_n$ is a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. This means that there are finitely many $l_1, \dots, l_r \in R_1$ such that $R = R_0[l_1, \dots, l_r]$. We denote $R_+ = \bigoplus_{n\in\mathbb{N}} R_n$, the irrelevant ideal of R, and that $\mathfrak{m} = \mathfrak{m}_0 \oplus R_+$, the graded maximal ideal of R. Assume also that $M = \bigoplus_{n\in\mathbb{Z}} M_n$ is a finitely generated graded R-module. For each $i \in \mathbb{N}_0$, let $H^i_{R_+}(M)$ denote the i-th local cohomology module of M with respect to R_+, furnished with its natural grading [2, Chapter 12]. For the unexplained terminology we refer to [2].

Brodmann, Fumasoli and Tajarod [3] proved that for each $i \in \mathbb{N}_0$ and $j = 0, 1$, the graded module $H^j_{\mathfrak{m}_0R}(H^i_{R_+}(M))$ is Artinian whenever $\dim R_0 \leq 1$. Later Brodmann, Rohrer and Sazeedeh [4]showed that $H^1_{\mathfrak{m}_0R}(H^i_{R_+}(M))$ is Artinian for each $i \in \mathbb{N}$ even if $\dim R_0 = 2$. Sazeedeh [8] proved that $\Gamma_{\mathfrak{m}_0R}(H^t_{R_+}(M))$ is Artinian whenever t is the least non-negative integer i such that $H^i_{R_+}(M)$ is not R_+-cofinite. The aim of this note is to show that $H^j_{\mathfrak{m}_0R}(H^t_{R_+}(M))$ is Artinian, whenever

$$t = \inf\{i \in \mathbb{N}_0 : H^i_{R_+}(M) \text{ is not finitely generated}\}$$

and $j = 0, 1$. This generalizes the corresponding result which is shown in [9, Theorem 2.2] for the special case $t = j = 1$ and which was already mentioned above. In addition, we show that if $\operatorname{cd}(R_+, M) = 1$, then for each $j, t \in \mathbb{N}_0$, $H^j_{\mathfrak{m}_0R}(H^t_{R_+}(M))$ is Artinian and also if $\operatorname{cd}(R_+, M) = 2$, then $H^j_{\mathfrak{m}_0R}(H^2_{R_+}(M))$ is Artinian if and only if $H^{j+2}_{\mathfrak{m}_0R}(H^1_{R_+}(M))$ is Artinian. This extends the main result which is shown in [9, Theorem 2.3].

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2 The Results

Theorem 2.1 *Let t be a non-negative integer and let $H^i_{R_+}(M)$ be a finitely generated R-module for all $i < t$. Then $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M))$ is Artinian for $j = 0, 1$.*

Proof By [6, Theorem 11.38], there is the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M).$$

Since $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $r \geq 2$, by [2, Exercise 2.1.9; Theorem 7.1.3] and our hypotheses we have that $E_r^{p,q}$ is Artinian for all $r \geq 2$, $p \geq 0$, and $q < t$. For each $r \geq 2$ and $p, q \geq 0$, let $Z_r^{p,q} = \ker(E_r^{p,q} \to E_r^{p+r,q-r+1})$ and $B_r^{p,q} = \operatorname{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})$. For each $r \geq 2$ and $p = 0, 1$ we have the exact sequences

$$0 \to B_r^{p,q} \longrightarrow Z_r^{p,q} \longrightarrow E_{r+1}^{p,q} \longrightarrow 0$$

and

$$0 \longrightarrow Z_r^{p,q} \longrightarrow E_r^{p,q} \longrightarrow B_r^{p+r,q-r+1} \longrightarrow 0. \tag{2.1}$$

Notice that $B_r^{p,t} = 0$ and $B_r^{p+r,t-r+1}$ is Artinian for all $r \geq 2$ and $p = 0, 1$. Hence we have that

$$Z_r^{p,t} \cong E_{r+1}^{p,t} \tag{2.2}$$

for all $r \geq 2$ and $p = 0, 1$.

Now $E_\infty^{p,t}$ is isomorphic to a subquotient of $H^{p+t}_{\mathfrak{m}}(M)$ and thus is Artinian for all $p \geq 0$. Since $E_\infty^{p,t} \cong E_r^{p,t}$ for r sufficiently large, we have that $E_r^{p,t}$ is Artinian for all $p \geq 0$ and all large r. Fix r and suppose $E_{r+1}^{p,t}$ is Artinian for $p = 0, 1$. From the isomorphism (2.2) we have that $Z_r^{p,t}$ is Artinian for $p = 0, 1$. From the exact sequence (2.1) we get that $E_r^{p,t}$ is Artinian. Continuing in this fashion we see that $E_r^{p,t}$ is Artinian for all $r \geq 2$ and $p = 0, 1$. In particular, $E_2^{p,t} = H^p_{\mathfrak{m}_0 R}(H^t_{R_+}(M))$ is Artinian for $p = 0, 1$. ■

The following corollaries immediately follow by Theorem 2.1.

Corollary 2.2 ([9, Theorem 2.2]) *The graded module $H^j_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian for $j = 0, 1$.*

Corollary 2.3 *Let t be a non-negative integer such that* $\operatorname{grade}(R_+, M) = t$. *Then $H^j_{\mathfrak{m}_0 R}(H^t_{R_+}(M))$ is Artinian for $j = 0, 1$.*

Proposition 2.4 *Let t be a non-negative integer and let $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$ be Artinian for all $j \neq t$ and for all i. Then $H^i_{\mathfrak{m}_0 R}(H^t_{R_+}(M))$ is Artinian for all i.*

Proof Consider the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M).$$

For each $r \geq 2$, we consider the exact sequence

$$0 \longrightarrow \ker d_r^{p,t} \longrightarrow E_r^{p,t} \xrightarrow{d_r^{p,t}} E_r^{p+r,t-r+1}. \tag{2.3}$$

It follows from our hypotheses that the R-module $E_r^{p+r,t-r+1}$ is Artinian. Note that $E_r^{p,q}$ is a subquotient of $E_2^{p,q}$ for all $p, q \geq 0$. There is an integer s such that $E_\infty^{p,q} = E_r^{p,q}$ for all p, q and all $r \geq s$. Also, for each $n \geq 0$, there is a finite filtration

$$0 = \phi^{n+1}H^n \subseteq \phi^n H^n \subseteq \cdots \subseteq \phi^1 H^n \subseteq \phi^0 H^n = H^n$$

of the module $H^n = H^n_{\mathfrak{m}}(M)$ such that $E_\infty^{p,n-p} \cong \phi^p H^n / \phi^{p+1} H^n$ for all $0 \leq p \leq n$.

Thus $E_\infty^{p,q}$ is Artinian for all $p, q \geq 0$. Since $E_s^{p,t} \cong \ker d_{s-1}^{p,t} / \operatorname{im} d_{s-1}^{p-s+1,t+s-2}$, it follows that $\ker d_{s-1}^{p,t}$ is Artinian for all $p \geq 0$. Hence by using the exact sequence (2.3) for $r = s-1$, we deduce that $E_{s-1}^{p,t}$ is Artinian for all $p \geq 0$. By continuing this argument repeatedly for integer $s-1, s-2, \ldots, 3$ instead of s, we obtain that $E_2^{p,t}$ is Artinian for $p \geq 0$. ■

Hellus [5, Example 1.1] showed that there exists an ideal of cohomological dimension 1 which is not principal. Hence the following consequence is a generalization of [9, Proposition 2.6].

Corollary 2.5 *Let* $\operatorname{cd}(R_+, M) = 1$. *Then* $H^i_{\mathfrak{m}_0 R}(H^j_{R_+}(M))$ *is Artinian for all* i, j.

Proof This is clear by Proposition 2.4. ■

Corollary 2.6 *Let* $\operatorname{cd}(R_+, M) = 2$. *Then* $H^i_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ *is Artinian for all* i *if and only if* $H^i_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ *is Artinian for all* i.

Proof By Proposition 2.4 and this fact that $H^i_{\mathfrak{m}_0 R}(\Gamma_{R_+}(M))$ is Artinian for all i, the result easily follows. ■

Aghapournahr and Melkersson [1, Theorem 2.18] proved that if $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of R such that $R/\mathfrak{a} + \mathfrak{b}$ is Artinian and $ara(\mathfrak{a}) = 2$, then the module $H^t_{\mathfrak{b}}(H^2_{\mathfrak{a}}(M))$ is Artinian if and only if the module $H^{t+2}_{\mathfrak{b}}(H^1_{\mathfrak{a}}(M))$ is Artinian for all t. Since the *arithmetic rank* is less than the *cohomological dimension*, the following result is an improvement of [1, Theorem 2.18].

Theorem 2.7 ([9, Theorem 2.3]) *Let* $\operatorname{cd}(R_+, M) = 2$ *and let* t *be a non-negative integer. Then* $H^t_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ *is Artinian if and only if* $H^{t+2}_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ *is Artinian.*

Proof By [2, Corollary 2.1.7] and [7, §1], we can assume that $\Gamma_{R_+}(M) = 0$. Consider the Grothendieck spectral sequence

$$E_2^{p,q} := H^p_{\mathfrak{m}_0 R}(H^q_{R_+}(M)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{m}}(M).$$

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For each $r \geq 2$, $p \geq 0$, and $q = 1,2$ let $Z_r^{p,q} = \ker(E_r^{p,q} \to E_r^{p+r,q-r+1})$ and $B_r^{p,q} = \operatorname{im}(E_r^{p-r,q+r-1} \to E_r^{p,q})$. Notice that $B_r^{p,q} = 0$ for all $r \geq 2$, $p \geq 0$, and $q \geq 2$ and $Z_r^{p,q} \cong E_r^{p,q}$ for all $r \geq 2$, $p \geq 0$, and $q = 1$. For all $r \geq 2$ and $p, q \geq 0$, we consider the exact sequence

$$0 \longrightarrow Z_r^{p,q} \longrightarrow E_r^{p,q} \longrightarrow B_r^{p+r,q-r+1} \longrightarrow 0. \tag{2.4}$$

Since $E_{r+1}^{p,q} = Z_r^{p,q}/B_r^{p,q}$ for all $r \geq 2$ and $p, q \geq 0$, it follows that

$$Z_r^{t,2} \cong E_{r+1}^{t,2}. \tag{2.5}$$

Hence from the exact sequence (2.4) and the isomorphism (2.5) we obtain that $Z_2^{t,2} \cong E_\infty^{t,2}$. On the other hand $E_2^{t+2,1} \cong Z_2^{t+2,1}$ and $B_r^{t+2,1} = 0$ for all $r \geq 3$. It therefore follows that $E_2^{t+2,1}/B_2^{t+2,1} \cong E_\infty^{t+2,1}$. Now from the exact sequence

$$0 \longrightarrow E_\infty^{t,2} \longrightarrow E_2^{t,2} \longrightarrow E_2^{t+2,1} \longrightarrow E_\infty^{t+2,1} \longrightarrow 0$$

the result follows. ∎

Remark Let $\operatorname{cd}(R_+, M) = 2$. Then $\Gamma_{\mathfrak{m}_0 R}(H^2_{R_+}(M))$ is Artinian if and only if $H^2_{\mathfrak{m}_0 R}(H^1_{R_+}(M))$ is Artinian.

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