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Statistical determinism in non-Lipschitz dynamical systems

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Abstract. We study a class of ordinary differential equations with a non-Lipschitz point singularity that admits non-unique solutions through this point. As a selection criterion, we introduce stochastic regularizations depending on a parameter ν: the regularized dynamics is globally defined for each $\nu > 0$, and the original singular system is recovered in the limit of vanishing ν. We prove that this limit yields a *unique statistical solution* independent of regularization when the deterministic system possesses a chaotic attractor having a physical measure with the convergence to equilibrium property. In this case, solutions become spontaneously stochastic after passing through the singularity: they are selected randomly with an intrinsic probability distribution.

Key words: singular dynamical systems, non-Lipschitz differential equations, spontaneous stochasticity
2020 Mathematics Subject Classification: 34F05 (Primary); 37H05, 37C83, 37D45 (Secondary)

> *'It is proposed that certain formally deterministic fluid systems which possess many scales of motion are observationally indistinguishable from indeterministic systems; specifically, that two states of the system differing initially by a small "observational error" will evolve into two states differing as greatly as randomly chosen states of the system within a finite time interval, which cannot be lengthened by reducing the amplitude of the initial error'.*
> — Edward N. Lorenz (1969)

1. *Introduction*

Consider a nonlinear ordinary differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d \tag{1.1}$$

for arbitrary dimension d. Local existence of solutions $\mathbf{x}(t)$ is guaranteed if the function $\mathbf{f} : \mathbb{R}^d \mapsto \mathbb{R}^d$ is continuous, while the Lipschitz continuity is required for its uniqueness by standard theorems. Breaking of the Lipschitz condition, and even the continuity condition, is remarkably abundant in dynamical systems modeling natural phenomena; for example, in the n-body problem [**17**] or the Kirchhoff–Helmholtz system of point vortices [**41**], where the forces diverge at vanishing distances. Other important examples arise in fluid dynamics, where particles are transported by shocks in compressible flows [**25**] or rough velocities in incompressible turbulence [**27**]. Many infinite-dimensional systems form singularities from smooth data in finite time; these often take the form of Hölderian cusps [**21**].

The problem of fundamental importance is: how to select a 'meaningful' solution after the singularity? A natural way to answer this question is to employ a regularization by which the system is modified (smoothed) very close to the singularity and the solution becomes well defined at longer times. However, this procedure is not robust in general; examples show it can be highly sensitive to the regularization details [**13**, **14**, **18**, **20**] although unique selection is possible in some notable situations [**40**]. In this work, we show that continuation as a stochastic process can accommodate such non-uniqueness in a natural and robust manner if the deterministic system has a chaotic attractor having a physical measure with the convergence to equilibrium property.

1.1. *Model.* We consider systems in equation (1.1) with the right-hand side of the form

$$\mathbf{f}(\mathbf{x}) = |\mathbf{x}|^{\alpha}\mathbf{F}\left(\frac{\mathbf{x}}{|\mathbf{x}|}\right), \quad \mathbf{F}(\mathbf{y}) = \mathbf{F}_{\mathsf{s}}(\mathbf{y}) + F_{\mathsf{r}}(\mathbf{y})\mathbf{y}, \tag{1.2}$$

where $\alpha < 1$ and $\mathbf{F} : \mathbb{S}^{d-1} \mapsto \mathbb{R}^d$ is a C^1-function on the unit sphere $\mathbb{S}^{d-1} = \{\mathbf{y} \in \mathbb{R}^d : |\mathbf{y}| = 1\}$ decomposed into the tangential spherical component $\mathbf{F}_{\mathsf{s}} : \mathbb{S}^{d-1} \mapsto T\mathbb{S}^{d-1}$ and the radial component $F_{\mathsf{r}} : \mathbb{S}^{d-1} \mapsto \mathbb{R}$. The field $\mathbf{f} : \mathbb{R}^d \mapsto \mathbb{R}^d$ defined by equation (1.2) is continuously differentiable away from the origin. At the origin, it is only α-Hölder continuous for $\alpha \in (0, 1)$, discontinuous for $\alpha = 0$, or divergent if $\alpha < 0$. Solutions of the system in equation (1.2) with non-zero initial condition $\mathbf{x}(0) = \mathbf{x}_0$ may reach the non-Lipschitz singularity in a *finite time*:

$$\lim_{t \nearrow t_b} \mathbf{x}(t) = \mathbf{0}, \quad 0 < t_b < +\infty, \tag{1.3}$$

after which the solution is generally non-unique.

The system in equation (1.2) is invariant under the space-time scaling

$$\mathbf{x} \mapsto \frac{\mathbf{x}}{\nu}, \quad t \mapsto \frac{t}{\nu^{1-\alpha}} \tag{1.4}$$

for any constant $\nu > 0$, e.g. if $x(t)$ is a solution to equation (1.2), then so is $x(\nu^{1-\alpha}t)/\nu$. This symmetry reflects, in a simplified form, the fundamental property of scale invariance in multi-scale systems [**21**], which feature finite-time singularities (often called *blowup*). Thus, models in equation (1.2) represent a rather large class of singular dynamical systems that can be seen as a toy model for blowup phenomena. Following this analogy, we refer to equation (1.3) as blowup, interpreting $|\mathbf{x}|$ as the 'scale' of solution, and $\mathbf{y} = \mathbf{x}/|\mathbf{x}|$ as its scale-invariant (angular) part.

For the dynamical system approach to models in equation (1.2), we define the auxiliary system for the variables $\mathbf{y} \in \mathbb{S}^{d-1}$ and $w \in \mathbb{R}^+$ as

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = w\mathbf{F}_{\mathsf{s}}(\mathbf{y}), \quad \frac{\mathrm{d}w}{\mathrm{d}\tau} = w + (\alpha - 1)F_{\mathsf{r}}(\mathbf{y})w^2. \tag{1.5}$$

Systems in equations (1.1)–(1.2) and (1.5) are related by the transformation

$$\mathbf{x} = R_t(\mathbf{y}, w) := \left(\frac{t}{w}\right)^{1/(1-\alpha)} \mathbf{y}, \quad t = e^{\tau}, \tag{1.6}$$

where $R_t : \mathbb{S}^{d-1} \times \mathbb{R}^+ \mapsto \mathbb{R}^d$ is the time-dependent map defined for $t > 0$. Relations in equation (1.6) are motivated by the scaling symmetry in equation (1.4), which becomes the time-translation symmetry $\tau \mapsto \tau + \tau_0$ in the autonomous system in equation (1.5) with the relation $\tau_0 = (\alpha - 1) \log \nu$. By changing the time as $\mathrm{d}s = w\mathrm{d}\tau$, we reduce the first equation in equation (1.5) to the form

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s} = \mathbf{F}_{\mathsf{s}}(\mathbf{y}). \tag{1.7}$$

It was shown in [20] that fixed-point and limit-cycle attractors of the system in equation (1.7) impose fundamental restrictions on solutions $\mathbf{x}(t)$ selected by generic regularization schemes. We now extend these results for chaotic attractors leading to a conceptually different mechanism: the long-time behavior of the system in equation (1.7) expressed in terms of its physical measure will define solutions selected randomly near the non-Lipschitz singularity in the systems in equations (1.1) and (1.2).

1.2. *Assumptions*

1.2.1. *On physical measures.* For each attractor $\mathcal{A} \subset \mathbb{S}^{d-1}$ of the system in equation (1.7), we denote its topological basin of attraction by $\mathcal{B}(\mathcal{A}) \subset \mathbb{S}^{d-1}$ and by $X^s : \mathbb{S}^{d-1} \mapsto \mathbb{S}^{d-1}$ the flow of the system in equation (1.7). (A compact set $\mathcal{A}$ is an *attractor* with respect to the flow X^s if there exists a compact (*trapping*) region U such that $\mathcal{A}$ is contained in the interior of U and so that $X^s(U) \subset (U)$ for all $s \geq S_0$ (S_0 fixed) and $\mathcal{A} = \bigcap_{s\geq 0} X^s(U)$ [42]. The topological basin $\mathcal{B}(\mathcal{A})$ is the set of points that converge to $\mathcal{A}$ under the forward flow.) We now recall the definition of a physical measure μ_{phys}. Define the basin $\mathcal{B}_{\mu_{\mathsf{phys}}}(\mathcal{A})$ with respect to the measure μ_{phys} as being the set of points $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A})$ such that

$$\lim_{s\to+\infty} \frac{1}{s} \int_0^s \varphi(X^{s_1}(\mathbf{y}_0))\, \mathrm{d}s_1 = \int \varphi(\mathbf{y})\, \mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y}) \tag{1.8}$$

holds for all continuous functions $\varphi : \mathcal{B}(\mathcal{A}) \mapsto \mathbb{R}$. Then the measure μ_{phys} is *physical* if the basin $\mathcal{B}_{\mu_{\mathsf{phys}}}(\mathcal{A})$ has positive Lebesgue measure. We will say that the physical measure has a *full* basin if the Lebesgue measure of $\mathcal{B}_{\mu_{\mathsf{phys}}}(\mathcal{A})$ coincides with the Lebesgue measure of $\mathcal{B}(\mathcal{A})$. In particular, having a full basin by the definition in equation (1.8) implies the uniqueness of the physical measure with respect to the attractor $\mathcal{A}$. Let us observe that ergodic Sinai–Bowen–Ruelle (SRB) measures without zero Lyapunov exponents are

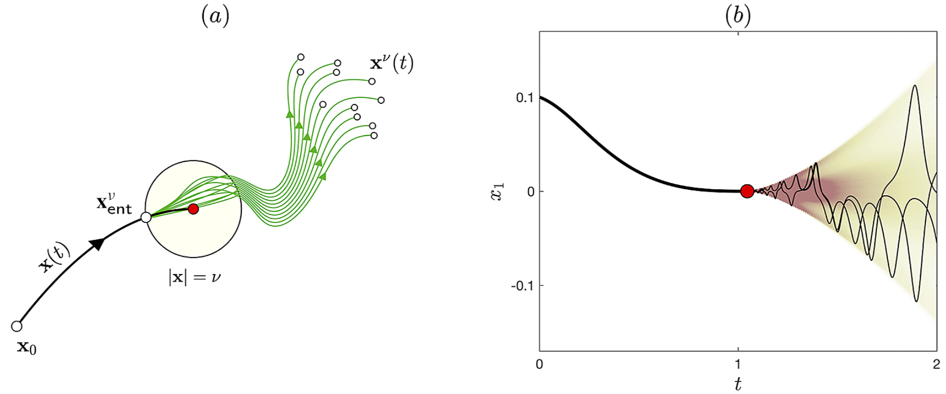

FIGURE 1. (a) Schematic representation of the stochastic regularization procedure in the phase space $\mathbf{x} \in \mathbb{R}^d$. The solution $\mathbf{x}(t)$ (the black curve) starts at $\mathbf{x}_0 = \mathbf{x}(0)$ and reaches the singularity at $\mathbf{x}(t_b) = \mathbf{0}$ in finite time. Regularized solutions (thin green curves) are given by dynamical systems smoothed in a small ball B_ν centered at the singularity. These regularizations are chosen randomly, therefore, the regularized solution is described by a time-dependent probability measure $\mathbf{x}^\nu(t) \sim \mu_t^\nu$. (b) Numerical results for the example from §3. Solid lines are random realizations of component $x_1(t)$ in the regularized system with $\nu = 10^{-5}$. Color shows the probability distribution in equation (1.15). Solutions become spontaneously stochastic passing through the non-Lipschitz singularity (red dot).

also physical measures [48]. Hyperbolic attractors [11] and the Lorenz attractor [3] give examples of systems having a unique physical (SRB) measure with a full basin. In our formulation, we assume the existence of:

(a) a fixed-point attractor $\mathcal{A}_- = \{\mathbf{y}_-\}$ with the *focusing* property $F_\mathsf{r}(\mathbf{y}_-) < 0$;

(b) a transitive attractor $\mathcal{A}_+$ having an ergodic physical measure μ_{phys} and the *defocusing* property, $F_\mathsf{r}(\mathbf{y}) > 0$ for any $\mathbf{y} \in \mathcal{A}_+$.

We also suppose that:

(c) the physical measure μ_{phys} has a full basin.

We note that the chaotic form of $\mathcal{A}_+$ is crucial for our study, while the fixed-point form of $\mathcal{A}_-$ is taken for simplicity. The system in equation (1.7) may have other attractors in addition to $\mathcal{A}_-$ and $\mathcal{A}_+$, but they will not affect our results.

The central part of our formulation refers to a class of regularized systems, which are defined by modifying equations (1.1) and (1.2) in a small ball $|\mathbf{x}| < \nu$ as shown schematically in Figure 1(a). Unlike usual deterministic regularizations, we assume that our regularization contains a random uncertainty, which is characterized by an absolutely continuous probability measure. We assume certain geometrical properties of this measure related to the attractors $\mathcal{A}_-$ and $\mathcal{A}_+$. The exact definition of such regularizations is given in §2 under the name of *stochastic regularization of type* $\mathcal{A}_- \to \mathcal{A}_+$. The regularized system provides a unique measure-valued (stochastic) solution, $\mathbf{x}^\nu(t) \sim \mu_t^\nu$, where μ_t^ν is a probability measure depending on time t and small regularization parameter ν.

We will prove that the auxiliary system in equation (1.5) has the property of *generalized synchronization*: in the limit $s \to +\infty$, a time-independent asymptotic relation exists between the variables as $w = G(\mathbf{y})$. Generalized synchronization originates from applications in nonlinear physics and communication [29], where the variables $\mathbf{y}$ and w

are referred to as a drive and response. In our case, it yields an expression for the physical measure in the system in equation (1.5).

PROPOSITION 1.1. (Generalized synchronization) *The system in equation (1.5) has an attractor*

$$\mathcal{A}'_+ = \{(\mathbf{y}, w) : \ w = G(\mathbf{y}), \ \mathbf{y} \in \mathcal{A}_+\}, \tag{1.9}$$

where $G : \mathcal{A}_+ \to \mathbb{R}^+$ *is a continuous function given by*

$$G(\mathbf{y}) = \int_0^{+\infty} \exp\left[(\alpha - 1)\int_0^{s_1} F_{\mathsf{r}}(X^{-s_2}(\mathbf{y}))\, \mathrm{d}s_2\right]\mathrm{d}s_1. \tag{1.10}$$

This attractor has the basin

$$\mathcal{B}(\mathcal{A}'_+) := \{(\mathbf{y}, w) : (\mathbf{y}, w) \in \mathcal{B}(\mathcal{A}_+) \times \mathbb{R}^+\}, \tag{1.11}$$

and a Borel physical measure given by

$$\mathrm{d}\mu'_{\mathsf{phys}}(\mathbf{y}, w) = \frac{\delta(w - G(\mathbf{y}))}{c\, G(\mathbf{y})}\, \mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y})\, \mathrm{d}w, \quad c = \int \frac{\mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y})}{G(\mathbf{y})}, \tag{1.12}$$

where δ *is the Dirac delta and* c *is the normalization factor.*

1.2.2. *On convergence to equilibrium.* Consider an attractor $\mathcal{A}$ for a flow X^s with a physical measure μ_{phys} having a full basin. We will say that the attractor $\mathcal{A}$ has the *convergence to equilibrium* property with respect to the measure μ_{phys} when

$$\lim_{s\to+\infty} \int \varphi \circ X^s \, \mathrm{d}\mu(\mathbf{y}) = \int \varphi \, \mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y}) \tag{1.13}$$

for all absolutely continuous probability measures μ supported in the basin $\mathcal{B}(\mathcal{A})$ and all bounded continuous functions $\varphi : \mathcal{B}(\mathcal{A}) \to \mathbb{R}$. Notice that the condition in equation (1.13) refers to statistical averages for an ensemble of solutions at long times, unlike the condition in equation (1.8) on the physical measure, which is applied to temporal averages along specific solutions. The convergence to equilibrium property is guaranteed, e.g. for hyperbolic flows [**11**, Theorem 5.3]. Now let $Y^\tau : \mathcal{B}(\mathcal{A}'_+) \mapsto \mathcal{B}(\mathcal{A}'_+)$ be the flow of the system in equation (1.5) in the basin $\mathcal{B}(\mathcal{A}'_+)$ given by equation (1.11). We will assume that:

(d) the physical measure μ'_{phys} of the attractor $\mathcal{A}'_+$ given by equation (1.12) in Proposition 1.1 has the property of convergence to equilibrium.

It is then natural to ask what are the conditions on the vector field on the sphere in equation (1.7), having the attractor $\mathcal{A}_+$ and the physical measure μ_{phys}, so that the above assumption is satisfied. Certain sufficient conditions are established in §4, which are now summarized. Let us suppose that the attractor $\mathcal{A}_+$ of the system in equation (1.7) with the physical measure μ_{phys} satisfies the convergence to equilibrium property. Consider a closed subset $\mathcal{V} \subset \mathbb{S}^{d-1}$ such that its complement $\mathbb{S}^{d-1} \setminus \mathcal{V}$ contains $\mathcal{A}_-$ and is contained in the interior of $\mathcal{B}(\mathcal{A}_-)$. For example, in the case when the basin $\mathcal{B}(\mathcal{A}_-)$ is open, then one can take $\mathcal{V} := S^{d-1} \setminus \mathcal{B}(\mathcal{A}_-)$. We will further assume that:

(i) there exists a constant $F_0 > 0$ such that $F_{\mathsf{r}}(\mathbf{y}) = F_0$ for any $\mathbf{y} \in \mathcal{V}$;

(ii) $\|\nabla \mathbf{F}_{\mathsf{s}}\| < (1 - \alpha)F_0$ for the operator norm of the Jacobian matrix and any $\mathbf{y} \in \mathcal{V}$.

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The hypotheses (i) and (ii) guarantee the existence of a center manifold in the auxiliary system using classical results from dynamical systems. In particular, under these hypotheses, the results of §4 and Proposition 4.1 state that the physical measure of the attractor $\mathcal{A}'_+$ in the system in equation (1.5) also has convergence to equilibrium. As is discussed in §4, this permits to conclude the existence of examples satisfying the assumptions (a)–(d). It would be interesting to relax these assumptions, since spontaneous stochasticity appears to generically occur in this class of examples.

1.3. *Formulation of the main result.* We continue to suppose the assumptions of the previous section. With respect to an attractor $\mathcal{A} \subset \mathbb{S}^{d-1}$ of the system in equation (1.7), we introduce the corresponding *domain of attraction* in the full phase space as the cone

$$\mathcal{D}(\mathcal{A}) = \{r\mathbf{y} : \mathbf{y} \in \mathcal{B}(\mathcal{A}),\ r > 0\} \subset \mathbb{R}^d. \tag{1.14}$$

As shown in §2, all solutions of equations (1.1) and (1.2) with initial condition $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$ reach the non-Lipschitz singularity at the origin in finite time. In contrast, solutions in $\mathcal{D}(\mathcal{A}_+)$ remain non-zero for arbitrarily long times.

Define the measure μ_t by the relation

$$\mu_t = (R_{t-t_b})_*\mu'_{\mathsf{phys}}, \tag{1.15}$$

with the measure μ'_{phys} from equation (1.12) and the map R_t introduced for $t > 0$ in equation (1.6). The measures μ_t are supported in $\mathcal{D}(\mathcal{A}_+)$ and satisfy the dynamic relation

$$\mu_{t_2} = (\Phi^{t_2-t_1})_*\mu_{t_1} \quad \text{for any } t_2 > t_1 > t_b, \tag{1.16}$$

where the asterisk denotes the pushforward and Φ^t is the flow of the system in equations (1.1) and (1.2). Moreover, as the measure μ'_{phys} has compact support, it follows from the expression of the map R_t in equation (1.6) that

$$\lim_{t\searrow t_b} \mu_t = \delta^d \tag{1.17}$$

converges to the Dirac mass at $\mathbf{0}$. This convergence corresponds to the limit at the blowup time being the deterministic singular state $\mathbf{x}(t_b) = \mathbf{0}$.

THEOREM 1.1. (Spontaneous stochasticity) *Given an arbitrary initial condition $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$, there exists a finite time $t_b > 0$ such that the solution $\mathbf{x}(t)$ of the system in equations (1.1) and (1.2) is non-zero in the interval $t \in [0, t_b)$ and reaches the singularity $\mathbf{x}(t_b) = \mathbf{0}$. For any $t > t_b$, the measure-valued solution μ_t satisfies*

$$\mu_t = \lim_{\nu\searrow 0} \mu_t^{\nu}. \tag{1.18}$$

In other words, μ_t is a weak limit of the regularization procedure and this limit is independent of the regularization.

There are two fundamental implications of Theorem 1.1. First, it shows that the limit $\nu \searrow 0$ of a stochastically regularized solution exists. This limit yields a stochastic solution for the original singular system in equations (1.1) and (1.2): even though the random perturbation formally vanishes in the limit $\nu \searrow 0$, a random path is selected

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at $t > t_b$; see Figure 1(b) demonstrating numerical results from the example presented in §3. Such behavior substantiates the fundamental role of infinitesimal randomness in the regularization procedure of non-Lipschitz systems, and this phenomenon is termed *spontaneous stochasticity*.

The second implication is that the spontaneously stochastic solution is insensitive to a specific choice of the stochastic regularization, within the class of regularizations under consideration. The reason, which is also an underlying idea of the proof, is the following: we show that an interval between t_b and any finite time $t > t_b$ in the system in equations (1.1) and (1.2) can be represented by an infinitely large time interval for the system in equation (1.5) as $\nu \searrow 0$. As a result, a random uncertainty introduced by the infinitesimal regularization develops into the unique physical measure. This relates the spontaneous stochasticity in our system with chaos or, more specifically, with the convergence to equilibrium property for a chaotic attractor.

We remark that, in the case when $\mathcal{A}_+$ is a fixed point, the analogous theory was developed previously in [20]. In this case, a unique deterministic solution is selected at times $t > t_b$ independently of regularization. In the present work, we only focus on a chaotic attractor for $\mathcal{A}_+$ leading to spontaneously stochastic solutions.

1.4. *Spontaneous stochasticity in models of fluid dynamics.* Our work provides a class of relatively simple mathematical models, where one can access sophisticated aspects of spontaneous stochasticity: its detailed mechanism, dependence on regularization, and robustness. We regard these models as toy descriptions of the spontaneous stochasticity phenomenon in hydrodynamic turbulence, where singularities and small noise are known to play important roles [23, 33, 43]. Below we provide a short survey guiding an interested reader through more sophisticated models from this field.

First, we would like to mention the prediction of Lorenz [34] (see the epigraph above), in which he envisioned that the role of uncertainty in multi-scale fluid models may be fundamentally different from usual chaos. Spontaneous stochasticity can be encountered in the Kraichnan model for a passive scalar advected by a Hölder continuous (non-Lipschitz) Gaussian velocity [8]. Here, the statistical solution emerges in a suitable zero-noise limit and describes non-unique particle trajectories [19, 22, 30–32]; see also related studies for one-dimensional vector fields with Hölder-type singularities [6, 7, 26, 46]. Similar behavior is encountered for particle trajectories in Burgers solutions at points of shock singularities [25] and quantum systems with singular potentials [24]. The uniqueness of statistical solutions has been tested numerically for shell models of turbulence [9, 35–37] and in the dynamics of singular vortex layers [45]. We note that the prior work on shell models together with recent numerical studies [12, 16] demonstrate chaotic behavior near non-Lipschitz singularities, when solutions are represented in renormalized variables and time. This is similar to our model, in which the spontaneous stochasticity is related to chaos in a smooth renormalized dynamical system in equation (1.7). For recent advances in discrete but infinite dimensional models, see [38, 39].

1.5. *Structure of the paper.* Section 1 contains the introduction and formulation of the main result. Section 2 describes the basic properties of solutions and defines the stochastic regularization. Section 3 contains a numerical example inspired by the Lorenz system. Section 4 provides further developments with the focus on the construction of theoretical examples having robust spontaneous stochasticity. All proofs are collected in §5.

2. *Definition of regularized solutions*

First let us show how non-vanishing solutions $\mathbf{x}(t)$ of the singular system in equations (1.1) and (1.2) are described in terms of solutions $\mathbf{y}(s)$ for the system in equation (1.7).

PROPOSITION 2.1. *Let $\mathbf{y}(s)$ solve equation (1.7) for $s \geq 0$ with initial condition $\mathbf{y}(0) = \mathbf{y}_0$ and let*

$$t_b = \lim_{s\to+\infty} t(s), \quad t(s) = \int_0^s r^{1-\alpha}(s_1)\,\mathrm{d}s_1, \quad r(s) = r_0 \exp\int_0^s F_{\mathsf{r}}(\mathbf{y}(s_1))\,\mathrm{d}s_1 \tag{2.1}$$

for any given $r_0 > 0$. Then, the solution $\mathbf{x}(t)$ of equations (1.1) and (1.2) for $t \in [0, t_b)$ with initial data $\mathbf{x}(0) = r_0\mathbf{y}_0$ is given by $\mathbf{x}(t) = r(s(t))\mathbf{y}(s(t))$, where $s : [0, t_b) \mapsto \mathbb{R}^+$ is the inverse of the function $t(s)$ defined in equation (2.1). If t_b is finite, the solution has the blowup property in equation (1.3).

This statement can be checked by the direct substitution into equations (1.1) and (1.2); see [**20**] for details. The next statement, also proved in [**20**], refers to the focusing and defocusing attractors of the system in equation (1.7), $\mathcal{A}_-$ and $\mathcal{A}_+$, which were introduced in §1.2.

PROPOSITION 2.2. *Solutions $\mathbf{x}(t)$ of the system in equations (1.1) and (1.2) with initial conditions $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$ have the blowup property in equation (1.3) with $\mathrm{dist}(\mathbf{x}/|\mathbf{x}|, \mathcal{A}_-) \to 0$ as $t \nearrow t_b$. Solutions with $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_+)$ remain in $\mathcal{D}(\mathcal{A}_+)$ at all times $t > 0$ with increasing $|\mathbf{x}|$ and $\mathrm{dist}(\mathbf{x}/|\mathbf{x}|, \mathcal{A}_+) \to 0$ as $t \to +\infty$.*

Let us illustrate these properties with the two-dimensional example [**20**] for $\alpha = 1/3$ and

$$\mathbf{F}(\mathbf{y}) = \begin{pmatrix} y_1^2 + y_1y_2 + y_1y_2^2 \\ y_1y_2 + y_2^2 - y_1^2y_2 \end{pmatrix}, \quad \mathbf{F}_{\mathsf{s}}(\mathbf{y}) = (y_1y_2^2, -y_1^2y_2), \quad F_{\mathsf{r}}(\mathbf{y}) = y_1 + y_2, \tag{2.2}$$

where $\mathbf{y} = (y_1, y_2) \in \mathbb{S}^1$ belongs to the unit circle on the plane. Dynamics on a circle of the scale-invariant system in equation (1.7) is shown in Figure 2(a) and the corresponding solutions of the singular system in equations (1.1) and (1.2) in Fig. 2(b). The focusing fixed-point attractor at $(-1, 0)$ features blowup solutions, which occupy the corresponding domain $\mathcal{D}(\mathcal{A}_-) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\}$. There is also a defocusing fixed-point attractor at $(1, 0)$. Its domain $\mathcal{D}(\mathcal{A}_+) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ comprises solutions growing indefinitely in time. This example demonstrates the strong non-uniqueness for all solutions starting in the left half-plane: they can be extended beyond the singularity in uncountably many ways.

2.1. *Regularized sytem.*

Let us consider a class of *ν-regularized systems*

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}^\nu(\mathbf{x}), \quad \mathbf{f}^\nu(\mathbf{x}) := \begin{cases} |\mathbf{x}|^\alpha \mathbf{F}(\mathbf{x}/|\mathbf{x}|), & \mathbf{x} \notin B_\nu, \\ \nu^\alpha \mathbf{H}(\mathbf{x}/\nu), & \mathbf{x} \in B_\nu, \end{cases} \tag{2.3}$$

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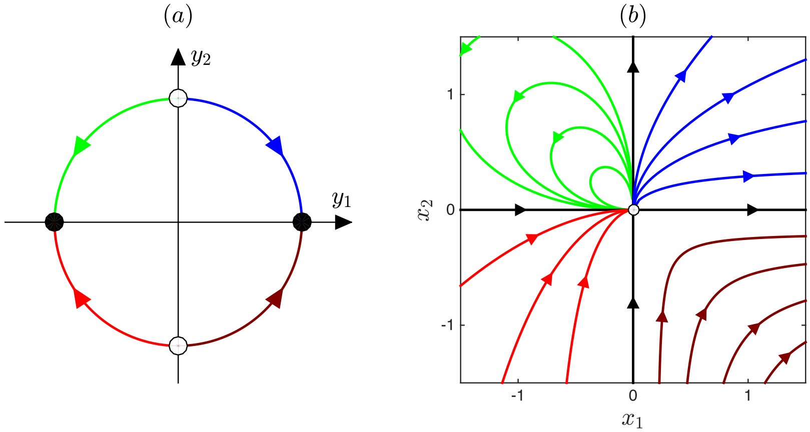

FIGURE 2. (a) Dynamics of the scale-invariant system in equation (1.7) on the unit circle for the example in equation (2.2). There are two attractors (black dots): focusing on the left and defocusing on the right. (b) Solutions of the system in equations (1.1) and (1.2). Colored curves correspond to solutions of the same color in panel (a).

where $\nu > 0$ is the regularization parameter and $B_\nu = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \leq \nu\}$ is the ball of radius ν; recall that $\alpha < 1$. Here, $\mathbf{H} : B_1 \mapsto \mathbb{R}^d$ is a C^1-function in the unit ball such that $H(x) = F(x)$ for $|x| = 1$. Then $\mathbf{f}^\nu$ is $C^1(\mathbb{R}^d)$ for all $\nu > 0$. Note that the described choice of regularization leaves large freedom due to its dependence on the function $\mathbf{H}$. The regularized field $\mathbf{f}^\nu$ recovers the original singular system in equation (1.2) by taking the limit $\nu \searrow 0$. Motivated by the conceptual similarity with the viscous regularization acting at small scales in fluid dynamics [27], we call ν the *viscous parameter* and the limit $\nu \searrow 0$ the *inviscid limit*.

The scaling symmetry in equation (1.4) extends to the system in equation (2.3) as follows. Let us denote the flow of the regularized system in equation (2.3) by $\Phi_\nu^t : \mathbb{R}^d \mapsto \mathbb{R}^d$; it is defined for $t \geq 0$, $\nu > 0$ and $\alpha < 1$. The regularized flows for arbitrary $\nu > 0$ and $\nu = 1$ are related by

$$\Phi_\nu^t(\mathbf{x}) = \nu \, \Phi_1^{t/\nu^{1-\alpha}}\left(\frac{\mathbf{x}}{\nu}\right). \tag{2.4}$$

Using this map for a deterministically or randomly chosen function $\mathbf{H}$, we now introduce the two types of regularizations: deterministic and stochastic.

2.2. *Deterministic regularization of type $\mathcal{A}_- \to \mathcal{A}_+$.* Consider any initial condition $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$ in the domain of the focusing attractor. The corresponding solution $\mathbf{x}(t)$ of the system in equation (1.2) reaches the origin in finite time t_b; see Proposition 2.2. Let us consider the solution $\mathbf{x}^\nu(t)$ of the regularized system in equation (2.3) with the same initial condition for a given viscous parameter $\nu > 0$, provided ν is small enough so that the initial data are outside $B_\nu(\mathbf{0})$. This solution exists and is unique globally in time. The two solutions $\mathbf{x}(t)$ and $\mathbf{x}^\nu(t)$ coincide up until the first time when the solution enters the ball B_ν; see Figure 3(a). We denote this *entry time* by t_{ent}^ν, which has the properties

$$t_{\text{ent}}^\nu < t_b, \quad \lim_{\nu \searrow 0} t_{\text{ent}}^\nu = t_b. \tag{2.5}$$

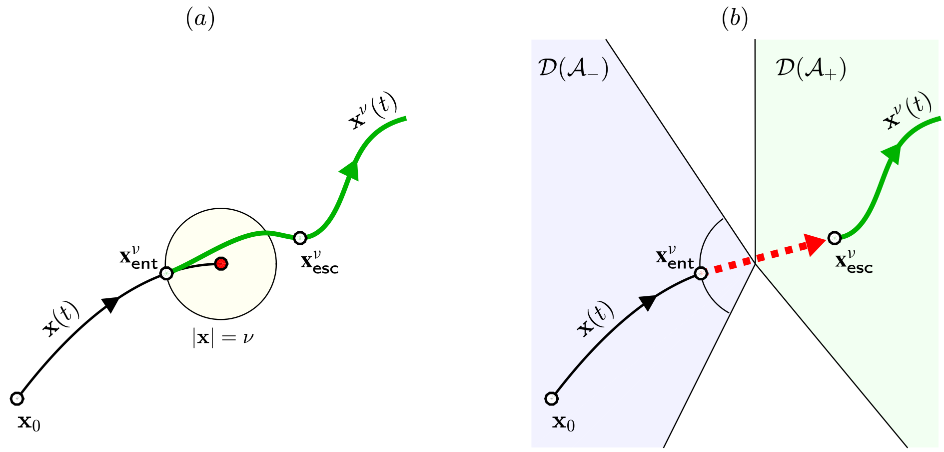

FIGURE 3. Schematic representation of the regularization procedure in the phase space $\mathbf{x} \in \mathbb{R}^d$. (a) The blowup solution $\mathbf{x}(t)$ (black curve) starts at $\mathbf{x}_0 = \mathbf{x}(0)$ and reaches the singularity at $\mathbf{x}(t_b) = \mathbf{0}$ in finite time. The regularized solution $\mathbf{x}^\nu(t)$ (thick green curve) is given by the dynamical system modified in a small ball B_ν centered at the singularity. The solutions $\mathbf{x}(t)$ and $\mathbf{x}^\nu(t)$ coincide until and differ after the point $\mathbf{x}^\nu_{\mathsf{ent}}$. (b) This regularization procedure is formalized by considering the two segments: the original solution $\mathbf{x}(t)$ until the *entry* point $\mathbf{x}^\nu_{\mathsf{ent}}$, and the regularized solution $\mathbf{x}^\nu(t)$ after the *escape* point $\mathbf{x}^\nu_{\mathsf{esc}}$. The two points $\mathbf{x}^\nu_{\mathsf{ent}}$ and $\mathbf{x}^\nu_{\mathsf{esc}}$ are related via the regularization map Ψ_{D} represented by the bold dashed arrow. For the regularization of type $\mathcal{A}_- \to \mathcal{A}_+$, the first segment belongs to the cone $\mathcal{D}(\mathcal{A}_-)$, while the second segment belongs to the cone $\mathcal{D}(\mathcal{A}_+)$.

We introduce an *escape time* $t^\nu_{\mathsf{esc}} > t^\nu_{\mathsf{ent}}$ as

$$t^\nu_{\mathsf{esc}} := \sup_{t > t^\nu_{\mathsf{ent}}} \{t : \mathbf{x}^\nu(t) \in B_\nu\}. \tag{2.6}$$

Observe that the entering orbit need not necessarily escape the regularized region, e.g. it may be that $t^\nu_{\mathsf{esc}} = +\infty$. However, we will give conditions under which finite t^ν_{esc} exist. The corresponding *entry* and *escape points* are denoted by

$$\mathbf{x}^\nu_{\mathsf{ent}} = \mathbf{x}(t^\nu_{\mathsf{ent}}), \quad \mathbf{x}^\nu_{\mathsf{esc}} = \mathbf{x}(t^\nu_{\mathsf{esc}}), \tag{2.7}$$

and have $|\mathbf{x}^\nu_{\mathsf{ent}}| = \nu$ and $|\mathbf{x}^\nu_{\mathsf{esc}}| > \nu$; see Figure 3(a). The following definition of a deterministic regularization ensures the existence of escape times.

Definition 1. (Deterministic regularization) Let $\mathcal{U}_-$ be a neighborhood around $\mathcal{A}_-$ in $\mathbb{S}^{d-1}$ so that the following holds. Suppose that there exists a constant $T > 0$ so that, with respect to the map Φ^T_1, we have (i) $|\Phi^T_1(\mathbf{y})| > 1$ for all $\mathbf{y} \in \mathcal{U}_-$ and (ii) $\Phi^T_1(\mathcal{U}_-) \subset \mathcal{D}(\mathcal{A}_+)$. Then the continuous map

$$\Phi^T_1 : \mathcal{U}_- \mapsto \mathcal{D}(\mathcal{A}_+) \tag{2.8}$$

will be called a *regularization of type* $\mathcal{A}_- \to \mathcal{A}_+$.

We remark that it is simple to construct families of vector fields $\mathbf{H}$ such that deterministic regularizations of the form in equation (2.3) are of type $\mathcal{A}_- \to \mathcal{A}_+$, in particular, when the vector field $\mathbf{F}$ satisfies our hypotheses (a) and (b). Assume now we have a regularization of type $\mathcal{A}_- \to \mathcal{A}_+$ given by Φ^T_1. Consider the initial condition $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$, a fixed $\nu > 0$

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(small enough) and an *entry time* t^{ν}_{ent} into the ball B_ν. Observe that by Proposition 2.2, $|\mathbf{x}^{\nu}_{\mathrm{ent}}/\nu - \mathbf{y}_-| \to 0$ as $\nu \searrow 0$ for the fixed-point attractor $\mathcal{A}_- = \{\mathbf{y}_-\}$. Then for ν small enough, $\mathbf{x}^{\nu}_{\mathrm{ent}}/\nu \in \mathcal{U}_-$. We will argue that an upper bound for the the escape time $t^{\nu,*}_{\mathrm{esc}}$ is given by $t\nu_{\mathrm{ent}} + \nu^{1-\alpha}T$. Using equation (2.4), we have

$$\mathbf{x}^{\nu}(t^{\nu,*}_{\mathrm{esc}}) = \Phi_{\nu}^{(t^{\nu,*}_{\mathrm{esc}} - t^{\nu}_{\mathrm{ent}})}(\mathbf{x}^{\nu}_{\mathrm{ent}}) = \nu \Phi_1^T\left(\frac{\mathbf{x}^{\nu}_{\mathrm{ent}}}{\nu}\right). \tag{2.9}$$

Since $\mathbf{x}^{\nu}_{\mathrm{ent}}/\nu \in \mathcal{U}_-$, then by the definition of the deterministic regularization $\Phi_1^T(\mathbf{x}^{\nu}_{\mathrm{ent}}/\nu) \in \mathcal{D}(\mathcal{A}_+)$ with norm bigger than one. By Proposition 2.2, $\Phi_1^t(\mathbf{x}^{\nu}_{\mathrm{ent}}/\nu)$ will stay in $\mathcal{D}(\mathcal{A}_+)$ for all $t \geq T$ and still will have norm bigger than one. Therefore, going back to equation (2.9), we may conclude that

$$t^{\nu}_{\mathrm{esc}} < t^{\nu}_{\mathrm{ent}} + \nu^{1-\alpha}T, \tag{2.10}$$

bounding above the escape time thereby ensuring it is finite. Having the escape point and time, one defines the regularized solution

$$\mathbf{x}^{\nu}(t) = \Phi^{t - t^{\nu}_{\mathrm{esc}}}(\mathbf{x}^{\nu}_{\mathrm{esc}}), \qquad t \geq t^{\nu}_{\mathrm{esc}}, \tag{2.11}$$

where Φ^t is the flow of the original singular system in equations (1.1) and (1.2). In the limit $\nu \searrow 0$, we will not be interested in the solution inside the vanishing interval $t \in (t^{\nu}_{\mathrm{ent}}, t^{\nu}_{\mathrm{esc}})$, see Figure 3(b). Therefore, for our purposes, the regularization process is conveniently represented by the single map Φ_1^T in the Definition 1 and hence we do not need to explicitly specify the regularizing field $\mathbf{H}$ which generated this map.

2.3. *Stochastic regularization.* It is known that, in general, solutions $\mathbf{x}^{\nu}(t)$ with deterministic regularization do not converge in the inviscid limit $\nu \searrow 0$ [20]. The limits may exist along some subsequences $\nu_n \searrow 0$ but need not be unique. We now introduce a different type of regularization by assuming that escape points are known up to some random uncertainty; see Figure 4.

For this purpose, one may consider a family of regularized systems in equation (2.3) with the field $\mathbf{H}_{\mathbf{a}}$ depending on a vector of parameters $\mathbf{a} \in \mathbb{R}^N$. Specifically, we consider

$$\frac{d\mathbf{x}_{\mathbf{a}}}{dt} = \mathbf{f}^{\nu}(\mathbf{x}_{\mathbf{a}}), \quad \mathbf{f}^{\nu}(\mathbf{x}_{\mathbf{a}}) := \begin{cases} |\mathbf{x}|^{\alpha}\mathbf{F}(\mathbf{x}/|\mathbf{x}|), & \mathbf{x} \notin B_\nu, \\ \nu^{\alpha}\mathbf{H}_{\mathbf{a}}(\mathbf{x}/\nu), & \mathbf{x} \in B_\nu. \end{cases} \tag{2.12}$$

We initialize this system at some deterministic initial condition $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$. For $\nu > 0$ and a fixed time $t > 0$, we call the corresponding flowmap $\Phi^t_{\nu}(\mathbf{x}_0; \mathbf{a}) : \mathbb{R}^d \times \mathbb{R}^N \to \mathbb{R}^d$, which now also depends on the parameter $\mathbf{a}$. Let us impose a probability distribution μ on values of these parameters $\mathbf{a} \in \mathbb{R}^N$ for $N \geq d+1$. Then the measure μ can be used to define a measure $\mu^{\nu}_{(t,\mathbf{x}_0)}$ on $\mathbb{R}^d$ via a pushforward by $\Phi^t_{\nu}(\mathbf{x}_0; \cdot)$ with fixed t, ν, and $\mathbf{x}_0$ as

$$\mu^{\nu}_{(t,\mathbf{x}_0)} = [\Phi^t_{\nu}(\mathbf{x}_0; \cdot)]_* \mu. \tag{2.13}$$

To define the stochastic regularization we assume the following.

(1) For each $\mathbf{a} \in \mathbb{R}^N$, $\mathbf{H}_{\mathbf{a}}$ is a regularization of type $\mathcal{A}_- \to \mathcal{A}_+$ in the sense of Definition 1. The neighborhood $\mathcal{U}_-$ and the time $T > 0$ do not depend on $\mathbf{a}$.

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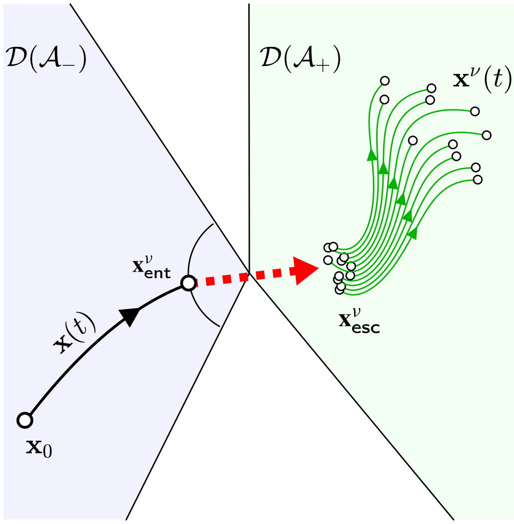

FIGURE 4. Schematic representation of the stochastic regularization procedure in the phase space $\mathbf{x} \in \mathbb{R}^d$. The solution contains two segments: the original deterministic solution $\mathbf{x}(t)$ until the entry point $\mathbf{x}^\nu_{\mathsf{ent}}$, and the regularized solution $\mathbf{x}^\nu(t)$ emanating from the random escape point $\mathbf{x}^\nu_{\mathsf{esc}}$. The probability distribution of $\mathbf{x}^\nu_{\mathsf{esc}}$ is related to the entry point $\mathbf{x}^\nu_{\mathsf{ent}}$ via the regularization map Ψ_{R}. For the regularization of type $\mathcal{A}_- \to \mathcal{A}_+$, the first segment belongs to the domain $\mathcal{D}(\mathcal{A}_-)$ and the second to $\mathcal{D}(\mathcal{A}_+)$.

(2) For $\nu = 1$ and any point $\mathbf{x}_0 \in \mathcal{U}_-$, the measure $\mu^1_{(T,\mathbf{x}_0)}$ has an absolutely continuous (with respect to Lebesgue) density $f(1, T, \mathbf{x}_0; y)$ depending on the variable y and supported in $\mathcal{D}(\mathcal{A}_+) \cap B_1^c$.

The above hypotheses allow to define the function

$$\Psi_{\mathsf{R}} : \mathcal{U}_- \longrightarrow L^1(\mathcal{D}(\mathcal{A}_+)), \tag{2.14}$$

where a point $\mathbf{x}_0$ is mapped to the function $f(1, T, \mathbf{x}_0; y)$. Then adding a continuity condition, we propose the following definition.

Definition 2. (Stochastic regularization) A *stochastic regularization of type* $\mathcal{A}_- \to \mathcal{A}_+$ is given by a continuous map Ψ_{R} in equation (2.14), constructed as above.

Now for $\mathbf{x}_0 \in \mathcal{D}(\mathcal{A}_-)$, consider a sufficiently small $\nu > 0$ so that the entry point $\mathbf{x}^\nu_{\mathsf{ent}}/\nu \in \mathcal{U}_-$. The entry point is independent of the parameter $\mathbf{a}$ by assumptions and we have, by our assumptions, a uniform bound on the escape time $t^\nu_{\mathsf{esc}} < t^\nu_{\mathsf{ent}} + \nu^{1-\alpha}T$. We shall denote

$$t^{\nu,*}_{\mathsf{esc}} := t^\nu_{\mathsf{ent}} + \nu^{1-\alpha}T, \tag{2.15}$$

which serves as a time by which all orbits have left the regularized region. The measure $\mu^\nu_{(t,\mathbf{x}_0)}$ also has an absolutely continuous density, which we call $f(\nu, t, \mathbf{x}_0; y)$. Using equation (2.4) and also equation (2.9), but having in mind the dependence on the parameter $\mathbf{a}$, we obtain

$$\mathbf{x}^{\nu,\mathbf{a}}_{\mathsf{esc}} = \Phi_\nu^{(t^{\nu,*}_{\mathsf{esc}} - t^\nu_{\mathsf{ent}})}(\mathbf{x}^\nu_{\mathsf{ent}}; \mathbf{a}) = \nu \Phi_1^T\left(\frac{\mathbf{x}^\nu_{\mathsf{ent}}}{\nu}; \mathbf{a}\right). \tag{2.16}$$

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With respect to the density, $f(\nu, t, \mathbf{x}_0; y)$, we can identify the variable y with $\mathbf{x}^{\nu,\mathbf{a}}_{\mathsf{esc}}$, and then equation (2.16) implies that

$$f(\nu, (t^{\nu,*}_{\mathsf{esc}} - t^{\nu}_{\mathsf{ent}}), \mathbf{x}^{\nu}_{\mathsf{ent}}; \mathbf{x}^{\nu,\mathbf{a}}_{\mathsf{esc}}) = f\bigg(1, T, \frac{\mathbf{x}^{\nu}_{\mathsf{ent}}}{\nu}; \frac{\mathbf{x}^{\nu,\mathbf{a}}_{\mathsf{esc}}}{\nu}\bigg). \tag{2.17}$$

For simplicity, we will denote the measure $\mu^{\nu}_{(t^{\nu}_{\mathsf{esc}},\mathbf{x}_0)}$ as the escape measure μ^{ν}_{esc}. The stochastic regularization map Ψ_{R} defines a probability density function we call the escape density f^{ν}_{esc},

$$f^{\nu}_{\mathsf{esc}} := \Psi_{\mathsf{R}}\bigg(\frac{\mathbf{x}^{\nu}_{\mathsf{ent}}}{\nu}\bigg) = f\bigg(1, T, \frac{\mathbf{x}^{\nu}_{\mathsf{ent}}}{\nu}; y\bigg) \in L^1(\mathcal{D}(\mathcal{A}_+)). \tag{2.18}$$

Using equation (2.17) and a change of variables $\mathbf{y} = \mathbf{x}/\nu$ in $\mathbb{R}^d$, we can conclude that the density of μ^{ν}_{esc} is given by

$$d\mu^{\nu}_{\mathsf{esc}}(\mathbf{x}) = f^{\nu}_{\mathsf{esc}}\bigg(\frac{\mathbf{x}}{\nu}\bigg)\frac{d\mathbf{x}}{\nu^d}. \tag{2.19}$$

Remark. (Construction of stochastic regularizations) A simple and explicit numerical example arising from a specific choice of $\mathbf{H}$ with random parameters is given in §3. More generally, we sketch here a construction of such a regularization. We work with the $\nu = 1$ rescaled system and let $\mathcal{A}_- = \{\mathbf{x}_*\}$ be the attracting fixed point. For $0 < t_1 < t_2$, let $E \subset \mathcal{B}(\mathcal{A}_+) \times [t_1, t_2]$ be an open connected subset. The set E represents the collection of exit points for trajectories that have traversed the regularized region. Let $f_0 \in C_0^{\infty}(E)$ be absolutely continuous with respect to Lebesgue. For each $\mathbf{a} = (\mathbf{x}_0, t_0) \in \mathrm{supp}(f_0)$, let $\mathbf{H}_{(\mathbf{x}_0,t_0)}$ be an autonomous vector field with the property that $\dot{\mathbf{z}}(t) = \mathbf{H}_{(\mathbf{x}_0,t_0)}(\mathbf{z}(t))$ with $\mathbf{z}(0) = \mathbf{x}_*$ and $\mathbf{z}(t_0) = \mathbf{x}_0$. This field can be built, for example, by taking it tangent to any simple curve connecting $\mathbf{x}_*$ and $\mathbf{x}_0$, properly rescaled to traverse in time t_0, and subsequently extending it to $B_1(\mathbf{0})$ smoothly. Such an extension is obviously highly non-unique. The parameterized collection $\{\mathbf{H}_{(\mathbf{x}_0,t_0)}\}_{(\mathbf{x}_0,t_0)\in\mathrm{supp}(f_0)}$ can be viewed as a random family of regularization vector fields with law inherited by their parameterization $(\mathbf{x}_0, t_0) \sim f_0$. Now, since $\mathcal{A}_+$ is expelling and $\mathrm{supp}(f_0) \subset \mathcal{B}(\mathcal{A}_+)$, trajectories starting their support leave the regularized region and the distribution f_0 is pushed forward by the dynamics in equation (2.12) to define the distribution f^{ν}_{esc} appearing in equation (2.18). Note finally that, in rescaled variables, the entry point $\mathbf{x}^{\nu}_{\mathsf{ent}}/|\mathbf{x}^{\nu}_{\mathsf{ent}}| \to \mathbf{x}_*$ as $\nu \to 0$. As such, by continuity of the above construction, for ν sufficiently small, the behavior is a slight perturbation of the scenario discussed.

We define the measure-valued *stochastically regularized solution* $\mathbf{x}^{\nu}(t) \sim \mu^{\nu}_t$ as

$$\mu^{\nu}_t = (\Phi^{t-t^{\nu,*}_{\mathsf{esc}}})_*\mu^{\nu}_{\mathsf{esc}}, \quad t \geq t^{\nu,*}_{\mathsf{esc}}, \tag{2.20}$$

where the asterisk denotes the push-forward of measure μ^{ν}_{esc} by the flow Φ^t of the original singular system in equations (1.1) and (1.2). Similarly to equation (2.11), the solution is now defined at all times except for a short interval $(t^{\nu}_{\mathsf{ent}}, t^{\nu,*}_{\mathsf{esc}})$ vanishing as $\nu \searrow 0$.

Definition 2 completes the formulation of our main result in Theorem 1.1. This theorem states that when the randomness of regularization is removed in the limit $\nu \searrow 0$, the

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limiting solution exists. This limit is independent of regularization and intrinsically random (spontaneously stochastic): different solutions are selected randomly at times $t > t_b$ with the uniquely defined probability distribution.

3. *Spontaneous stochasticity with Lorenz attractor: numerical example*

In this section, we design an explicit example of the singular system in equation (1.2) with the exponent chosen as $\alpha = 1/3$, and observe numerically the spontaneously stochastic behavior. We consider this example for the dimension $d = 4$, which is the lowest dimension allowing chaotic dynamics in equation (1.7) on the unit sphere, $\mathbf{y} = (y_0, y_1, y_2, y_3) \in S^3$. The radial field is chosen as $F_\mathrm{r}(\mathbf{y}) = -y_0$. The tangent vector field $\mathbf{F}_\mathrm{s}$ is defined as the interpolation between two specific fields $\mathbf{F}_-$ and $\mathbf{F}_+$ in the form

$$\mathbf{F}_\mathrm{s}(\mathbf{y}) = S_1(\xi)\mathbf{F}_-(\mathbf{y}) + (1 - S_1(\xi))\mathbf{F}_+(\mathbf{y}), \quad \xi = 2y_0 - 1/2, \tag{3.1}$$

where S_1 the is the smoothstep (the cubic Hermite) interpolation function

$$S_1(\xi) = \begin{cases} 0, & \xi \le 0, \\ 3\xi^2 - 2\xi^3, & 0 \le \xi \le 1, \\ 1, & 1 \le \xi. \end{cases} \tag{3.2}$$

The function $\mathbf{F}_\mathrm{s}$ coincides with $\mathbf{F}_-$ in the upper region $y_0 \ge 0.75$ and with $\mathbf{F}_+$ in the lower region $y_0 \le 0.25$; see Figure 5. We take $\mathbf{F}_-(\mathbf{y}) = \mathrm{P}_s(0, -y_1, -2y_2, -3y_3)$, where P_s is the operator projecting on a tangent space of the unit sphere. This field has the fixed-point attractor $\mathcal{A}_- = \{\mathbf{y}_-\}$ at the 'North Pole' $\mathbf{y}_- = (1, 0, 0, 0)$, which is the node with eigenvalues -1, -2, and -3. This attractor is focusing because $F_\mathrm{r}(\mathbf{y}_-) = -1$. We choose the field $\mathbf{F}_+(\mathbf{y})$ such that its flow is diffeomorphic to the flow of the Lorenz system

$$\dot{x} = 10(y - x), \quad \dot{y} = x(28 - z) - y, \quad \dot{z} = xy - 8z/3 \tag{3.3}$$

by the scaled stereographic projection

$$x = \frac{40y_1}{1 - y_0}, \quad y = \frac{40y_2}{1 - y_0}, \quad z = 38 + \frac{40y_3}{1 - y_0}. \tag{3.4}$$

This projection is designed such that the lower hemisphere, $y_0 < 0$, contains the Lorenz attractor $\mathcal{A}_+$; see Figure 5. It is defocusing, because $F_\mathrm{r}(\mathbf{y}) = -y_0 > 0$.

In the system in equation (2.3), we use the regularized field

$$\mathbf{H}(\mathbf{x}) = S_1(\eta)\mathbf{H}_0 + (1 - S_1(\eta))\mathbf{f}(\mathbf{x}), \quad \eta = 2|\mathbf{x}| - 1/2, \tag{3.5}$$

which interpolates smoothly between the original singular field $\mathbf{f}(\mathbf{x})$ for $|\mathbf{x}| \ge 3/4$ and the constant field $\mathbf{H}_0$ for $|\mathbf{x}| \le 1/4$. The latter is chosen as $\mathbf{H}_0 = (X_0, X_1, X_2, X_3 - 1)$, where X_i are time-independent random numbers uniformly distributed in the interval $[-1/2, 1/2]$. We confirmed numerically that such a field induces the stochastic regularization of type $\mathcal{A}_- \to \mathcal{A}_+$ according to Definition 2.

It is expected but not known whether the flow of the Lorenz system has the property of convergence to equilibrium, as required in Theorem 1.1. Therefore, with the present

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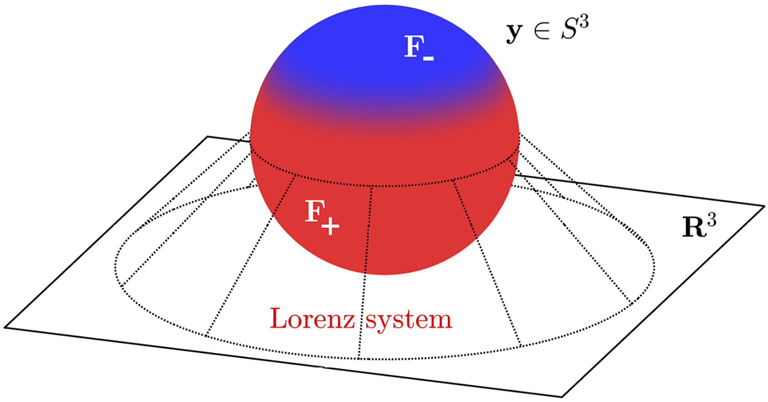

FIGURE 5. Schematic structure of the spherical field $\mathbf{F_s}(\mathbf{y})$ in our example. It is composed of the field $\mathbf{F}_-$ in the blue region, which has the fixed-point attractor at the 'North Pole', and the field $\mathbf{F}_+$ in the red region, which is diffeomorphic to the Lorenz system. The fields are patched together using a smooth interpolation.

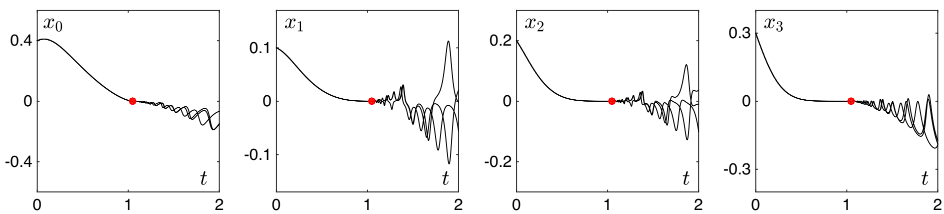

FIGURE 6. Components (x_0, x_1, x_2, x_3) of regularized solutions $\mathbf{x}^\nu(t)$ for $\nu = 10^{-5}$ for three random choices of vector $\mathbf{H}_0$ in the regularized field in equation (3.5). These solutions are different after the blowup time $t_b \approx 1.046$; the blowup point is indicated by the red dot.

example, we verify numerically that the concept of spontaneous stochasticity extends to such systems. We perform high-accuracy numerical simulations of the systems in equations (1.1), (1.2), and (2.3) with the Runge–Kutta fourth-order method. The initial condition is chosen as $\mathbf{x}_0 = (0.4, 0.1, 0.2, 0.3)$. The solution $\mathbf{x}(t)$ of the singular system in equations (1.1) and (1.2) reaches the origin at $t_b \approx 1.046$ (blowup). Figure 6 shows regularized solutions for three random realizations of the regularized system with the tiny $\nu = 10^{-5}$. One can see that these solutions are distinct at post-blowup times.

To observe the spontaneous stochasticity, we compute numerically the probability density for the regularized solution projected on the plane (x_1, x_2) at two post-blowup times: $t = 1.6$ and 2.0. This is done by considering an ensemble of 10^5 random realizations of the regularized field, and the results are shown in Figure 7. Here the magnitude of the probability density is shown by the color: darker regions correspond to larger probabilities. For a better visual effect, the color intensity was taken proportional to the logarithm of the probability density. The presented results demonstrate the spontaneously stochastic behavior, because the probability density is almost identical for two very small values of the regularization parameter: $\nu = 10^{-5}$ (first row) and $\nu = 10^{-7}$ (second row). This provides convincing numerical evidence that the inviscid limit exists and it is spontaneously stochastic. The probability distributions have similar form at different times

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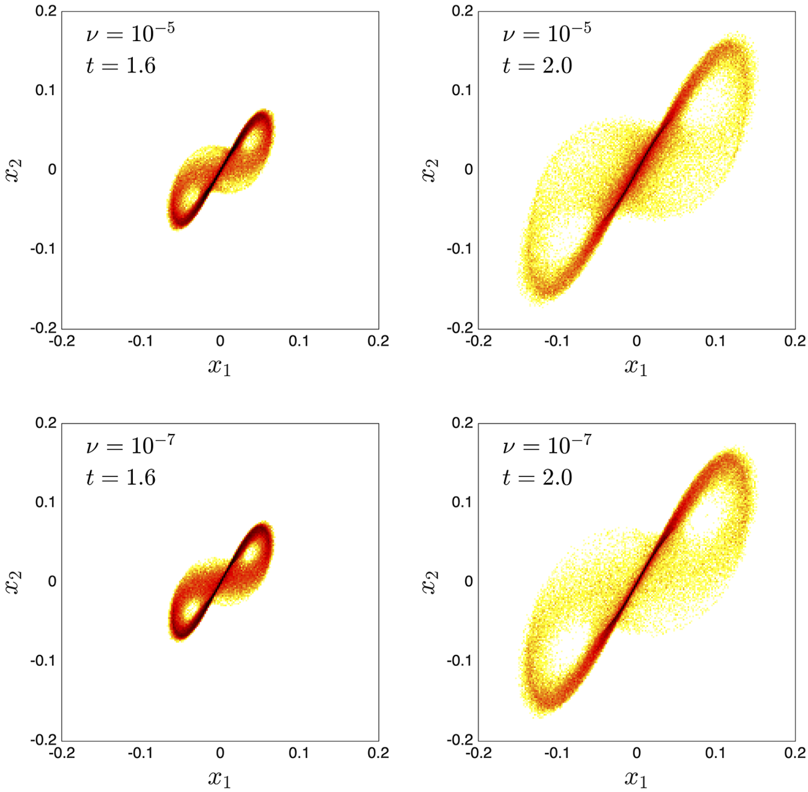

FIGURE 7. Probability density computed numerically at times $t = 1.6$ (left) and $t = 2.0$ (right) using the statistical ensemble of 10^5 regularized solutions. The darker color indicates the higher density. The first row corresponds to $\nu = 10^{-5}$ and the second row to $\nu = 10^{-7}$, confirming the spontaneous stochasticity in the inviscid limit.

up to a proper scaling, in agreement with the self-similar limit in equation (1.15) from Theorem 1.1; see also Figure 1(b). The Supplementary Video shows the evolution of the probability density with time.

4. *Robust spontaneous stochasticity*

The major difficulty in applications of Theorem 1.1 to specific systems is how to verify the assumption of convergence to equilibrium in equation (1.13), which is formulated for the attractor $\mathcal{A}'_+$ from Proposition 1.1. In this section, we discuss how specific and robust examples of systems satisfying this assumption can be constructed.

Recall that the system in equation (1.7) must have a fixed point attractor $\mathcal{A}_-$. Let us choose a closed subset $\mathcal{V} \subset \mathbb{S}^{d-1}$, such that its complement $\mathbb{S}^{d-1} \setminus \mathcal{V}$ contains $\mathcal{A}_-$ and is contained in the interior of $\mathcal{B}(\mathcal{A}_-)$. The subset $\mathcal{V}$ contains basins of all the other attractors, in particular, $\mathcal{B}(\mathcal{A}_+) \subset \mathcal{V}$. It is convenient to use a diffeomorphism $h : \mathcal{V} \mapsto \hat{\mathcal{V}}$, which maps to a closed subset $\hat{\mathcal{V}} \subset \mathbb{R}^{d-1}$ and defines the new variable $\hat{\mathbf{y}} = h(\mathbf{y})$. One can verify that the systems in equations (1.5) and (1.7) keep the same form in terms of $\hat{\mathbf{y}}$ if we substitute $\mathbf{F}_\mathsf{s}$ and F_r by the conjugated vector field $\hat{\mathbf{F}}_\mathsf{s} : \hat{\mathcal{V}} \mapsto \mathbb{R}^{d-1}$ and $\hat{F}_\mathsf{r} = F_\mathsf{r} \circ h^{-1} : \hat{\mathcal{V}} \mapsto \mathbb{R}$. For simplicity, we will omit the hats in the notation below, therefore, assuming in

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all the relations that $\mathbf{y} \in \mathcal{V} \subset \mathbb{R}^{d-1}$. Although $\mathcal{V}$ is not forward invariant by the flow, this will not be necessary in what follows.

Consider now the attractor $\mathcal{A}_+$ of the system in equation (1.7) with the physical measure μ_{phys}. Let us assume that it satisfies the *convergence to equilibrium* property in equation (1.13).

Definition 3. We say that the convergence to equilibrium property is C^k-robust if there exists $\varepsilon > 0$ and a closed neighborhood $\mathcal{U}$ of the attractor, $\mathcal{A}_+ \subset \mathcal{U} \subset \mathcal{B}(\mathcal{A}_+)$, such that the following holds: for any ε-perturbation of $\mathbf{F}_{\mathsf{s}}$ in the C^k-topology, the corresponding system in equation (1.7) has an attractor contained in $\mathcal{U}$ having a physical measure and the convergence to equilibrium property.

This definition extends naturally from the angular dynamics in equation (1.7) to the full auxiliary system in equation (1.5) by considering perturbations of both $\mathbf{F}_{\mathsf{s}}$ and F_{r}. The following proposition provides a criterion that can be used for satisfying the condition in equation (1.13) in specific examples.

PROPOSITION 4.1. *Let us assume that the attractor $\mathcal{A}_+$ in the system in equation (1.7) has convergence to equilibrium and there exists a constant $F_0 > 0$ such that $F_{\mathsf{r}}(\mathbf{y}) = F_0$ for any $\mathbf{y} \in \mathcal{V}$.*

(i) *If, for any $\mathbf{y} \in \mathcal{V}$,*

$$\|\nabla \mathbf{F}_{\mathsf{s}}\| < (1-\alpha)F_0, \tag{4.1}$$

where $\|\nabla \mathbf{F}_{\mathsf{s}}\|$ is the operator norm of the Jacobian matrix $\nabla \mathbf{F}_{\mathsf{s}}$ at the point $\mathbf{y}$, then the attractor $\mathcal{A}'_+$ in the system in equation (1.5) has convergence to equilibrium.

(ii) *If the convergence to equilibrium of $\mathcal{A}_+$ is C^k-robust and, for any $\mathbf{y} \in \mathcal{V}$,*

$$\|\nabla \mathbf{F}_{\mathsf{s}}\| < \frac{(1-\alpha)F_0}{k}, \tag{4.2}$$

then the attractor $\mathcal{A}'_+$ has C^k-robust convergence to equilibrium.

Notice that the conditions in equations (4.1) and (4.2) of Proposition 4.1 can always be satisfied by a proper choice of the function F_{r}. This suggests a constructive way for designing the specific systems in equations (1.1) and (1.2) having spontaneous stochasticity. For a system to have C^k-robust spontaneous stochasticity, one should also impose that the fixed-point attractor $\mathcal{A}_-$ is hyperbolic, that is, it persists under small perturbations of the system.

Since the crucial hypothesis in this construction is that the attractor $\mathcal{A}_+$ has (C^k-robust) convergence to equilibrium, let us discuss examples of attractors having this property. The classical results on the ergodic theory of hyperbolic flows show that a C^2-hyperbolic attractor satisfying the C-dense condition of Bowen–Ruelle (density of the stable manifold of some orbit) has C^2-robust convergence to equilibrium, see [**11**, Theorem 5.3]. In the last decades, many statistical properties have been studied for the larger class of singular hyperbolic attractors, which includes the hyperbolic and the Lorenz attractors; see for example [**3**] as a basic reference and [**1**, **2**, **4**, **5**] for more recent advances. Robust convergence to equilibrium was naturally conjectured for such attractors [**10**, Problem E.4]. Although the general proof is not available yet, recently in [**5**, Corollary B and §4] were given examples of singular hyperbolic attractors having robust convergence to equilibrium, which include perturbations of the Lorenz attractor. In particular, it was shown there exists an arbitrary small C^2-perturbation of the Lorenz attractor so that the resulting system has C^2-robust convergence to equilibrium with respect to C^1-observables.

Having in mind the above discussion, assume that the attractor $\mathcal{A}_+$ has C^2-robust convergence to equilibrium (for example, a hyperbolic attractor as in [**11**] or the Lorenz attractor as in [**5**]). Supposing moreover the hypothesis of Proposition 4.1, we obtain that $\mathcal{A}'_+$ has robust convergence to equilibrium. Then as a consequence of Theorem 1.1, we conclude that these examples are robustly spontaneously stochastic.

COROLLARY 4.1. *There exist examples exhibiting C^2-robust spontaneous stochasticity.*

5. *Proofs*

The central idea of the proofs is to reduce post-blowup dynamics of the stochastically regularized equations to the evolution of the system in equation (1.5) over a time interval, which tends to infinity in the inviscid limit $\nu \searrow 0$. In this way, the inviscid limit is linked to the attractor and physical measure of the system in equation (1.5).

For the analysis of equation (1.5), we transform them to a unidirectionally coupled dynamical system, whose decoupled part is the scale-invariant equation (1.7). Let us introduce the new temporal variable

$$s(\tau) = \int_0^\tau w(\tau_1)\, \mathrm{d}\tau_1. \tag{5.1}$$

Then, the system in equation (1.5) reduces to the so-called master-slave configuration

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}s} = \mathbf{F}_\mathsf{s}(\mathbf{y}), \tag{5.2}$$

$$\frac{\mathrm{d}w}{\mathrm{d}s} = 1 + (\alpha - 1) F_\mathsf{r}(\mathbf{y}) w, \tag{5.3}$$

where the functions $\mathbf{y}(s)$ and $w(s)$ are written in terms of the new temporal variable s. Note that the right-hand side of equation (5.3) is unity for $w = 0$, which prevents $w(s)$ from changing the sign. Hence, s in equation (5.1) is a monotonically increasing function of τ. Since $\mathbf{F}_\mathsf{s}$ and F_r are bounded functions, solutions of the system in equations (5.2) and (5.3) are defined globally in time s.

Notice that the new temporal variable in equation (5.1) is solution-dependent. This is a minor problem for the analysis of physical measures, which are related to temporal averages in equation (1.8). However, this is a serious obstacle for the property of convergence to equilibrium, which is associated with the ensemble average in equation (1.13) at a fixed time.

5.1. *Proof of Proposition 1.1.*

By the assumptions, the system in equation (5.2) has the attractor $\mathcal{A}_+$. Therefore, we need to understand the dynamics of the second equation (5.3). The function $F_\mathsf{r} : \mathbb{S}^{d-1} \to \mathbb{R}$ is continuous and therefore has an upper bound, $F_\mathsf{r}(\mathbf{y}) < F_M$.

Recall that the attractor $\mathcal{A}_+$ is a compact set with the defocusing property, $F_r(\mathbf{y}) > 0$ for any $\mathbf{y} \in \mathcal{A}_+$. Hence, we can choose a trapping neighborhood $\mathcal{U}_+$ of $\mathcal{A}_+$ (recall this is a neighborhood in the sphere) and a positive constant F_m such that

$$0 < F_m < F_r(\mathbf{y}) < F_M \quad \text{for } \mathbf{y} \in \mathcal{U}_+. \tag{5.4}$$

We define the two quantities

$$w_m = \frac{1}{(1-\alpha)F_M} > 0, \quad w_M = \frac{1}{(1-\alpha)F_m} > w_m. \tag{5.5}$$

For any $\mathbf{y} \in \mathcal{U}_+$, the derivative in equation (5.3) satisfies the inequalities $dw/ds > 0$ for $0 < w \leq w_m$ and $dw/ds < 0$ for $w \geq w_M$. Thus, the region

$$\mathcal{U}'_+ = \{(\mathbf{y}, w) : \ \mathbf{y} \in \mathcal{U}_+, \ w \in (w_m, w_M)\} \tag{5.6}$$

is trapping for the system in equations (5.2) and (5.3), and it attracts any solution starting in $\mathcal{B}(\mathcal{A}_+) \times \mathbb{R}^+$.

LEMMA 5.1. *The function*

$$G(\mathbf{y}) = \int_0^{+\infty} \exp\left[(\alpha - 1)\int_0^{s_1} F_r(X^{-s_2}(\mathbf{y}))\, \mathrm{d}s_2\right] \mathrm{d}s_1 \tag{5.7}$$

is continuous on the attractor $\mathcal{A}_+$.

Proof. Convergence of the integral in equation (5.7) follows from the existence of positive lower bound F_m in equation (5.4) and the condition $\alpha < 1$. For $p > 0$, we split the integral in equation (5.7) into two segments for $s_1 \in [0, p]$ and $s_1 \in [p, +\infty)$ with an arbitrary parameter $p > 0$. This yields

$$G(\mathbf{y}) = G_p(\mathbf{y}) + R_p(\mathbf{y}), \tag{5.8}$$

where

$$G_p(\mathbf{y}) = \int_0^{p} \exp\left[(\alpha - 1)\int_0^{s_1} F_r(X^{-s_2}(\mathbf{y}))\, \mathrm{d}s_2\right] \mathrm{d}s_1, \tag{5.9}$$

$$R_p(\mathbf{y}) = \int_p^{+\infty} \exp\left[(\alpha - 1)\int_0^{s_1} F_r(X^{-s_2}(\mathbf{y}))\, \mathrm{d}s_2\right] \mathrm{d}s_1. \tag{5.10}$$

The positive function R_p can be bounded using the property $F_r(\mathbf{y}) > F_m > 0$ from equation (5.4) as

$$R_p(\mathbf{y}) < \int_p^{+\infty} \exp[(\alpha - 1)F_m s_1]\, \mathrm{d}s_1 = \frac{\exp[(\alpha - 1)F_m p]}{(1-\alpha)F_m}. \tag{5.11}$$

By choosing p sufficiently large, we have that $R_p(\mathbf{y}) < \varepsilon/4$ and this bound is valid for any $\mathbf{y} \in \mathcal{A}_+$. Then

$$|G(\mathbf{y}') - G(\mathbf{y})| < |G_p(\mathbf{y}') - G_p(\mathbf{y})| + \frac{\varepsilon}{2}. \tag{5.12}$$

The function $G_p(\mathbf{y})$ in equation (5.9) contains integration over finite intervals and, therefore, it is a continuous function defined for any $\mathbf{y} \in \mathbb{S}^{d-1}$. One can choose $\delta > 0$

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such that $|G_p(\mathbf{y}') - G_p(\mathbf{y})| < \varepsilon/2$ for any $\mathbf{y}$ and $\mathbf{y}' \in \mathbb{S}^{d-1}$ with $|\mathbf{y}' - \mathbf{y}| < \delta$. This yields the desired property as the consequence of equation (5.12). □

Let $X^s : \mathbb{S}^{d-1} \mapsto \mathbb{S}^{d-1}$ denote the flow of the system in equation (5.2) and the pair (X^s, X_w^s) with $X_w^s : \mathbb{S}^{d-1} \times \mathbb{R}^+ \mapsto \mathbb{R}^+$ denote the flow of the system in equations (5.2) and (5.3). We will show the following properties, observing that the first expression in equation (5.13) is that of generalized synchronization whereas $w(s)$ gets synchronized with the evolution of $\mathbf{y}(s)$ [29].

LEMMA 5.2

(i) *For any* $\mathbf{y} \in \mathcal{A}_+$ *and* $w_0 > 0$,

$$G(\mathbf{y}) = \lim_{s \to +\infty} X_w^s(X^{-s}(\mathbf{y}), w_0). \tag{5.13}$$

(ii) *Convergence of the above limit is uniform in the region*

$$\mathbf{y} \in \mathcal{A}_+, \quad w_0 \in (w_m, w_M). \tag{5.14}$$

(iii) *For any solution* $\mathbf{y}(s)$ *of equation (5.2) belonging to the attractor* $\mathcal{A}_+$, *the function* $w(s) = G(\mathbf{y}(s))$ *solves equation (5.3).*

Proof. Let us verify that equation (5.3) has the explicit solution in the form

$$\begin{aligned} w(s) = X_w^s(\mathbf{y}_0, w_0) = w_0 \exp\left[(\alpha - 1)\int_0^s F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right] \\ + \int_0^s \exp\left[(\alpha - 1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1. \end{aligned} \tag{5.15}$$

It is easy to see that $w(0) = w_0$. The change of integration variable $\tilde{s}_2 = s_2 - s$ yields

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2 = F_{\mathrm{r}}(X^s(\mathbf{y}_0)), \tag{5.16}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2 = F_{\mathrm{r}}(X^s(\mathbf{y}_0)) - F_{\mathrm{r}}(X^{s-s_1}(\mathbf{y}_0)). \tag{5.17}$$

Taking the derivative of equation (5.15) and using equations (5.16) and (5.17), we have

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}s} &= w_0(\alpha - 1)F_{\mathrm{r}}(X^s(\mathbf{y}_0)) \exp\left[(\alpha - 1)\int_0^s F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right] \\ &+ \exp\left[(\alpha - 1)\int_0^s F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right] \\ &+ (\alpha - 1)F_{\mathrm{r}}(X^s(\mathbf{y}_0))\int_0^s \exp\left[(\alpha - 1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1 \\ &- (\alpha - 1)\int_0^s F_{\mathrm{r}}(X^{s-s_1}(\mathbf{y}_0)) \exp\left[(\alpha - 1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1. \end{aligned} \tag{5.18}$$

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The term in the last line is integrated explicitly with respect to s_1 as

$$-\exp\left[(\alpha-1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\Bigg|_{s_1=0}^{s_1=s} = 1-\exp\left[(\alpha-1)\int_0^{s} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]. \tag{5.19}$$

Combining the expressions in equations (5.15), (5.18), and (5.19) with $\mathbf{y}(s) = X^s(\mathbf{y}_0)$, one verifies that equation (5.3) is indeed satisfied.

Note that $X^s(\mathbf{y}_0) \in \mathcal{A}_+$ for any $s \in \mathbb{R}$ and initial point on the attractor, $\mathbf{y}_0 \in \mathcal{A}_+$. Because of the positive lower bound F_m in equation (5.4) and $\alpha < 1$, the first term on the right-hand side of equation (5.15) vanishes in the limit $s \to +\infty$ uniformly for all initial points $\mathbf{y}_0 \in \mathcal{A}_+$ and $w_0 \in (w_m, w_M)$. For the same reason, the limit $s \to +\infty$ of the last term in equation (5.15) converges uniformly in this region. Therefore, taking the limit $s \to +\infty$ in equation (5.15) with $\mathbf{y}_0 = X^{-s}(\mathbf{y})$ yields the equivalence of relations in equations (5.7) and (5.13), proving items (i) and (ii) of the lemma.

To prove item (iii), consider the solution in equation (5.15) with $w_0 = G(\mathbf{y}_0)$ given by equation (5.7). This yields

$$\begin{aligned} w(s) = &\int_0^{+\infty} \exp\left[(\alpha-1)\int_{-s}^{s_1} F_{\mathrm{r}}(X^{-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1 \\ &+ \int_0^{s} \exp\left[(\alpha-1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1, \end{aligned} \tag{5.20}$$

where we combined the product of two exponents in the first term into the single one. After changing the integration variables $s_1 = s_1' - s$ and $s_2 = s_2' - s$ in the first integral term of equation (5.20), the full expression reduces to the simple form

$$w(s) = \int_0^{+\infty} \exp\left[(\alpha-1)\int_0^{s_1} F_{\mathrm{r}}(X^{s-s_2}(\mathbf{y}_0))\,\mathrm{d}s_2\right]\mathrm{d}s_1 = G(\mathbf{y}(s)), \tag{5.21}$$

where $G(\mathbf{y})$ is given by equation (5.7) and $\mathbf{y}(s) = X^s(\mathbf{y}_0)$. □

Lemma 5.2 shows that $\mathcal{A}_+'$ from equation (1.9) is the invariant set for the system in equations (5.2) and (5.3). This set has the same structure of orbits as the attractor $\mathcal{A}_+$ of the system in equation (5.2). We need to show that $\mathcal{A}_+'$ is an attractor with the trapping neighborhood in equation (5.6). Since $\mathcal{A}_+$ is the attractor of the first equation (5.2), it is sufficient to prove that

$$\lim_{s\to+\infty} |w(s) - G(\mathbf{y}(s))| = 0 \tag{5.22}$$

uniformly for all initial conditions $\mathbf{y}_0 \in \mathcal{A}_+$ and $w_0 \in (w_m, w_M)$. Since $\mathbf{y}(s) = X^s(\mathbf{y}_0)$ and $w(s) = X_w^s(\mathbf{y}_0, w_0)$, we rewrite equation (5.22) as

$$\lim_{s\to+\infty} |X_w^s(X^{-s}(\mathbf{y}(s)), w_0) - G(\mathbf{y}(s))| = 0. \tag{5.23}$$

The uniform convergence in this expression follows from Lemma 5.2.

It remains to prove the relations

$$\mathrm{d}\mu'_{\mathsf{phys}}(\mathbf{y}, w) = \frac{\delta(w - G(\mathbf{y}))}{c\,G(\mathbf{y})}\,\mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y})\,\mathrm{d}w, \quad c = \int \frac{\mathrm{d}\mu_{\mathsf{phys}}(\mathbf{y})}{G(\mathbf{y})}. \tag{5.24}$$

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Because of the synchronization condition in equation (5.13), the physical measure μ_{syn} for the attractor $\mathcal{A}'_+$ of the system in equations (5.2) and (5.3) is obtained from the physical measure μ_{phys} of attractor $\mathcal{A}_+$ as

$$d\mu_{\mathsf{syn}}(\mathbf{y}, w) = \delta(w - G(\mathbf{y}))\, d\mu_{\mathsf{phys}}(\mathbf{y})\, dw. \tag{5.25}$$

This measure corresponds to the dynamics of the system in equations (5.2) and (5.3). The time change $ds = G(\mathbf{y})d\tau$ following from equation (5.1) with $w = G(\mathbf{y})$ transforms equation (5.25) to the physical measure in equation (5.24) for the system ni equation (1.5); see [**15**, Ch. 10].

5.2. *Proof of Theorem 1.1.* Let us consider the variables

$$w = (t - t^{\nu})|\mathbf{x}|^{\alpha-1}, \quad \tau = \log(t - t^{\nu}), \tag{5.26}$$

where the temporal shift t^{ν}, specified later in equation (5.33), depends on the regularization parameter $\nu > 0$. Observe that t^{ν} was not present in the original definition of equation (1.6), but it does not affect the system in equation (1.5): at times $t > t^{\nu}$, each non-vanishing solution $\mathbf{x}(t)$ of equations (1.1) and (1.2) is uniquely related to the solution $\mathbf{y}(\tau),\ w(\tau)$ of the system in equation (1.5) through the relations

$$\mathbf{x} = R_{t-t^{\nu}}(\mathbf{y}, w), \quad t = t^{\nu} + e^{\tau}. \tag{5.27}$$

Consider arbitrary times $t_2 > t_1 > t^{\nu}$ and denote

$$\mathbf{x}_i = \mathbf{x}(t_i), \quad \mathbf{y}_i = \mathbf{y}(t_i), \quad w_i = w(t_i), \quad \tau_i = \log(t_i - t^{\nu}), \quad i = 1, 2. \tag{5.28}$$

Recalling that Φ^t and Y^{τ} denote the flows of the systems in equations (1.1), (1.2), and (1.5), one has

$$\mathbf{x}_2 = \Phi^{t_2-t_1}(\mathbf{x}_1), \quad t_2 > t_1 > t^{\nu}, \tag{5.29}$$

and

$$(\mathbf{y}_2, w_2) = Y^{\tau_2-\tau_1}(\mathbf{y}_1, w_1), \quad \tau_2 > \tau_1. \tag{5.30}$$

The first expression in equation (5.27) yields

$$\mathbf{x}_1 = R_{t_1-t^{\nu}}(\mathbf{y}_1, w_1), \quad \mathbf{x}_2 = R_{t_2-t^{\nu}}(\mathbf{y}_2, w_2). \tag{5.31}$$

Equations (5.29)–(5.31) provide the conjugation relation between the flows as

$$\Phi^{t_2-t_1} = R_{t_2-t^{\nu}} \circ Y^{\tau_2-\tau_1} \circ R^{-1}_{t_1-t^{\nu}}, \tag{5.32}$$

where $(\mathbf{y}, w) = R_t^{-1}(\mathbf{x})$ is the inverse map.

Let us apply equations (5.28) and (5.32) for the stochastically regularized solution given by equations (2.19) and (2.20). We take

$$t_2 = t, \quad t_1 = t^{\nu,*}_{\mathsf{esc}}, \quad t^{\nu} = t^{\nu,*}_{\mathsf{esc}} - \nu^{1-\alpha} = t^{\nu}_{\mathsf{ent}} + (T-1)\nu^{1-\alpha} \tag{5.33}$$

for any given time $t > t_b$. Notice that $t_2 > t_1$ for sufficiently small $\nu > 0$. Then, we use equation (5.32) to rewrite equation (2.20) in the form of three successive measure

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pushforwards as

$$\mu_t^\nu(\mathbf{x}) = (\Phi^{t_2-t_1})_*\mu_{\mathsf{esc}}^\nu(\mathbf{x}) = (R_{t_2-t^\nu})_*(Y^{\tau_2-\tau_1})_*(R_{t_1-t^\nu}^{-1})_*\mu_{\mathsf{esc}}^\nu(\mathbf{x}). \tag{5.34}$$

For the first pushforward, equation (5.33) yields

$$(R_{t_1-t^\nu}^{-1})_*\mu_{\mathsf{esc}}^\nu(\mathbf{x}) = (R_{\nu^{1-\alpha}}^{-1})_*\mu_{\mathsf{esc}}^\nu(\mathbf{x}). \tag{5.35}$$

Notice from equation (1.6) that $R_{\nu^{1-\alpha}}^{-1}(\mathbf{x}) = R_1^{-1}(\mathbf{x}/\nu)$. Thus, applying equation (2.19), we reduce equation (5.35) to the form

$$(R_{t_1-t^\nu}^{-1})_*\mu_{\mathsf{esc}}^\nu(\mathbf{x}) = (R_1^{-1})_*\mu_f^\nu(\mathbf{x}), \quad d\mu_f^\nu(\mathbf{x}) = f_{\mathsf{esc}}^\nu(\mathbf{x})d\mathbf{x}, \tag{5.36}$$

where μ_f^ν denotes the absolutely continuous probability measure with the density f_{esc}^ν. Finally, using equations (5.28), (5.33), and (5.36) in equation (5.34) yields

$$\mu_t^\nu(\mathbf{x}) = (R_{t-t^\nu})_*(Y^{\tau^\nu})_*(R_1^{-1})_*\mu_f^\nu(\mathbf{x}) \tag{5.37}$$

with

$$\tau^\nu = \tau_2 - \tau_1 = \log\frac{t - t_{\mathsf{ent}}^\nu - (T-1)\nu^{1-\alpha}}{\nu^{1-\alpha}}. \tag{5.38}$$

In the inviscid limit, from equations (2.5), (5.33), and (5.38), one has

$$\lim_{\nu\searrow 0} t^\nu = t_b, \quad \lim_{\nu\searrow 0} \tau^\nu = +\infty. \tag{5.39}$$

It remains to take the limit $\nu \searrow 0$ in equation (5.37). The convergence of entry times from equation (2.5) and Proposition 2.2 yields

$$\lim_{\nu\searrow 0} \mathbf{y}_{\mathsf{ent}}^\nu = \mathbf{y}_-, \tag{5.40}$$

where $\mathcal{A}_- = \{\mathbf{y}_-\}$ denotes the fixed-point attractor and $\mathbf{y}_{\mathsf{ent}}^\nu = \mathbf{x}_{\mathsf{ent}}^\nu/\nu$ correspond to entry points. Since the map Ψ_{R} in equation (2.14) is continuous, the limit in equation (5.40) implies

$$f_{\mathsf{esc}}^\nu \xrightarrow{L^1} f_- \quad \text{as } \nu \searrow 0, \tag{5.41}$$

where

$$f_{\mathsf{esc}}^\nu = \Psi_{\mathsf{R}}(\mathbf{y}_{\mathsf{ent}}^\nu), \quad f_- = \Psi_{\mathsf{R}}(\mathbf{y}_-). \tag{5.42}$$

Using this limiting function, we rewrite equation (5.37) as

$$\mu_t^\nu(\mathbf{x}) = (R_{t-t^\nu})_*[(Y^{\tau^\nu})_*(R_1^{-1})_*\mu_-(\mathbf{x}) + (Y^{\tau^\nu})_*(R_1^{-1})_*\Delta\mu_f^\nu(\mathbf{x})], \tag{5.43}$$

where we introduced the probability measure $d\mu_-(\mathbf{x}) = f_-(\mathbf{x})d\mathbf{x}$ and the signed measure for the difference $\Delta\mu_f^\nu(\mathbf{x}) = \mu_f^\nu(\mathbf{x}) - \mu_-(\mathbf{x})$. Now we can take the inviscid limit $\nu \searrow 0$ for the expression in square parentheses of equation (5.43), where the times of pushforwards behave as equation (5.39). Since the measure $(R_1^{-1})_*\mu_-(\mathbf{x})$ does not depend on ν, the first term in square parentheses converges to μ'_{phys} by the convergence to equilibrium property. The remaining term vanishes in the limit $\nu \searrow 0$, because the flow conserves the L^1 norm of the density function, and this norm vanishes by the property in equation (5.41). This yields the limit in equation (1.18) with the measure in equation (1.15).

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5.3. *Proof of Proposition 4.1.* We will formulate the proof for the first part of the proposition, such that it can be extended later for C^k-perturbed systems.

The system in equation (1.5) considered for $(\mathbf{y}, w) \in \mathcal{V} \times \mathbb{R}^+$ with $F_r(\mathbf{y}) \equiv F_0$ takes the form

$$\frac{d\mathbf{y}}{d\tau} = w\mathbf{F}_s(\mathbf{y}), \quad \frac{dw}{d\tau} = w + (\alpha - 1)F_0 w^2. \tag{5.44}$$

The second equation in equation (5.44) has the fixed point attractor $w = W_0 := [(1-\alpha)F_0]^{-1} > 0$ with the basin $w > 0$. Recall that $\mathcal{B}(\mathcal{A}_+) \subset \mathcal{V} \subset \mathbb{R}^{d-1}$. Equation (1.10)
with $F_r(\mathbf{y}) \equiv F_0$ defines the function $G : \mathcal{B}(\mathcal{A}_+) \mapsto \mathbb{R}^+$ as

$$G(\mathbf{y}) \equiv W_0. \tag{5.45}$$

We define the corresponding graph as

$$\mathcal{G}(\mathcal{A}_+) = \{(\mathbf{y}, w) : \mathbf{y} \in \mathcal{B}(\mathcal{A}_+),\ w = G(\mathbf{y})\}, \tag{5.46}$$

which is the invariant manifold for the system in equation (5.44). The attractor $\mathcal{A}'_+ \subset \mathcal{G}(\mathcal{A}_+)$ is given by equation (1.9).

Linearization of the system in equation (5.44) at any point of $\mathcal{G}(\mathcal{A}_+)$ takes the form

$$\frac{d}{d\tau}\begin{pmatrix} \delta\mathbf{y} \\ \delta w \end{pmatrix} = \begin{pmatrix} W_0 \nabla \mathbf{F}_s & \mathbf{F}_s(\mathbf{y}) \\ \mathbf{0} & -1 \end{pmatrix}\begin{pmatrix} \delta\mathbf{y} \\ \delta w \end{pmatrix}, \tag{5.47}$$

where $(\delta\mathbf{y}, \delta w) \in \mathbb{R}^{d-1} \times \mathbb{R}$ is an infinitesimal perturbation in the tangent space. It is straightforward to verify that the system in equation (5.47) has a solution

$$\begin{pmatrix} \delta\mathbf{y} \\ \delta w \end{pmatrix} = e^{-\tau}\begin{pmatrix} \mathbf{F}_s(\mathbf{y}) \\ -1 \end{pmatrix}, \tag{5.48}$$

which provides the eigenvalue -1 with the corresponding eigenvector. The eigenvector defines the linear space E^{ss} transversal to the graph $\mathcal{G}(\mathcal{A}_+)$, and it will play the role of strong stable (contracting) direction. Remaining eigenvalues are determined by the Jacobian matrix $W_0 \nabla \mathbf{F}_s$ with the corresponding linear invariant space $E^c = \mathbb{R}^{d-1} \times \{0\}$ tangent to the graph $\mathcal{G}(\mathcal{A}_+)$. Assumptions in equation (4.1) imply that eigenvalues of $W_0 \nabla \mathbf{F}_s$ with $W_0 = [(1-\alpha)F_0]^{-1}$ have absolute values smaller than unity.

We showed that, at each point of the graph in equation (5.46), there exists a splitting $E^{ss} \oplus E^c$ of the tangent space, which is invariant for the linearized system in equation (5.47) and such that E^{ss} dominates (contracts stronger than) the so-called central directions in E^c. It follows from the stable manifold theorem that each point of $\mathcal{G}(\mathcal{A}_+)$ has a one-dimensional strong stable invariant manifold, which is tangent to E^{ss}; for background on the invariant manifold theory, see [**44**, Ch. 6] for discrete systems and [**47**, §4.5] for flows. Such a structure can be described locally by a homeomorphism $\rho : \mathcal{U} \times (-\delta, \delta) \mapsto \mathcal{U}'$, where $\mathcal{U}$ and $\mathcal{U}'$ are respectively some trapping neighborhoods of the attractors $\mathcal{A}_+$ and $\mathcal{A}'_+$, and $\delta > 0$ is some (small) number. Here, the fibers $\rho(\mathbf{y}, \xi)$ for fixed $\mathbf{y}$ are local C^1-parameterizations of the strong stable manifolds starting on the graph $\rho(\mathbf{y}, 0) \in \mathcal{G}(\mathcal{A}_+)$.

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Let Y^τ be the flow of the system in equation (5.44). We denote by $Y^\tau_\rho = \rho^{-1} \circ Y^\tau \circ \rho$ the flow, which is defined in $\mathcal{U} \times (-\delta, \delta)$ and conjugated to Y^τ. By construction, this new flow Y^τ_ρ has the attractor $\mathcal{A}_\rho = \{(\mathbf{y}, 0) : \mathbf{y} \in \mathcal{A}_+\}$ with the physical measure

$$d\mu_\rho(\mathbf{y}, \xi) = d\mu_{\mathsf{phys}}(\mathbf{y})\, \delta(\xi) d\xi, \tag{5.49}$$

where $\delta(\xi)$ is the Dirac delta-function and μ_{phys} is the physical measure of the attractor $\mathcal{A}_+$. Straight segments $(\mathbf{y}, \xi)$ with fixed $\mathbf{y}$ and $\xi \in (-\delta, \delta)$ correspond to strong stable manifolds for the new flow Y^τ_ρ. Moreover, since strong stable manifolds have constant eigenvalue -1, Y^τ_ρ has uniform contraction along strong stable manifolds to the plane $\xi = 0$ in a sufficiently small neighborhood $\mathcal{U} \times (-\delta, \delta)$.

Now, the property of convergence to equilibrium for the flow Y^τ follows from the same property for Y^τ_ρ, where the latter is established as follows. The condition of convergence to equilibrium in equation (1.13) for the new system becomes

$$\lim_{\tau \to +\infty} \int \varphi \circ Y^\tau_\rho \, d\mu(\mathbf{y}, \xi) = \int \varphi \, d\mu_\rho(\mathbf{y}, \xi) = \int \varphi(\mathbf{y}, 0)\, d\mu_{\mathsf{phys}}(\mathbf{y}), \tag{5.50}$$

where we used equation (5.49) and integrated the Dirac delta-function. It is enough to verify this condition for absolutely continuous probability measures $\mu(\mathbf{y}, \xi)$ supported in the trapping neighborhood $\mathcal{U} \times (-\delta, \delta)$. Using properties of strong stable manifolds for the flow Y^τ_ρ, the integral in the left-hand side of equation (5.50) can be written as

$$\int \varphi \circ Y^\tau_\rho \, d\mu(\mathbf{y}, \xi) = \int \varphi(Y^\tau_\rho(\mathbf{y}, 0))\, d\mu(\mathbf{y}, \xi) + \int \varphi_1 \circ Y^\tau_\rho \, d\mu(\mathbf{y}, \xi), \tag{5.51}$$

where we introduced the function $\varphi_1(\mathbf{y}, \xi) = \varphi(\mathbf{y}, \xi) - \varphi(\mathbf{y}, 0)$. Since the flow Y^τ_ρ has the property of uniform contraction to the plane $\xi = 0$, where $\varphi_1 = 0$, the last integral in equation (5.51) vanishes in the limit $\tau \to +\infty$. For the first integral on the right-hand side of equation (5.51), we write

$$\int \varphi(Y^\tau_\rho(\mathbf{y}, 0))\, d\mu(\mathbf{y}, \xi) = \int \varphi(Y^\tau_\rho(\mathbf{y}, 0))\, d\mu_{\mathsf{int}}(\mathbf{y}), \tag{5.52}$$

where $\mu_{\mathsf{int}}(\mathbf{y})$ is obtained from the measure $\mu(\mathbf{y}, \xi)$ by integration with respect to ξ. The last integral in equation (5.52) corresponds to the flow Y^τ_ρ restricted to the invariant plane $\xi = 0$, and it is conjugate to the original flow Y^τ restricted to the graph in equation (5.46) with the constant function in equation (5.45). The latter becomes the flow X^s of the system in equation (1.7) after the scaling of time with the constant factor W_0. Therefore, we reduced equation (5.50) to the analogous condition of convergence to equilibrium for the system in equation (1.7), which holds by our assumptions. This proves the first part of the proposition.

For the proof of the C^k-robust convergence to equilibrium, we will need the following lemma.

LEMMA 5.3. *Consider an attractor $\mathcal{A}_+$ of the system in equation (1.7) with C^k-functions $\mathbf{F}_{\mathsf{s}} : \mathcal{V} \mapsto \mathbb{R}^{d-1}$ and $F_{\mathsf{r}} : \mathcal{V} \mapsto \mathbb{R}$ satisfying the conditions*

$$\|\nabla \mathbf{F}_{\mathsf{s}}\| < M, \quad F_{\mathsf{r}}(\mathbf{y}) > m > 0 \tag{5.53}$$

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for any $\mathbf{y} \in \mathcal{B}(\mathcal{A}_+)$ *and positive constants* m *and* M *such that* $M < (1-\alpha)m/k$. *Then, we have the following.*

(i) *Equation (1.10) defines the* C^k*-differentiable function* $G : \mathcal{B}(\mathcal{A}_+) \mapsto \mathbb{R}^+$.

(ii) *Let* $\mathbf{y}(\tau)$ *be the solution of the equation*

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = G(\mathbf{y})\mathbf{F}_\mathsf{s}(\mathbf{y}) \tag{5.54}$$

for arbitrary initial condition $\mathbf{y}_0 \in \mathcal{B}(\mathcal{A}_+)$. *Then* $w(\tau) = G(\mathbf{y}(\tau))$ *satisfies equation (1.5).*

(iii) *Sufficiently small* C^k*-perturbations of* $\mathbf{F}_\mathsf{s}$ *and* F_r *yield small* C^k*-perturbations of* G.

Proof. The above lemma is related to the general statements of the invariant manifold theory as stated in [**28**] for discrete systems and in [**47**] for flows. Below, for completeness, we present a direct proof for arbitrary functions F_r satisfying equation (5.53). Let us first consider the case $k = 1$.

(i) Changing signs of the integration variables s_1 and s_2 in equation (1.10) yields

$$G(\mathbf{y}) = \lim_{s\to-\infty} G_s(\mathbf{y}), \quad G_s(\mathbf{y}) = \int_s^0 \exp\left[(\alpha-1)\int_{s_1}^0 F_\mathsf{r} \circ X^{s_2}(\mathbf{y})\,\mathrm{d}s_2\right]\mathrm{d}s_1, \tag{5.55}$$

where we introduced the function $G_s : \mathcal{V} \mapsto \mathbb{R}^+$. By construction, G_s is a C^1-function for any s. The second condition in equation (5.53) implies the uniform convergence of the limit in equation (5.55) for $\mathbf{y} \in \mathcal{B}(\mathcal{A}_+)$. Hence, the limiting function G is continuous in $\mathcal{B}(\mathcal{A}_+)$.

Computing the Jacobian matrix ∇G_s in equation (5.55) at a given point $\mathbf{y}$ yields

$$\nabla G_s = (\alpha-1)\int_s^0 \left(\int_{s_1}^0 \nabla(F_\mathsf{r} \circ X^{s_2})\,\mathrm{d}s_2\right) \exp\left[(\alpha-1)\int_{s_1}^0 F_\mathsf{r} \circ X^{s_2}(\mathbf{y})\,\mathrm{d}s_2\right]\mathrm{d}s_1, \tag{5.56}$$

where

$$\nabla(F_\mathsf{r} \circ X^{s_2}) = (\nabla F_\mathsf{r})_{X^{s_2}(\mathbf{y})} \nabla X^{s_2}, \tag{5.57}$$

and $(\nabla F_\mathsf{r})_{X^{s_2}(\mathbf{y})}$ denotes the gradient vector ∇F_r computed at $X^{s_2}(\mathbf{y})$.

Since X^s is the flow of the system in equation (1.7), by the classical theory of ordinary differential equations, the Jacobian matrix ∇X^s satisfies the linear Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}s}\nabla X^s = (\nabla \mathbf{F}_\mathsf{s})_{X^s(\mathbf{y})} \nabla X^s, \quad \nabla X^0 = \mathbf{I}, \tag{5.58}$$

where $\mathbf{I}$ is the identity matrix and $(\nabla \mathbf{F}_\mathsf{s})_{X^s(\mathbf{y})}$ is the Jacobian matrix $\nabla \mathbf{F}_\mathsf{s}$ at $X^s(\mathbf{y})$. Using equation (5.58) for negative s, the first bound of equation (5.53), and Grönwall's inequality, we estimate

$$\|\nabla X^s\| \le e^{-Ms}, \quad s \le 0. \tag{5.59}$$

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Let $M_r = \max_{\mathbf{y}\in\mathcal{V}} \|\nabla F_r\| \geq 0$. Using equations (5.56), (5.57), and (5.59) with the bounds in equation (5.53) and recalling that $\alpha < 1$, $s_1 \leq 0$ and $s_2 \leq 0$, we obtain

$$\|\nabla G_s - \nabla G_{s'}\| \leq (1-\alpha) \int_s^{s'} \bigg(\int_{s_1}^0 M_r e^{-Ms_2}\, \mathrm{d}s_2 \bigg) e^{(1-\alpha)ms_1}\, \mathrm{d}s_1 \tag{5.60}$$

for any $s < s' < 0$. Integrating the right-hand side of equation (5.60) and taking into account that

$$(1-\alpha)m > (1-\alpha)m - M > 0 \tag{5.61}$$

by the conditions of the lemma, one can show the Cauchy convergence of the gradients ∇G_s in the limit $s \to -\infty$. Since the bound in equation (5.60) does not depend on $\mathbf{y}$, the convergence is uniform in $\mathcal{B}(\mathcal{A}_+)$. This proves the continuity of the limiting gradient ∇G in equation (5.55).

(ii) Consider the pair of functions $\mathbf{y}(\tau)$ and $w(\tau) = G(\mathbf{y}(\tau))$, where $\mathbf{y}(\tau)$ satisfies equation (5.54). Obviously, these functions satisfy the first equation of equation (1.5). The second equation in equation (1.5) can be transformed to the form in equation (5.3) with the time change in equation (5.1). Then, this equation is verified as in Lemma 5.2, taking into account that the integrals converge uniformly for all $\mathbf{y} \in \mathcal{B}(\mathcal{A}_+)$.

(iii) Using the uniform bound in equation (5.60), one proves that the convergence of integrals in equation (5.56) as $s \to -\infty$ is uniform not only with respect to $\mathbf{y}$, but also with respect to sufficiently small C^1-perturbations of the functions $\mathbf{F}_s(\mathbf{y})$ and $F_r(\mathbf{y})$. This implies that such perturbations lead to C^1-perturbations of $G(\mathbf{y})$.

This proof extends to the C^k case for $k > 1$ by computing high-order derivatives of G in the way similar to equations (5.55) and (5.56). Generalizing equation (5.59), one can show that kth-order derivatives of $X^s(\mathbf{y})$ are bounded by $c \exp(-kMs)$ for $s \leq 0$ and some coefficient $c > 0$. We leave details of this rather straightforward derivation to the interested reader. □

Consider now a perturbed system in equation (1.5) with $\tilde{\mathbf{F}}_s$ close to $\mathbf{F}_s$ and $\tilde{F}_r$ close to F_r in the C^k-metric; here and below, the tildes denote properties of the perturbed system. Conditions of Definition 3 ensure that the perturbed system in equation (1.7) has an attractor $\tilde{\mathcal{A}}_+$ with the physical measure and the convergence to equilibrium property. In turn, the perturbed system in equation (1.5) has the attractor $\tilde{\mathcal{A}}'_+$ given by the graph $w = \tilde{G}(\mathbf{y})$ of $\mathbf{y} \in \tilde{\mathcal{A}}_+$; see Proposition 1.1. Conditions in equation (4.2) remain valid if the perturbation is sufficiently small. Hence, one can choose m and M satisfying conditions of Lemma 5.3, establishing that the function $\tilde{G}(\mathbf{y})$ is C^k-close to the constant from equation (5.45), and also the graph $w = \tilde{G}(\mathbf{y})$ with $\mathbf{y} \in \mathcal{B}(\tilde{\mathcal{A}}_+)$ is invariant under the flow of the perturbed system in equation (1.5).

Restriction of equation (1.5) to the invariant hyper-surface $w = \tilde{G}(\mathbf{y})$ yields

$$\frac{\mathrm{d}\mathbf{y}}{\mathrm{d}\tau} = \tilde{G}(\mathbf{y})\tilde{\mathbf{F}}_s(\mathbf{y}). \tag{5.62}$$

This system is C^k-close to $\mathrm{d}\mathbf{y}/\mathrm{d}\tau = W_0\mathbf{F}_s(\mathbf{y})$, where the latter is equivalent to $\mathrm{d}\mathbf{y}/\mathrm{d}s = \mathbf{F}_s(\mathbf{y})$ up to the constant time scaling. Since the attractor $\mathcal{A}_+$ of the unperturbed system in equation (1.7) is assumed to have a physical measure with the C^k-robust convergence to equilibrium, the attractor $\tilde{\mathcal{A}}_+$ of the perturbed system in equation (5.62) has a physical measure with the property of convergence to equilibrium, provided that the perturbation is sufficiently small. For concluding the proof, one should notice that all arguments in the first part of the proof (based on the invariant manifold theory) remain valid for small C^k perturbations of the system and of the graph in equation (5.46).

Supplementary material. The Supplementary Video is available online at https://doi.org/10.1017/etds.2023.74. The video shows the time evolution, where the green points represent a few specific solutions and the color indicates the probability density obtained with a statistical ensemble of 10^5 solutions regularized with $\nu = 10^{-5}$.

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