Bull. Aust. Math. Soc. **101** (2020), 118–129
doi:10.1017/S0004972719000686

WEAKENING OF THE HARDY PROPERTY FOR MEANS

PAWEŁ PASTECZKA

(Received 9 February 2019; accepted 15 May 2019; first published online 9 July 2019)

Abstract

The aim of this paper is to find a broad family of means defined on a subinterval of $I \subset [0, +\infty)$ such that

$$\sum_{n=1}^{\infty} \mathcal{M}(a_1, \ldots, a_n) < +\infty \quad \text{for all } a \in \ell_1(I).$$

Equivalently, the averaging operator $(a_1, a_2, a_3, \ldots) \mapsto (a_1, \mathcal{M}(a_1, a_2), \mathcal{M}(a_1, a_2, a_3), \ldots)$ is a selfmapping of $\ell_1(I)$. This property is closely related to the so-called Hardy inequality for means (which additionally requires boundedness of this operator). We prove that these two properties are equivalent in a broad family of so-called Gini means. Moreover, we show that this is not the case for quasi-arithmetic means, that is functions $f^{-1}(\sum f(a_i)/n)$, where $f\colon I \to \mathbb{R}$ is continuous and strictly monotone, $n \in \mathbb{N}$ and $a \in I^n$. However, the weak Hardy property is localisable for this family.

2010 *Mathematics subject classification*: primary 26E60; secondary 26D15.

Keywords and phrases: Hardy mean, quasi-arithmetic mean, selfmapping of ℓ_1, bounded operator.

1. Introduction

A mean $\mathcal{M}$ on an interval $I \subset [0, +\infty)$ (that is, a function $\mathcal{M}\colon \bigcup_{n=1}^{\infty} I^n \to I$ satisfying $\min(a) \le \mathcal{M}(a) \le \max(a)$ for every admissible vector a) is said to be a *Hardy mean* if there exists a finite constant C such that

$$\sum_{n=1}^{\infty} \mathop{\mathcal{M}}_{k=1}^{n}(a_k) \le C \sum_{n=1}^{\infty} a_n \quad \text{for all } a \in \ell_1(I),$$

where $\mathcal{M}_{k=1}^{n}(a_k)$ stands for $\mathcal{M}(a_1, \ldots, a_n)$. The smallest extended real number C satisfying the inequality above is called the *Hardy constant* of $\mathcal{M}$ and denoted by $\mathcal{H}(\mathcal{M})$.

These definitions were introduced recently by Páles–Persson [17] and Páles–Pasteczka [15], respectively. In fact they are closely related as a mean is Hardy if and only if its Hardy constant is finite.

There are a number of earlier results which can be expressed in terms of the Hardy mean and Hardy constant. These properties were studied for power means $\mathcal{P}_\alpha$ in the papers [2, 6, 7, 10]. The result (in a unified form) can be expressed as

$$\mathcal{H}(\mathcal{P}_\alpha) = \begin{cases} (1-\alpha)^{-1/\alpha} & \text{for } \alpha \in (-\infty, 0) \cup (0, 1), \\ e & \text{for } \alpha = 0, \\ +\infty & \text{for } \alpha \in [1, \infty). \end{cases}$$

The history of Hardy-type inequalities is sketched in the surveys by Pečarić–Stolarsky [20] and Duncan–McGregor [4] and in a book of Kufner–Maligranda–Persson [9]. Further examples of Hardy means (with known Hardy constant) were given recently by Pasteczka [19] and Páles–Pasteczka [15, 16]. Some negative results were obtained in [18] (see Proposition 2.1 below).

The Hardy property of a mean $\mathcal{M}$ on I can be expressed in terms of the $\mathcal{M}$-*averaging operator* defined by

$$I^{\mathbb{N}} \ni (a_1, a_2, \ldots) \mapsto (a_1,\ \mathcal{M}(a_1, a_2),\ \mathcal{M}(a_1, a_2, a_3),\ \ldots) \in I^{\mathbb{N}}.$$

Indeed, a mean $\mathcal{M}$ is a Hardy mean if and only if the $\mathcal{M}$-averaging operator is a bounded operator from $\ell_1(I)$ to itself. In fact, its norm equals $\mathcal{H}(\mathcal{M})$. Motivated by these preliminaries, we will be dealing with a more general definition. Namely, we call a mean $\mathcal{M}$ on I a *weak Hardy mean* if

$$\sum_{n=1}^{\infty} \underset{k=1}{\overset{n}{\mathcal{M}}}(a_k) < +\infty \quad \text{for all } a \in \ell_1(I).$$

Equivalently, the $\mathcal{M}$-averaging operator is a selfmap of $\ell_1(I)$ (with no boundedness assumption).

REMARK 1.1. The $\mathcal{M}$-averaging operator is a selfmapping of $\ell_1(I)$ if and only if the conjugated operator

$$(I^{1/p})^{\mathbb{N}} \ni (a_1, a_2, a_3, \ldots) \mapsto (a_1, \mathcal{M}(a_1^p, a_2^p)^{1/p}, \mathcal{M}(a_1^p, a_2^p, a_3^p)^{1/p}, \ldots)$$

is a selfmapping of $\ell_p(I^{1/p})$ for $p \in (1, +\infty)$. In this way the considerations in the present paper can be generalised easily to ℓ_p spaces.

Obviously, each Hardy mean is a weak Hardy mean, but in general the converse is not valid. We are interested in families of means where all weak Hardy means are Hardy means. For example, it is easy to verify that this is the case for power means. We prove this property for Gini means (Section 3.2). On the other hand we show that it is not the case for quasi-arithmetic means (Section 3.1). In general this problem remains open (compare with Remark 3.7).

We conclude the paper with some results in a family of quasi-arithmetic means. In particular we prove that in this case the weak Hardy property is determined by values of the mean in a neighbourhood of zero.

1.1. Basic properties of means. We recall some notions from [15]. We say that $\mathcal{M}$ is *symmetric* and *(strictly) increasing* if for all $n \in \mathbb{N}$ the n-variable restriction $\mathcal{M}|_{I^n}$ is respectively symmetric and (strictly) increasing in each of its variables. If $I = \mathbb{R}_+$, we can analogously define the notion of homogeneity of $\mathcal{M}$. *Monotonicity* of a mean is associated with its increasingness. Finally, the mean $\mathcal{M}$ is called *repetition invariant* if for all $n, m \in \mathbb{N}$ and $(a_1, \ldots, a_n) \in I^n$ the following identity is satisfied:

$$\mathcal{M}(\underbrace{a_1, \ldots, a_1}_{m \text{ times}}, \ldots, \underbrace{a_n, \ldots, a_n}_{m \text{ times}}) = \mathcal{M}(a_1, \ldots, a_n).$$

2. Necessary conditions for weak Hardy property

In this section we give some necessary conditions for a mean to be weak Hardy. Such a result was obtained for the Hardy property in [18].

Proposition 2.1 ([18], Theorem 1.1). *Let $I \subset \mathbb{R}_+$ be an interval,* $\inf I = 0$*. Let $\mathcal{M}$ be a mean defined on I and $(a_n)_{n=1}^{\infty}$ be a sequence of numbers in I satisfying $\sum_{n=1}^{\infty} a_n = +\infty$. If $\lim_{n\to\infty} a_n^{-1} \mathcal{M}_{k=1}^{n}(a_k) = +\infty$ then $\mathcal{M}$ is not a Hardy mean.*

Our aim is to establish an analogue of this result for the weak Hardy property. We first introduce some technical notation.

We say that a sequence (a_n) of positive numbers is *nearly increasing* if there exists $\varepsilon > 0$ such that $\varepsilon a_m \le a_n$ for all m, $n \in \mathbb{N}$ with $m \le n$. Notice that nearly increasing sequences inherit some properties which are characteristic for monotone sequences. For example it is easy to verify that every such sequence is either divergent or bounded (in fact $\liminf a_n \ge \varepsilon \sup a_n$). On the other hand, a bounded sequence is nearly increasing if and only if it is separated from zero. Therefore, this definition is only interesting for divergent sequences.

Our main result is the following theorem.

Theorem 2.2. *Let $\mathcal{M}$ be a homogeneous and monotone mean defined on $\mathbb{R}_+$. If there exists a sequence (a_n) of positive numbers such that*

(1) $\sum_{n=1}^{\infty} a_n = +\infty$;
(2) *the sequence $(a_n^{-1} \mathcal{M}_{k=1}^{n}(a_k))_{n=1}^{\infty}$ is nearly increasing and divergent;*
(3) $\sum_{n=1}^{\infty} a_n^{1+s}(\mathcal{M}_{k=1}^{n}(a_k))^{-s}$ *is finite for some $s \in \mathbb{R}_+$;*

then $\mathcal{M}$ is not a weak Hardy mean.

Proof. Let $b_n := a_n^{-1} \mathcal{M}_{k=1}^{n}(a_k)$ and $\varepsilon > 0$ be the parameter which appears in the definition of nearly increasingness in the second assumption. We can rewrite (3) as

$$\sum_{n=1}^{\infty} a_n b_n^{-s} < +\infty \quad \text{for some } s > 0. \tag{2.1}$$

Since $\mathcal{M}$ is homogeneous and monotone and (b_n) is nearly increasing,

$$\sum_{n=1}^{\infty} \mathop{\mathcal{M}}_{k=1}^{n}(a_k b_k^{-s}) \ge \sum_{n=1}^{\infty} \mathop{\mathcal{M}}_{k=1}^{n}(a_k \varepsilon^s b_n^{-s}) = \sum_{n=1}^{\infty} \varepsilon^s b_n^{-s} \mathop{\mathcal{M}}_{k=1}^{n} a_k = \sum_{n=1}^{\infty} \varepsilon^s a_n b_n^{1-s}. \tag{2.2}$$

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

We prove the theorem by induction with respect to s (or, more precisely, with respect to $\lceil s \rceil$). For $s \in (0,1]$, from $\sum_{n=1}^{\infty} a_n = +\infty$, $\lim_{n\to\infty} b_n = +\infty$ and (2.2),

$$\sum_{n=1}^{\infty} \mathop{\mathcal{M}}_{k=1}^{n}(a_k b_k^{-s}) = +\infty.$$

By property (2.1), $\mathcal{M}$ is not a weak Hardy mean.

For $s > 1$, either the rightmost sum of (2.2) is infinite and, consequently, $\mathcal{M}$ does not admit the weak Hardy property, or

$$\sum_{n=1}^{\infty} a_n b_n^{1-s} < +\infty,$$

which is exactly the third condition with s replaced by $s-1$. By the inductive assumption, $\mathcal{M}$ is not a weak Hardy mean. □

In the special case $a_n = 1/n$ and for an arbitrary positive D ($D = 2/s$), Theorem 2.2 implies the following corollary.

Corollary 2.3. *Let $\mathcal{M}$ be a homogeneous and monotone mean defined on $\mathbb{R}_+$. If $(\mathcal{M}_{k=1}^{n}(n/k))_{n=1}^{\infty}$ is nearly increasing and there exist $C, D \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}$ such that*

$$\mathop{\mathcal{M}}_{k=1}^{n}\Big(\frac{1}{k}\Big) \geq \frac{C(\ln n)^D}{n} \quad \text{for all } n \geq n_0, \tag{2.3}$$

then $\mathcal{M}$ is not a weak Hardy mean.

Here, nearly increasingness of the sequence $(\mathcal{M}_{k=1}^{n}(n/k))_{n=1}^{\infty}$ seems to be a most restrictive condition. Luckily we have the following result.

Proposition 2.4. *Let $\mathcal{M}$ be a homogeneous, monotone and repetition-invariant mean defined on $\mathbb{R}_+$. Then the sequence $(\mathcal{M}_{k=1}^{n}(n/k))_{n=1}^{\infty}$ is nearly increasing (with $\varepsilon = \frac{1}{2}$).*

Proof. Let $d_n := \mathcal{M}_{k=1}^{n}(n/k)$. We prove that $d_m \leq 2d_n$ for all $m \leq n$. The proof is divided into two parts:

(i) $d_p \leq 2d_q$ for $p \in \mathbb{N}$ and $q \in \{p, \dots, 2p-1\}$;
(ii) $d_p \leq d_{2p}$ for $p \in \mathbb{N}$.

Then we can use simple induction to obtain the final assertion.

As the first inequality for $p = q$ is trivial, fix $p \in \mathbb{N}$ and $q \in \{p+1, \dots, 2p-1\}$. Consider two sequences of length pq:

$$a = \Big(\frac{q}{\lceil k/p \rceil}\Big)_{k=1}^{pq} \quad \text{and} \quad b = \Big(\frac{p}{\lceil k/q \rceil}\Big)_{k=1}^{pq}.$$

For $k \leq p$, we get $a_k/b_k = q/p \geq \frac{1}{2}$. Similarly for $k > p$, from $\lceil k/p \rceil \leq 2k/p$,

$$\frac{a_k}{b_k} = \frac{q\lceil k/q \rceil}{p\lceil k/p \rceil} \geq \frac{q \cdot k/q}{p \cdot 2k/p} = \frac{1}{2}.$$

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

Consequently $b_k \le 2a_k$ for all $k \in \{1, \dots, pq\}$. Thus, by monotonicity, homogeneity and repetition invariance of $\mathcal{M}$,

$$d_p = \mathop{\mathcal{M}}_{k=1}^{p}\Big(\frac{p}{k}\Big) = \mathop{\mathcal{M}}_{k=1}^{pq} b_k \le 2 \mathop{\mathcal{M}}_{k=1}^{pq} a_k = 2 \mathop{\mathcal{M}}_{k=1}^{q}\Big(\frac{q}{k}\Big) = 2d_q,$$

which is (i).

The second inequality is significantly simpler. Indeed, for every $p \in \mathbb{N}$,

$$d_p = \mathop{\mathcal{M}}_{k=1}^{p}\Big(\frac{p}{k}\Big) = \mathop{\mathcal{M}}_{k=1}^{2p}\Big(p\Big\lceil\frac{k}{2}\Big\rceil^{-1}\Big) \le \mathop{\mathcal{M}}_{k=1}^{2p}\Big(p \cdot \Big(\frac{k}{2}\Big)^{-1}\Big) = \mathop{\mathcal{M}}_{k=1}^{2p}\Big(\frac{2p}{k}\Big) = d_{2p}.$$

Define $s \in \mathbb{N} \cup \{0\}$ and $\theta \in [1, 2)$ such that $n = 2^s\theta m$. Applying (ii) iteratively and then (i),

$$d_m \le d_{2m} \le \cdots \le d_{2^s m} \le 2d_{2^s\theta m} = 2d_n,$$

as required. □

Combining this result with Corollary 2.3 gives the following result.

Corollary 2.5. *Let $\mathcal{M}$ be a homogeneous, monotone and repetition-invariant mean. If there exist $C, D \in \mathbb{R}_+$ and $n_0 \in \mathbb{N}$ such that condition (2.3) holds, then $\mathcal{M}$ is not a weak Hardy mean.*

3. Applications

In the subsequent sections we discuss the weak Hardy property for several families of means.

3.1. Quasi-arithmetic means. Quasi-arithmetic means were introduced in a series of papers [3, 7, 8, 14] in the 1920s and 30s as a generalisation of the family of power means. For a continuous and strictly monotone function $f : I \to \mathbb{R}$ (hereafter I is an interval and $\mathcal{CM}(I)$ the family of all continuous and monotone functions on I) and a vector $a = (a_1, a_2, \dots, a_n) \in I^n$, $n \in \mathbb{N}$, we define

$$\mathcal{A}^{[f]}(a) := f^{-1}\Big(\frac{f(a_1) + f(a_2) + \cdots + f(a_n)}{n}\Big).$$

For a subinterval $J \subset I$ we denote by $\mathcal{A}^{[f]}|_J$ the restriction of the quasi-arithmetic mean to an interval J, that is, $\mathcal{A}^{[f]}|_J := \mathcal{A}^{[f]}|_{\bigcup_{n=1}^{\infty} J^n}$. It is easy to verify that for $I = \mathbb{R}_+$ and $f = \pi_p$, where $\pi_p(x) := x^p$ if $p \ne 0$ and $\pi_0(x) := \ln x$, the mean $\mathcal{A}^{[f]}$ coincides with the pth power mean.

The Hardy property for this family was characterised by Mulholland [13]. He proved that $\mathcal{A}^{[f]}$ is a Hardy mean if and only if there exist $\alpha < 1$ and $C > 0$ such that $\mathcal{A}^{[f]}(a) \le C \cdot \mathcal{P}_\alpha(a)$ for every $a \in \bigcup_{n=1}^{\infty} I^n$. Next we consider the weak Hardy property. First, we present a result which provides localisability of the weak Hardy property for quasi-arithmetic means.

Theorem 3.1. *Let I be an interval with* $\inf I = 0$ *and let* $f \in \mathcal{CM}(I)$*. If there exists* $\varepsilon \in I$*, such that* $\mathcal{A}^{[f]}|_{(0,\varepsilon)}$ *is a weak Hardy mean, then* $\mathcal{A}^{[f]}$ *is a weak Hardy mean.*

We also establish a much stronger result under somewhat different assumptions.

Theorem 3.2. *Let I be an interval with* $\inf I = 0$ *and let* $f \in \mathcal{CM}(I)$*. If there exists* $\varepsilon \in I$*, such that* $\mathcal{A}^{[f]}|_{(0,\varepsilon)}$ *is a Hardy mean, then there exists a function* $c_f \colon I \to \mathbb{R}_+$ *such that*

$$\sum_{n=1}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(a_k) \le c_f(\|a\|_\infty) \cdot \|a\|_1 \quad \textit{for all } a \in \ell_1(I).$$

Proofs of these theorems are postponed until Sections 4.1 and 4.2, respectively.

We conjecture that the weak Hardy property for quasi-arithmetic means is equivalent to the fact that its restriction to some interval $(0, \varepsilon)$ (for $\varepsilon \in I$) is a Hardy mean. It is worth mentioning that this property does not depend on the choice of ε. More precisely we have the following result.

Corollary 3.3. *Let I be an interval with* $\inf I = 0$ *and let* $f \in \mathcal{CM}(I)$*. If there exists* $\varepsilon \in I$ *such that* $\mathcal{A}^{[f]}|_{(0,\varepsilon)}$ *is a Hardy mean then* $\mathcal{A}^{[f]}|_{(0,s)}$ *is a Hardy mean for all* $s \in I$*.*

Proof. If $s \le \varepsilon$ the statement is trivial. From now on, assume that $s > \varepsilon$. By Theorem 3.2, there exists a constant $C := c_f(s)$ such that

$$\sum_{n=1}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(a_k) \le C \cdot \|a\|_1 \quad \text{for all } a \in \ell_1(I) \text{ with } \|a\|_\infty = s.$$

Now take $v \in \ell_1(0, s)$. If $\|v\|_\infty \le \varepsilon$, then

$$\sum_{n=1}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(v_k) \le \mathcal{H}(\mathcal{A}^{[f]}|_{(0,\varepsilon)}) \sum_{n=1}^{\infty} v_n.$$

For $\|v\|_\infty \in (\varepsilon, s]$, let us add the artificial element $v_0 = s$. Then, as $v_0 \ge v_i$ for all $i \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(v_k) &\le \sum_{n=1}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(v_k) \le \sum_{n=0}^{\infty} \mathcal{A}_{k=1}^{n}{}^{[f]}(v_k) \le c_f(s) \sum_{n=0}^{\infty} v_n = s c_f(s) + c_f(s) \sum_{n=1}^{\infty} v_n \\ &\le c_f(s) \cdot \frac{s}{\varepsilon} \cdot \sup_{n \in \{1,2,\dots\}} v_n + c_f(s) \sum_{n=1}^{\infty} v_n \le \Big(1 + \frac{s}{\varepsilon}\Big) c_f(s) \sum_{n=1}^{\infty} v_n.\end{aligned}$$

Thus $\mathcal{A}^{[f]}$ restricted to $(0, s)$ is a Hardy mean with Hardy constant majorised by

$$\max\Big(\mathcal{H}(\mathcal{A}^{[f]}|_{(0,\varepsilon)}), \Big(1 + \frac{s}{\varepsilon}\Big) c_f(s)\Big).$$

This completes the proof. □

We conclude this section with a simple example showing that in a family of quasi-arithmetic means not every weak Hardy mean is a Hardy mean.

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

EXAMPLE 3.4. Let $f\colon (0,+\infty) \to \mathbb{R}$ be given by

$$f(x) := \begin{cases} \ln x & \text{if } x \in (0,1], \\ x-1 & \text{if } x \in (1,\infty). \end{cases}$$

Since $\mathcal{A}^{[f]}$ restricted to $(0,1]$ is a geometric mean $(\mathcal{P}_0)$, by Theorem 3.2, $\mathcal{A}^{[f]}$ is a weak Hardy mean. We prove that it is not a Hardy mean.

Indeed, fix $N \in \mathbb{N}$ arbitrarily and define $a_n := N^2/n^2$. Then

$$\sum_{n=1}^{\infty} \mathop{\mathcal{A}}_{k=1}^{n}{}^{[f]}(a_k) \le \mathcal{H}(\mathcal{A}^{[f]}) \cdot \sum_{n=1}^{\infty} a_n = N^2 \cdot \mathcal{H}(\mathcal{A}^{[f]}) \cdot \frac{\pi^2}{6}. \tag{3.1}$$

On the other hand, $a_n \ge 1$ for all $n \le N$ and, as $\mathcal{A}^{[f]}$ restricted to $[1,\infty)$ coincides with the arithmetic mean,

$$\mathop{\mathcal{A}}_{k=1}^{n}{}^{[f]}(a_k) = \frac{1}{n}\sum_{k=1}^{n} a_k \ge \frac{a_1}{n} = \frac{N^2}{n} \quad (n \le N).$$

Thus, using the well-known estimate for the harmonic sequence,

$$\sum_{n=1}^{\infty} \mathop{\mathcal{A}}_{k=1}^{n}{}^{[f]}(a_k) \ge \sum_{n=1}^{N} \mathop{\mathcal{A}}_{k=1}^{n}{}^{[f]}(a_k) \ge \sum_{n=1}^{N} \frac{N^2}{n} \ge N^2 \ln N. \tag{3.2}$$

If we now combine (3.1) and (3.2) we obtain $N^2 \ln N \le N^2 \cdot \mathcal{H}(\mathcal{A}^{[f]}) \cdot \pi^2/6$, which simplifies to $\mathcal{H}(\mathcal{A}^{[f]}) \ge (6/\pi^2)\ln N$. Letting $N \to \infty$ gives $\mathcal{H}(\mathcal{A}^{[f]}) = +\infty$, which proves that $\mathcal{A}^{[f]}$ is not a Hardy mean.

3.2. Gini means. Another generalisation of Power Means was proposed in 1938 by Gini [5]. For $p, q \in \mathbb{R}$, the *Gini means* form a two-parameter family defined on $\mathbb{R}_+$ by

$$\mathcal{G}_{p,q}(a_1,\dots,a_n) := \begin{cases} \Big(\sum_{i=1}^{n} a_i^p \Big/ \sum_{i=1}^{n} a_i^q\Big)^{1/(p-q)} & \text{if } p \ne q, \\ \exp\Big(\sum_{i=1}^{n} a_i^p \ln a_i \Big/ \sum_{i=1}^{n} a_i^p\Big) & \text{if } p = q. \end{cases}$$

For $q = 0$ one easily identifies here the pth power mean. It is known that $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$ and Gini means are nondecreasing with respect to p and q (see [1, page 249]). Furthermore, from [17, 18],

$$\mathcal{G}_{p,q} \text{ is a Hardy mean} \iff \min(p,q) \le 0 \text{ and } \max(p,q) < 1.$$

We prove that the weak Hardy and Hardy properties coincide for Gini means.

PROPOSITION 3.5. *A Gini mean is a weak Hardy mean if and only if it is a Hardy mean.*

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

PROOF. First, it was proved in [18] that for all $q < 0$ the mean $\mathcal{G}_{1,q}$ satisfies inequality (2.3). Furthermore, by the results of Losonczi [11, 12], $\mathcal{G}_{p,q}$ is monotone if and only if $pq \le 0$. As both homogeneity and repetition invariance are easy to check, by Corollary 2.5, $\mathcal{G}_{1,q}$ is a weak Hardy mean for no $q < 0$. Since $\mathcal{G}_{p,q} \le \mathcal{G}_{p',q'}$ for every $p \le p'$ and $q \le q'$, it follows that $\mathcal{G}_{p,q}$ is not a weak Hardy mean whenever $\max(p, q) \ge 1$. In other words,

$$\mathcal{G}_{p,q} \text{ is a weak Hardy mean} \implies \max(p, q) < 1. \tag{3.3}$$

Now suppose that p, $q \in (0, 1)$, $p \ne q$. For $a_n := 2^{1-n}$,

$$\mathcal{G}_{p,q}(a_1, \ldots, a_n) = \left(\frac{1 + 2^{-p} + \cdots + 2^{(1-n)p}}{1 + 2^{-q} + \cdots + 2^{(1-n)q}}\right)^{1/(p-q)} = \left(\frac{1 - 2^{-np}}{1 - 2^{-nq}} \cdot \frac{1 - 2^{-q}}{1 - 2^{-p}}\right)^{1/(p-q)}$$

Thus

$$\lim_{n\to\infty} \mathcal{G}_{p,q}(a_1, \ldots, a_n) = \left(\frac{1 - 2^{-q}}{1 - 2^{-p}}\right)^{1/(p-q)} > 0.$$

This shows that $\mathcal{G}_{p,q}$ is not a weak Hardy mean for $(p, q) \in (0, 1)^2$, $p \ne q$. Moreover, for $p \in (0, 1)$, it is easy to see that $\mathcal{G}_{p,p} \ge \mathcal{G}_{p,p/2}$ which implies that $\mathcal{G}_{p,p}$ is not a weak Hardy mean. These facts together with (3.3) yield

$$\mathcal{G}_{p,q} \text{ is weak Hardy} \implies (\min(p, q) \le 0 \text{ and } \max(p, q) < 1) \implies \mathcal{G}_{p,q} \text{ is Hardy.}$$

As the converse implication is trivial, the proof is complete. □

REMARK 3.6. We can use the same argument to prove that the Gaussian product of power means has the same property (see [18, section 3.1] and Corollary 2.5 above).

REMARK 3.7. It remains an open question whether it is possible to verify the equivalence of the Hardy and weak Hardy properties without verifying these properties separately.

4. Proofs of Theorems 3.1 and 3.2

Theorems 3.1 and 3.2 are closely related, but neither is a consequence of the other. In fact, Theorem 3.2 is a stronger result under more restrictive assumptions. This motivates us to combine the two proofs in a rather unconventional way.

In the first subsection we prove Theorem 3.1. Then we prove Theorem 3.2. However, as this theorem has stronger assumptions, all the intermediate steps and notations in Section 4.1 remain valid in Section 4.2. Consequently, we may refer to them as it they were an intrinsic part of the proof.

4.1. Proof of Theorem 3.1. First, choose an arbitrary sequence $a \in \ell_1(I)$. If we define $f(0) := \lim_{x\to 0^+} f(x) \in [-\infty, +\infty]$ and apply the Cesàro limit principle and continuity of f^{-1} on $f(I \cup \{0\})$, we obtain, as $a_n \to 0$,

$$0 = f^{-1}(f(0)) = f^{-1}(\lim_{n\to\infty} f(a_n)) = f^{-1}\Big(\lim_{n\to\infty} \frac{f(a_1) + \cdots + f(a_n)}{n}\Big)$$
$$= \lim_{n\to\infty} f^{-1}\Big(\frac{f(a_1) + \cdots + f(a_n)}{n}\Big) = \lim_{n\to\infty} \mathcal{A}^{[f]}(a_1, \ldots, a_n).$$

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

Let $n_0 \in \mathbb{N}$ be the smallest natural number such that

$$\mathcal{A}^{[f]}(a_1, \ldots, a_n) \le \varepsilon \quad \text{and} \quad a_n \le \varepsilon \quad \text{for all } n \ge n_0.$$

Define the sequence $(b_n)_{n=1}^{\infty}$ by

$$b_n := \begin{cases} \varepsilon & \text{for } n \le n_0, \\ a_n & \text{for } n > n_0. \end{cases}$$

Then

$$\|b\|_1 \le \varepsilon n_0 + \|a\|_1 . \tag{4.1}$$

Since $\mathcal{A}^{[f]}$ is associative and monotone,

$$\begin{aligned}\sum_{n=1}^{\infty} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k) &= \sum_{n=1}^{n_0} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k) + \sum_{n=n_0+1}^{\infty} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k) \\ &\le \sum_{n=1}^{n_0} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k) + \sum_{n=n_0+1}^{\infty} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(b_k).\end{aligned}$$

But $\|b_n\|_\infty \le \varepsilon$ and so

$$\sum_{n=1}^{\infty} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k) \le n_0 \|a\|_\infty + \sum_{n=1}^{\infty} \mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(b_k) < +\infty, \tag{4.2}$$

proving that $\mathcal{A}^{[f]}$ is a weak Hardy mean.

4.2. Proof of Theorem 3.2. As we remarked at the beginning of this section, all conventions and results from the previous subsection remain valid.

By Mulholland's result, as $\mathcal{A}^{[f]}|_{(0,\varepsilon)}$ is a Hardy mean, there exist $\alpha < 1$ and $C > 0$ such that

$$\mathcal{A}^{[f]}(v) \le C \cdot \mathcal{P}_\alpha(v) \quad \text{for every } v \in \bigcup_{n=1}^{\infty} (0, \varepsilon)^n,$$

Therefore we can put $c_f(x) := C \cdot \mathcal{H}(\mathcal{P}_\alpha)$ for $x \le \varepsilon$. From now on we assume that $\|a\|_\infty > \varepsilon$.

Following the idea of [15, Proposition 3.2], we may assume that the sequence (a_n) is nonincreasing, that is, $\|a\|_\infty = a_1$. Furthermore $a_n \le \|a\|_1 / n$. Then n_0 is the smallest natural number such that

$$\mathcal{A}^{[f]}(a_1, \ldots, a_n) \le \varepsilon \quad \text{for all } n \ge n_0.$$

Indeed, as (a_n) is nonincreasing, $a_1 > \varepsilon$ and the quasi-arithmetic mean is strict, we obtain $a_n < \mathcal{A}^{[f]}(a_1, \ldots, a_n) \le \varepsilon$ for all $n \ge n_0$. On the other hand, since $\|b\|_\infty = \varepsilon$,

$$\mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(b_k) \le C \cdot \mathop{\mathcal{P}_\alpha}_{k=1}^{n}(b_k) \quad \text{for all } n \in \mathbb{N}.$$

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

As $\mathcal{P}_\alpha$ is a Hardy mean, we obtain (in the spirit of Mulholland [13]),

$$\sum_{n=n_0+1}^{\infty} \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}(b_k) \le \sum_{n=1}^{\infty} \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}(b_k) \le C \cdot \sum_{n=1}^{\infty} \underset{k=1}{\overset{n}{\mathcal{P}_\alpha}}(b_k) \le C \cdot \mathcal{H}(\mathcal{P}_\alpha) \|b\|_1 .$$

Combining this with (4.2),

$$\sum_{n=1}^{\infty} \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}(a_k) \le n_0 \|a\|_\infty + C \cdot \mathcal{H}(\mathcal{P}_\alpha) \|b\|_1 . \tag{4.3}$$

We now estimate n_0. Since (a_n) is nonincreasing, $a_k \le \|a\|_1 /k$ for all $k \in \mathbb{N}$. Thus

$$\underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}(a_k) \le \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}\Big(\min\Big(\frac{\|a\|_1}{k}, \|a\|_\infty\Big)\Big).$$

Let $u(s,t)$ $(s \ge t \ge \varepsilon)$ be the smallest natural number such that

$$\underset{k=1}{\overset{u(s,t)}{\mathcal{A}^{[f]}}}\Big(\min\Big(\frac{s}{k}, t\Big)\Big) \le \varepsilon.$$

It follows that

$$\underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}(a_k) \le \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}\Big(\min\Big(\frac{\|a\|_1}{k}, \|a\|_\infty\Big)\Big) \le \varepsilon \quad \text{for } n \ge u(\|a\|_1, \|a\|_\infty).$$

Thus $n_0 \le u(\|a\|_1, \|a\|_\infty)$.

Define a weighted quasi-arithmetic mean of two variables by

$$\mathcal{A}^{[f]}((a_1, a_2), (w_1, w_2)) := f^{-1}\Big(\frac{w_1 f(a_1) + w_2 f(a_2)}{w_1 + w_2}\Big).$$

Let $K\colon I \cap (\varepsilon, +\infty) \to (0, +\infty)$ be the unique function such that

$$\mathcal{A}^{[f]}\Big(\Big(t, \frac{\varepsilon}{2}\Big), (1,\ K(t))\Big) = \varepsilon \quad \text{for } t \in I \cap (\varepsilon, +\infty).$$

Since $\mathcal{A}^{[f]}$ is monotone, for $n \ge \lceil 2s/\varepsilon \rceil$,

$$\begin{aligned}\underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}\Big(\min\Big(\frac{s}{k}, t\Big)\Big) &\le \mathcal{A}^{[f]}\Big(\Big(t, \frac{\varepsilon}{2}\Big), \Big(\Big\lceil \frac{2s}{\varepsilon} \Big\rceil,\ n - \Big\lceil \frac{2s}{\varepsilon} \Big\rceil\Big)\Big) \\ &= \mathcal{A}^{[f]}\Big(\Big(t, \frac{\varepsilon}{2}\Big), \Big(1,\ n\Big\lceil \frac{2s}{\varepsilon} \Big\rceil^{-1} - 1\Big)\Big).\end{aligned}$$

Now, for all $n \in \mathbb{N}$ such that $n < u(s,t)$ and $n \ge \lceil 2s/\varepsilon \rceil$,

$$\varepsilon \le \underset{k=1}{\overset{n}{\mathcal{A}^{[f]}}}\Big(\min\Big(\frac{s}{k}, t\Big)\Big) \le \mathcal{A}^{[f]}\Big(\Big(t, \frac{\varepsilon}{2}\Big), \Big(1,\ n\Big\lceil \frac{2s}{\varepsilon} \Big\rceil^{-1} - 1\Big)\Big).$$

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

Thus

$$n\left\lceil\frac{2s}{\varepsilon}\right\rceil^{-1}-1\le K(t)\qquad\text{for all } n\in\mathbb{N} \text{ such that } n<u(s,t) \text{ and } n\ge\left\lceil\frac{2s}{\varepsilon}\right\rceil.$$

In particular for $n := u(s,t)-1$,

$$(u(s,t)-1)\left\lceil\frac{2s}{\varepsilon}\right\rceil^{-1}-1\le K(t)\quad\text{ or }\quad u(s,t)\le\left\lceil\frac{2s}{\varepsilon}\right\rceil+1,$$

which implies

$$u(s,t)\le(K(t)+1)\left\lceil\frac{2s}{\varepsilon}\right\rceil+1.$$

Dividing both sides by s and take an upper limit as $s\to\infty$ yields

$$\limsup_{s\to\infty}\frac{u(s,t)}{s}\le\frac{2(K(t)+1)}{\varepsilon}.$$

Thus, as u is nondecreasing with respect to both variables, there exists a function $\Phi\colon(\varepsilon,+\infty)\to\mathbb{R}$ such that

$$u(s,t)\le\Phi(t)\cdot s\qquad\text{for all } s\ge t\ge\varepsilon.$$

In particular $n_0\le u(\|a\|_1,\|a\|_\infty)\le\Phi(\|a\|_\infty)\cdot\|a\|_1$. Combining this inequality with (4.3) and (4.1),

$$\begin{aligned}\sum_{n=1}^{\infty}\mathop{\mathcal{A}^{[f]}}_{k=1}^{n}(a_k)&\le n_0\,\|a\|_\infty+C\cdot\mathcal{H}(\mathcal{P}_\alpha)(\varepsilon n_0+\|a\|_1)\\&=(\|a\|_\infty+\varepsilon C\cdot\mathcal{H}(\mathcal{P}_\alpha))\cdot n_0+C\cdot\mathcal{H}(\mathcal{P}_\alpha)\cdot\|a\|_1\\&\le(\|a\|_\infty+\varepsilon C\cdot\mathcal{H}(\mathcal{P}_\alpha))\cdot\Phi(\|a\|_\infty)\cdot\|a\|_1+C\cdot\mathcal{H}(\mathcal{P}_\alpha)\cdot\|a\|_1\\&=((\|a\|_\infty+\varepsilon C\cdot\mathcal{H}(\mathcal{P}_\alpha))\cdot\Phi(\|a\|_\infty)+C\cdot\mathcal{H}(\mathcal{P}_\alpha))\cdot\|a\|_1\end{aligned}$$

To conclude the proof, take

$$c_f(x):=\begin{cases}C\cdot\mathcal{H}(\mathcal{P}_\alpha) & \text{for } x\le\varepsilon,\\(x+\varepsilon C\cdot\mathcal{H}(\mathcal{P}_\alpha))\cdot\Phi(x)+C\cdot\mathcal{H}(\mathcal{P}_\alpha) & \text{for } x>\varepsilon.\end{cases}$$

References

[1] P. S. Bullen, *Handbook of Means and their Inequalities*, Mathematics and its Applications, Vol. 560 (Kluwer Academic Publishers Group, Dordrecht, 2003).

[2] T. Carleman, 'Sur les fonctions quasi-analytiques', in: *Conférences faites au cinquième congrès des mathématiciens scandinaves, Helsinki (1922)* (Akadem, Buchh., Helsinki, 1923), 181–196.

[3] B. de Finetti, 'Sul concetto di media', *Giornale dell' Instituto, Italiano degli Attuarii* **2** (1931), 369–396.

[4] J. Duncan and C. M. Mcgregor, 'Carleman's inequality', *Amer. Math. Monthly* **110**(5) (2003), 424–431.

[5] C. Gini, 'Di una formula compressiva delle medie', *Metron* **13** (1938), 3–22.

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press

[6] G. H. Hardy, 'Note on a theorem of Hilbert', *Math. Z.* **6** (1920), 314–317.
[7] K. Knopp, 'Über Reihen mit positiven Gliedern', *J. Lond. Math. Soc.* **3** (1928), 205–211.
[8] A. N. Kolmogorov, 'Sur la notion de la moyenne', *Rend. Accad. dei Lincei (6)* **12** (1930), 388–391.
[9] A. Kufner, L. Maligranda and L. E. Persson, *The Hardy Inequality: About its History and Some Related Results* (Vydavatelskỳ servis, Pilsen, 2007).
[10] E. Landau, 'A note on a theorem concerning series of positive terms', *J. Lond. Math. Soc.* **1** (1921), 38–39.
[11] L. Losonczi, 'Subadditive Mittelwerte', *Arch. Math. (Basel)* **22** (1971), 168–174.
[12] L. Losonczi, 'Subhomogene Mittelwerte', *Acta Math. Acad. Sci. Hungar.* **22** (1971), 187–195.
[13] P. Mulholland, 'On the generalization of Hardy's inequality', *J. Lond. Math. Soc.* **7** (1932), 208–214.
[14] M. Nagumo, 'Über eine Klasse der Mittelwerte', *Jpn. J. Math.* **7** (1930), 71–79.
[15] Zs. Páles and P. Pasteczka, 'Characterization of the Hardy property of means and the best Hardy constants', *Math. Inequal. Appl.* **19** (2016), 1141–1158.
[16] Zs. Páles and P. Pasteczka, 'On the best Hardy constant for quasi-arithmetic means and homogeneous deviation means', *Math. Inequal. Appl.* **21** (2018), 585–599.
[17] Zs. Páles and L.-E. Persson, 'Hardy type inequalities for means', *Bull. Aust. Math. Soc.* **70**(3) (2004), 521–528.
[18] P. Pasteczka, 'On negative results concerning Hardy means', *Acta Math. Hungar.* **146**(1) (2015), 98–106.
[19] P. Pasteczka, 'On some Hardy type inequalities involving generalized means', *Publ. Math. Debrecen* **87**(1–2) (2015), 167–173.
[20] J. E. Pečarić and K. B. Stolarsky, 'Carleman's inequality: history and new generalizations', *Aequationes Math.* **61**(1–2) (2001), 49–62.

PAWEŁ PASTECZKA, Institute of Mathematics,
Pedagogical University of Cracow,
Podchorążych str. 2, 30-084 Kraków, Poland
e-mail: pawel.pasteczka@up.krakow.pl

https://doi.org/10.1017/S0004972719000686 Published online by Cambridge University Press