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ELEMENTARY AMENABLE GROUPS AND 4-MANIFOLDS WITH EULER CHARACTERISTIC 0

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Abstract

We extend earlier work relating asphericity and Euler characteristics for finite complexes whose fundamental groups have nontrivial torsion free abelian normal subgroups. In particular a finitely presentable group which has a nontrivial elementary amenable subgroup whose finite subgroups have bounded order and with no nontrivial finite normal subgroup must have deficiency at most 1, and if it has a presentation of deficiency 1 then the corresponding 2-complex is aspherical. Similarly if the fundamental group of a closed 4-manifold with Euler characteristic 0 is virtually torsion free and elementary amenable then it either has 2 ends or is virtually an extension of $\mathbb{Z}$ by a subgroup of $\mathbb{Q}$, or the manifold is aspherical and the group is virtually poly-$\mathbb{Z}$ of Hirsch length 4.

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Rosset has shown that if a group G has a nontrivial torsion free abelian normal subgroup A then the group ring $C[G]$ may be localized with respect to the multiplicative system $C[A]-\{0\}$, and that nontrivial stably free modules over the localization have well defined, strictly positive rank [23] (this was first proven for the unlocalized group ring by Kaplansky (cf. [16, page 122]). Such localizations have been used to strengthen the theorem of Gottlieb on Euler characteristics of finite aspherical complexes [11], to prove a converse for 2-complexes and to show that the closed 4-manifold obtained by surgery on a 2-knot whose group has such a subgroup is often aspherical [6, 7, 8, 10, 13, 14, 15, 24].

We wish to extend this technique to the case when "abelian" is relaxed to "elementary amenable". That such an extension might exist is suggested by the work of Cheeger and Gromov, who used L^2-cohomology to show that Gottlieb's theorem holds under the assumption that G has an infinite amenable normal subgroup [3]. (If G is the group of a finite aspherical complex such a subgroup must be torsion free.) Moreover Kropholler, Linnell and Moody have shown that the rational group ring of an elementary amenable group whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup has a classical ring of fractions which is a matrix algebra over a division ring [18]. (Linnell has also noticed that Rosset's theorem may be so extended [19], and we have used his work to strengthen our results below.)

Some restriction on the group is to be expected, for any connected graph is aspherical, and so the technique cannot apply to the group rings of non-abelian free groups. Elementary amenable groups do not have nonabelian free subgroups. (There are torsion groups which are not amenable [20], and amenable groups which are not elementary amenable [12].) It is noteworthy also that the class of elementary amenable groups is at present the largest class of groups over which the 4-dimensional TOP disk embedding theorem is known [9], and that the general problem can be reduced to the case of free groups [2].

In Section 1 we shall extend the notion of Hirsch length to elementary amenable groups (Theorem 1) and show that torsion free elementary amenable groups of small Hirsch length are solvable (Theorem 2). In Section 2 we show that if a (discrete) group G has a nontrivial elementary amenable normal subgroup N whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then $Q[G]$ embeds in a ring with the strong invariant basis number property which is flat as a $Q[G]$-algebra and such that the tensor product with the augmentation module Q is 0 (Theorem 3). In Section 3 we shall use this theorem and the result of [18] to extend the arguments of [15, Chapter 3]. Theorem 4 gives an extension of Gottlieb's result, and its corollary gives an application to combinatorial group theory. Theorems 5 and 6 give applications to 4-manifolds. In particular, we show that if M is a closed orientable 4-manifold with $\chi(M) = 0$ and $G = \pi_1(M)$ is torsion free and elementary amenable then either $G \cong \mathbb{Z}$ or G has a presentation $\langle a, t | tat^{-1} = a^n \rangle$ for some $n \neq 0$ or M is aspherical and G is virtually poly-$\mathbb{Z}$ of Hirsch length 4.

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1. Elementary amenable groups and Hirsch length

The class EG of elementary amenable groups is the smallest class which contains all finite groups and is closed under extension and increasing union, and the formation of sub- and quotient-groups. The class EG may also be described as follows. If X, Y are classes of groups, let XY denote the class of groups G which have a normal subgroup H in X such that the quotient G/H is in Y, and let LX denote the class of groups such that each finitely generated subgroup is contained in some X-subgroup. Let $X_0 = \{1\}$ and let X_1 be the class of finitely generated virtually abelian groups. If X_α has been defined for some ordinal α let $X_{\alpha+1} = (LX_\alpha)X_1$, and if X_α has been defined for all ordinals α less than some limit ordinal β let $X_\beta = \bigcup X_\alpha$. Then it is not hard to show by transfinite induction that each X_α is subgroup closed, that $X_\alpha X_\beta \subseteq X_{\alpha+\beta}$ and that $EG = \bigcup X_\alpha$ [18]. (Similarly it can be shown that each X_α is closed under taking quotients by normal subgroups.) If G is an elementary amenable group we shall let $\alpha(G) = \min\{\alpha | G \text{ is in } X_\alpha\}$.

Using transfinite induction it may also be shown that torsion groups in EG are locally finite, that finitely generated simple groups in EG are finite, and that no group in EG has a nonabelian free subgroup [4]. Every virtually solvable group is elementary amenable, but the converse is false.

EXAMPLE. Let $\mathbb{Z}^\infty$ be the free abelian group on generators $\{x_i | i \text{ in } \mathbb{Z}\}$ and let G be the subgroup of $\mathrm{Aut}(\mathbb{Z}^\infty)$ generated by $\{e_i | i \text{ in } \mathbb{Z}\}$, where $e_i(x_i) = x_i + x_{i+1}$ and $e_i(x_j) = x_j$ if $j \neq i$. As G is the increasing union of subgroups isomorphic to groups of upper triangular integer matrices it is locally nilpotent. However it has no nontrivial abelian normal subgroup. If we let ϕ be the automorphism of G defined by $\phi(e_i) = e_{i+1}$ for all i then the corresponding extension of $\mathbb{Z}$ by G is a finitely generated torsion free elementary amenable group which is not virtually solvable.

LEMMA 1. *Let G be a finitely generated infinite elementary amenable group. Then G has normal subgroups $K < H$ such that G/H is finite, H/K is a free abelian of positive rank and the action of G/H on H/K by conjugation is effective.*

PROOF. By transfinite induction on $\alpha(G)$ we may show that G has a normal subgroup K such that G/K is an infinite virtually abelian group

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(that is, in X_1). We may assume that G/K has no nontrivial finite normal subgroup. If H is a subgroup of G which contains K and is such that H/K is a maximal abelian normal subgroup of G/K then H and K satisfy the above conditions. □

We may extend the notion of Hirsch length to elementary amenable groups as follows. In general it shall be a nonnegative integer or ∞. If G is in X_1, that is, has a finitely generated abelian subgroup A of finite index, let $h(G) = \operatorname{rank} A$. Suppose that the Hirsch length has been defined for all groups in X_α and that $\alpha(G) = \alpha + 1$. If G is in LX_α let $h(G) = \operatorname{lub}\{h(F) | F$ is an X_α-subgroup of $G\}$. If G is not in LX_α but has a normal subgroup K in LX_α with quotient in X_1, let $h(G) = h(K) + h(G/K)$. In the following theorem we shall show that this sum is independent of the choice of such a normal subgroup.

THEOREM 1. *Let G be an elementary amenable group. Then*

(a) *$h(G)$ is well defined,*
(b) *if H is a subgroup of G then $h(H) \le h(G)$,*
(c) *$h(G) = \operatorname{lub}\{h(F) | F$ is a finitely generated subgroup of $G\}$, and*
(d) *if H is a normal subgroup of G then $h(G) = h(H) + h(G/H)$.*

PROOF. We shall prove all four assertions simultaneously by induction on $\alpha(G)$. They are clearly true when $\alpha(G) = 1$. Suppose that they hold for all groups in X_α and that $\alpha(G) = \alpha+1$. If G is in LX_α then so is any subgroup, and (a) and (b) are immediate, while (c) follows since it holds for groups in X_α and since each finitely generated subgroup of G is an X_α-subgroup. To prove (d) we may assume that $h(H)$ is finite, for otherwise both $h(G)$ and $h(H) + h(G/H)$ are ∞, by (b). Therefore by (c) there is a finitely generated subgroup J of H with $h(J) = h(H)$. Given a finitely generated subgroup Q of G/H, we may choose a finitely generated subgroup F of G containing J and whose image in G/H is Q. Since F is finitely generated it is in X_α and so $h(G) = h(H) + h(Q)$. Taking least upper bounds over all such Q we have $h(G) \ge h(H) + h(G/H)$. On the other hand if F is any X_α-subgroup of G then $h(F) = h(F \cap H) + h(FH/H)$, since (d) holds for F, and so $h(G) \le h(H) + h(G/H)$. Thus (d) holds for G also.

Now suppose that G is not in LX_α, but has a normal subgroup K in LX_α such that G/K is in X_1. If K_1 is another such normal subgroup then (d) holds for K and K_1 by the hypothesis of induction and so $h(K) = h(K \cap K_1) + h(KK_1/K_1)$ and $h(K_1) = h(K \cap K_1) + h(KK_1/K)$. Since we also have $h(G/K) = h(G/KK_1) + h(KK_1/K)$ and $h(G/K_1) = h(G/KK_1) + h(KK_1/K_1)$ it follows that $h(K_1) + h(G/K_1) = h(K) + h(G/K)$ and so $h(G)$ is well defined. As any subgroup of G is an extension of a subgroup of G/K

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by a subgroup of K, (b) follows easily. By the hypothesis of induction, (c) holds for K. Therefore if $h(K)$ is finite K has a finitely generated subgroup J with $h(J) = h(K)$. Since G/K is finitely generated there is a finitely generated subgroup F of G containing J and such that $FK/K = G/K$. Clearly $h(F) = h(G)$. If $h(K)$ is infinite then for every $n \geq 0$ there is a finitely generated subgroup J_n of K with $h(J_n) \geq n$. In either case, (c) also holds for G. If H is a normal subgroup of G then H and G/H are also in $X_{\alpha+1}$, while $H \cap K$ and $HK/H = K/H \cap K$ ae in LX_α and $HK/K = H/H \cap K$ and G/HK are in X_1. Therefore

$$\begin{aligned} h(H) + h(G/H) &= h(H \cap K) + h(HK/K) + h(HK/H) + h(G/HK) \\ &= h(H \cap K) + h(HK/H) + h(HK/K) + h(G/HK). \end{aligned}$$

Since K is in LX_α and G/K is in X_1 this sum equals $h(G) = h(K) + h(G/K)$ and so (d) holds for G. This completes the inductive step.

It is easy to see that a group is elementary amenable of Hirsch length 0 if and only if it is locally finite. More generally if G is elementary amenable and H is a locally finite normal subgroup of G then $h(G/H) = h(G)$. Our next theorem shows that elementary amenable groups of small Hirsch length are extensions of solvable groups by locally finite normal subgroups.

THEOREM 2. *Let G be an elementary amenable group and let T be its maximal locally finite normal subgroup. If $h(G)$ is finite then G is in $LX_{h(G)+1}$. If $h(G) \leq 3$ then G/T is solvable, of derived length at most 5. Moreover if $h(G) = 1$ or 2 and G is finitely generated then G/T is virtually torsion free.*

PROOF. The first assertion follows by induction on $h(G)$, from Lemma 1. To prove the other assertions we may assume that $T = 1$. We may also assume that G is finitely generated, for although a locally solvable group need not be solvable, it is solvable of derived length at most d if every finitely generated subgroup is solvable of derived length at most d. If $h(G) = 0$ then G is finite and hence trivial. Therefore we may also assume that G is infinite, and so by Lemma 1 it has normal subgroups $K < H$ with $H/K \cong \mathbb{Z}^r$ for some $0 < r \leq h(G)$ and G/H isomorphic to a subgroup of $GL(r, Z)$. Clearly $h(K) = h(G) - r$. If $h(G) = 1$ then K is trivial and $G = \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$; hence G is metabelian and virtually torsion free. If $h(G) = 2$ then either $H \cong \mathbb{Z}^2$ and G/H is a finite cyclic or dihedral group, or $h(K) = h(G/K) = 1$. In the latter case the Hirsch-Plotkin radical $\sqrt{K}$ of K is a torsion free rank 1 abelian subgroup of index at most 2 in K and

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$G/\sqrt{K}$ has an infinite cyclic subgroup of index at most 4. In both cases G has derived length at most 3 and is virtually torsion free. If $h(G) = 3$ then either $H \cong \mathbb{Z}^3$ and G/H has derived length at most 3, or K is metabelian and G/K has derived length at most 3, or K has derived length at most 3 and G/K is metabelian; in all cases G is solvable and of derived length at most 5.

The argument breaks down for larger values of h as $GL(h, \mathbb{Z})$ may then have nonsolvable finite subgroups. However this result is all that we shall need. (See [25] for a more general result.)

2. Localizing group rings

Kropholler, Linnell and Moody have shown that if G is an elementary amenable group with finite subgroups of bounded order and with no nontrivial finite normal subgroup then $Q[G]$ has a classical ring of fractions which is a matrix ring over a division ring [18]. That is, there is a division ring D and an embedding $i: Q[G] \to M_n(D)$ (where n is the l.c.m. of the orders of the finite subgroups of G) such that the image of every nonzero divisor of $Q[G]$ is invertible in $M_n(D)$ and every element of $M_n(D)$ can be uniquely expressed in the form $i(\delta)^{-1}i(\gamma)$ for some γ and δ in $Q[G]$. Since a finitely free left $M_n(D)$-module is a finite dimensional left D-vector space, every onto endomorphism of such a module is an isomorphism. A ring for which this holds is said to have the *strong invariant basis number* property, or to be an *SIBN*-ring. Equivalently, a ring R is an *SIBN*-ring if whenever a stably free left R-module M satisfies $M \oplus R^a \cong R^b$ for some integers a, $b \geq 0$ the difference $b - a$ depends only on M, and is 0 only if $M = 0$.

We shall show that certain localizations of group rings are *SIBN*-rings, by adapting Rosset's argument with the help of [18] and of another result of Linnell. He has shown that if N is an elementary amenable group in which the finite subgroups have bounded order then the action of each nonzero divisor of $C[N]$ by left multiplication on $L^2(N, \mu_c)$ is injective [19]. (Here μ_c is counting measure on N, that is, the Haar measure for the discrete topology on N. Whether a similar result is true of all torsion free groups remains an open question [5].)

THEOREM 3. *Let G be a group with an elementary amenable normal subgroup N whose finite subgroups have bounded order and with no nontrivial finite normal subgroup, and let M be the classical ring of fractions for $Q[N]$.*

Then $Q[G]_S = M \otimes_{Q[N]} Q[G]$ *is an SIBN-ring in which* $Q[G]$ *embeds and is flat as a* $Q[G]$*-algebra. If* N *is nontrivial then* $Q[G]_S \otimes_{Q[G]} Q = 0$.

PROOF. Since M is a classical ring of fractions for $Q[N]$ it is a direct limit of free right $Q[N]$-modules and so is flat as a $Q[N]$-algebra. Therefore $Q[G]_S = M \otimes_{Q[N]} Q[G]$ is flat as a right $Q[G]$-module. We may define a multiplication which makes this module into a $Q[G]$-algebra by

$$(s_1^{-1}r_1 \otimes \alpha)(s_2^{-1}r_2 \otimes \beta) = s_1^{-1}r_1(\alpha s_2 \alpha^{-1})^{-1}(\alpha r_2 \alpha^{-1}) \otimes \alpha\beta$$

for r_1 and r_2 in $Q[N]$, s_1 and s_2 in $Q[N]-\{1\}$ and α and β in G. The assumptions on N imply that if it is nontrivial it must have an element n of infinite order. Since $n-1$ is then a nonzero divisor in $Q[N]$ it is invertible in M and hence in $Q[G]_S$, and as it annihilates the augmentation module Q it follows that $Q[G]_S \otimes_{Q[G]} Q = 0$. That $Q[G]_S$ is an *SIBN*-ring follows as in [23], from [19, Theorem 4] instead of Rosset's 3.4.

Our original version of Theorem 3 assumed Linnell's result as one of the hypotheses. Theorem 3 is also implicit in the introduction to [19].

Goodearl used a more conservative localization, in which just products of terms of the form $q-\nu$ with q in Q and ν in $N-\{1\}$ were inverted [10]. It is easy to verify that such terms act injectively on $L^2(N, \mu_c)$: in effect we may reduce to the case when N is generated by ν. If N is abelian the multiplicative system S generated by such terms is a left Ore set in $C[G]$, that is, the direct limit of the right module homomorphisms $\{s: C[G] \to C[G] | s \text{ in } S\}$ forms a ring. However this is not clear in general.

3. Applications to 2-complexes and 4-manifolds

The arguments of Theorems 1 and 3 of Chapter 3 of [15] depend only on the existence of an embedding of the group ring into an *SIBN*-ring which is flat as an algebra over the group ring and such that the tensor product with the augmentation module is trivial. (This strategy was first applied, in a special case, in [13].) Thus the next two theorems follow from [18] and Theorem 3 without further argument. (Note however that Theorem 4 is formulated in terms of the "$[G, m]$-complexes" of [6] rather than 2-complexes as in its model in [15].)

THEOREM 4. *Let* X *be a finite* m*-dimensional cell complex with* $\pi_j(X) = 0$ *for* $2 \leq j \leq m-1$, *and let* $G = \pi_1(X)$. *Suppose that* G *has a nontrivial elementary amenable normal subgroup* U *whose finite subgroups have* *bounded order and which has no nontrivial finite normal subgroup. Then* $(-1)^m\chi(X) \geq 0$, *and* $\chi(X) = 0$ *if and only if* X *is aspherical.*

COROLLARY. *If a finitely presentable group* G *has such a subgroup it has deficiency at most* 1. *If* $\operatorname{def} G = 1$ *and* $G \neq \mathbb{Z}$ *then* c.d. $G = 2$ *and either* G *is metabelian or* $U \cong \mathbb{Z}$.

Is the corollary still true if we assume merely that G has an infinite normal subgroup U which has no nonabelian free subgroup?

Fornera has shown that if a connected finite complex X has a regular covering $\overline{X}$ whose automorphism group has a nontrivial abelian normal subgroup A which is not a torsion group and if for each $n \geq 0$ either there is an element of infinite order in A which acts nilpotently on $H_n(\overline{X}; Q)$ or $H_n(\overline{X}; Q)$ has finite dimension then $\chi(X) = 0$ [8]. (This extends earlier results of [6] and [7].) This may be further extended by allowing A to be an elementary amenable group which is not a torsion group and such that the finite subgroups of the quotient of A by its maximal locally finite normal subgroup have bounded order.

THEOREM 5. *Let* M *be a closed 4-manifold. Suppose that there are normal subgroups* $T < U$ *of* $G = \pi_1(M)$ *and a subring* R *of* Q *such that* $\operatorname{Hom}(T/T', R) = 0$, $H^s(G/T; R[G/T]) = 0$ *for* $s \leq 2$ *and* U/T *is a nontrivial elementary amenable group whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup. Then* $\chi(M) \geq 0$ *and the covering space* M_T *of* M *with group* T *is R-acyclic if and only if* $\chi(M) = 0$.

For another example showing that some condition is needed on the group, note that the manifold $S^1 \times S^2 \# S^1 \times S^3$ has fundamental group free of rank 2 and Euler characteristic -2.

If $h(G/T) > 2$ the cohomological condition is always satisfied. On the other hand, if $h(G/T) = 1$ or 2 then G/T has a nontrivial torsion free abelian subgroup which is characteristic and so we may be able to apply the *ad hoc* arguments of [15, Chapters 3 and 4].

LEMMA 2. *Let* G *be elementary amenable and let* W *be a free left* $R[G]$-*module, where* R *is a ring. Then* $H^s(G; W) = 0$ *for all* $s < h(G)$.

PROOF. We argue by transfinite induction. The result is well known for G in X_1. If it is true for all groups in X_α then it is true for all groups in LX_α by [22, Lemma 4.1]. The inductive step then follows for groups in $X_{\alpha+1}$ on applying the LHS spectral sequence.

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It follows easily from this lemma that $h(G) \leq \text{c.d.}_R G$, for any elementary amenable group G and ring R.

THEOREM 6. *Let M be a closed 4-manifold with $\chi(M) = 0$. Suppose that $G = \pi_1(M)$ is elementary amenable, and that if T is the maximal locally-finite normal subgroup of G then the finite subgroups of G/T have bounded order. Then either*

(a) *$G/T \cong Z$ or $(Z/2Z) * (Z/2Z)$ and T is finite (that is, G has two ends); or*

(b) *G/T is solvable and has a subgroup of finite index which is an extension of Z by a subgroup of Q; or*

(c) *G/T is virtually poly-$\mathbb{Z}$ and $h(G/T) = 4$. In this case, if T is finite then M is aspherical.*

PROOF. Since $\chi(M) = 0$ the fundamental group G must be infinite, and so $h(G/T) > 0$. Suppose first that $h(G/T) = 1$. After passing to a subgroup of finite index if need be, we may assume that $G/T \cong \mathbb{Z}$ and M is orientable. The argument of the first paragraph of Theorem 6 of Chapter 3 of [15] applies to show that T is finite. Therefore G has two ends, and so (a) holds. If $h(G/T) = 2$ then G/T is solvable and virtually torsion free by Theorem 2. It is easily checked that a finitely generated torsion free elementary amenable group of Hirsch length 2 is an extension of Z by a subgroup of Q. If $h(G/T) > 2$ then by Lemma 2 we may apply Theorem 5 (with $R = Q$) and so G/T is a PD_4^+-group over Q. In particular $h(G/T) \leq 4$, by Lemma 2. It now follows from Lemma 1 and Theorem 2 that G/T is virtually solvable. Therefore G/T must be virtually poly-$\mathbb{Z}$, by a result of Kropholler [17], and so $h(G/T) = \text{c.}d._Q G/T = 4$. The final remark follows from Theorem 5 as M then has a finite covering space whose group is a torsion free poly-$\mathbb{Z}$ group of Hirsch length 4.

In cases (b) and (c) must G be torsion free? (Cf. pages 64–70 and 104–105 of [15].)

COROLLARY. *If G is torsion free then either $G \cong \mathbb{Z}$ or G is an ascending HNN extension with a presentation $\langle a, t | tat^{-1} = a^n \rangle$ for some $n \neq 0$ or M is aspherical and G is virtually poly-$\mathbb{Z}$ of Hirsch length 4.*

PROOF. The cases when $h(G) = 1$ or $h(G) > 2$ are immediate from the theorem. Thus we may suppose that G has a normal subgroup H of finite index which is an extension of $\mathbb{Z}$ by a nontrivial subgroup of Q. In particular the Hirsch-Plotkin radical $\sqrt{G}$ is nontrivial. Since $\sqrt{G}$ is a

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torsion free locally nilpotent group of Hirsch length at most 2 it is abelian. If $h(\sqrt{G}) = h(G)$ then $G/\sqrt{G}$ is finite, so $\sqrt{G} \cong \mathbb{Z}^2$ and $G \cong \mathbb{Z}^2$ or $\mathbb{Z}\tilde{\times}\mathbb{Z}$. Otherwise $\sqrt{G}$ has rank 1 and $G/\sqrt{G}$ has 2 ends. Therefore G has a normal subgroup B containing $\sqrt{G}$ such that $G/B \cong \mathbb{Z}$ or $(\mathbb{Z}/2\mathbb{Z}) * (\mathbb{Z}/2\mathbb{Z})$ and $B/\sqrt{G}$ is finite. Since B is torsion free it must in fact be abelian, and so $B = \sqrt{G}$. Thus we may assume that H has index at most 2. Since G is finitely presentable so is H and therefore it is an ascending HNN extension over a finitely generated base [1]. The base must be $\mathbb{Z}$ and the associated subgroups $\mathbb{Z}$ and $n\mathbb{Z}$, for some $n \neq 0$. Thus H has a presentation of the above form. As we are assuming that $\sqrt{G}$ has rank 1 we must have $n \neq 1$ or -1. It is not hard to see that such a group cannot be a subgroup of index 2 in a torsion free group. Hence $H = G$ and the corollary is proved.

Theorem 6 and its corollary may be used to strengthen some of the results of [15, Chapter 6]. Note that Theorem 1 of that chapter needs correction—the third sentence is wrong, as a locally nilpotent group need not have a nontrivial abelian normal subgroup. (Compare with the example in Section 1 above.) However this is true in all the applications of the theorem later in the chapter, and Theorem 3 above may be used to show that the statement of the theorem is correct as it stands.

References

[1] R. Bieri and R. Strebel, 'Almost finitely presented soluble groups', *Comment. Math. Helv.* **53** (1978), 258–278.

[2] A. Casson and M. H. Freedman, 'Atomic surgery problems', in *Four-Manifold Theory*, edited by C. McA. Gordon and R. C. Kirby, (Contemp. Math. 35, Amer. Math. Soc., Providence, R. I., 1984).

[3] J. Cheeger and M. Gromov, 'L_2-cohomology and group cohomology', *Topology* **25** (1986), 189–215.

[4] C. Chou, 'Elementary amenable groups', *Illinois J. Math.* **24** (1980), 396–407.

[5] J. M. Cohen, 'Von Neumann dimension and the homology of covering spaces', *Quart. J. Math. Oxford Ser.* **30** (1979), 133–142.

[6] M. N. Dyer, 'Localization of group rings and applications to 2-complexes', *Comment. Math. Helv.* **62** (1987), 1–17.

[7] B. Eckmann, 'Nilpotent group actions and Euler characteristic', in *Algebraic Topology*, Barcelona 1986, edited by J. Aguadé and R. Kane, pp. 120–123 (Lecture Notes in Math. 1298, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[8] L. Fornera, 'Normal abelian subgroups and Euler characteristics', *J. Appl. Algebra* **60** (1989), 237–243.

[9] M. H. Freedman, 'The disk theorem for four-dimensional manifolds', in *Proceedings of the Internat Congress of Mathematicians*, vol. 1, Warsaw, 1983, pp. 647–663.

[10] K. R. Goodearl, 'Localized group rings, the invariant basis property and Euler characteristics', *Proc. Amer. Math. Soc.* **91** (1984), 523–528.

[11] D. H. Gottlieb, 'A certain subgroup of the fundamental group', *Amer. J.Math.* **87** (1965), 840–856.
[12] R. Grigorchuk, 'Degrees of growth of finitely generated groups and the theory of invariant means', *Math. USSR-Izv.* **25** (1985), 259–300.
[13] J. A. Hillman, 'Orientability, asphericity and two-knots', *Houston J. Math.* **6** (1980), 67–76.
[14] J. A. Hillman, 'Aspherical four-manifolds and the centres of 2-knot groups', *Comment. Math. Helv.* **56** (1981), 465–471.
[15] J. A. Hillman, *2-knots and their groups*, (Austral. Math. Soc. Lecture Series 5, Cambridge Univ. Press, Cambridge, New York, Melbourne, 1989).
[16] I. Kaplansky, *Fields and rings*, (Chicago Lectures in Math., Chicago Univ. Press, Chicago, London, 1969).
[17] P. H. Kropholler, Letter to J. A. Hillman (31 March 1987).
[18] P. H. Kropholler, P. A. Linnell and J. A. Moody, 'Applications of a new K-theoretic theorem to soluble group rings', *Proc. Amer. Math. Soc.* **104** (1988), 675–684.
[19] P. A. Linnell, 'Zero divisors and group von Neumann algebras', *Pacific J. Math.*, to appear.
[20] O. Ol′shanskii, 'On the problem of the existence of an invariant mean on a group', *Russian Math. Surveys* **35:4** (1980), 180–181.
[21] J.-P. Pier, *Amenable locally compact groups*, (John Wiley, New York, 1984).
[22] D. J. S. Robinson, 'On the cohomology of soluble groups of finite rank', *J. Pure Appl. Algebra* **6** (1975), 155–164.
[23] S. Rosset, 'A vanishing theorem for Euler characteristics', *Math. Z.* **185** (1984), 211–215.
[24] M. K. Smith, 'Central zero divisors in group algebras', *Proc. Amer. Math. Soc.* **91** (1984), 529–531.
[25] J. A. Hillman and P. A. Linnell, 'Elementary amenable groups of finite Hirsch length are locally-finite by virtually-solvable', *J. Austral. Math. Soc.*, to appear.

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