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STABILITY OF THE TRIANGULAR LAGRANGIAN SOLUTIONS OF THE PHOTO GRAVITATIONAL RESTRICTED THREE-BODY PROBLEM IN THE THREE-DIMENSIONAL CASE

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ABSTRACT

The stability of the triangular Lagrangian solutions for the photo-gravitational restricted three-body problem in the three-dimensional case is investigated for the case when the resonances are absent and also when the resonances are present. Stability is proved for most (in the sense of Lebesgue) initial conditions for all $\mu < \mu_0$ except for the resonance cases.

1. INTRODUCTION

This work is a generalisation over the work by Kumar and Choudhry (1987,1988) in the sense that here we have taken up three-dimensional case and a generalisation over Markeev's work (1972) in the sense that here we have taken photo-gravitational effects of the two finites bodies which have been assumed to be radiating ones as well.

Here we have studied the stability of the triangular Lagrangian solutions. Since we have two such solutions situated symmetrically, so we have taken up the study of the stability of L_4 alone and it is claimed that the nature of L_5 will be the same.

For the investigation of the stability we have tried to reduce our Hamiltonian to a form suitable for the application of Arnold-theory (1963). The equations of motion have been normalised by Birkhoff's transformations. The non-resonace case has been dealt in section 4 and the resonance cases in section 5. The resonance case when $\omega_1 = \omega_2$ has not been taken up.

2. THE HAMILTONIAN OF THE PERTURBED MOTION

We introduce the rotating coordinate system (0,xyz). Its origin coincides with 0, the centre of mass of the bodies P_1 and P_2, its Ox-axis is directed towards the body P_2, the Oy-axis lies in the rotation plane of the bodies P_1 and P_2 perpendicular to P_1P_2 and the Oz-axis completes a right-handed coordinate system with Ox and Oy. We adopt the units of measurement for time and length such that the angular rotational velocity of the bodies P_1 and P_2 and the distance between them are equal to unity. The sum of masses of P_1 and P_2 will be taken for unit mass. Both of the bodies P_1 and P_2 are taken to be radiating and the reduction factors α and β are assumed to be such that $0 < \alpha,\ \beta < 1$ as detailed in the work (1987).

If x,y,z be the coordinates of the body P and P_x, P_y, P_z the corresponding momenta, then the Hamilton function for the photo-gravitational restricted three body problem will take the form:

$$H = \frac{1}{2}\left(P_x^2 + P_y^2 + P_z^2\right) + P_x y - P_y x - \alpha(1-\mu)/P_1P - \beta\mu/P_zP \qquad (1)$$

$$P_1P^2 = (x+\mu)^2 + y^2 + z^2,$$

$$P_2P^2 = (x-1+\mu)^2 + y^2 + z^2.$$

In the (0,xyz) coordinate system the triangular solution corresponding to the equilibrium position L_4 is given as

$$x_o = \frac{\delta_1^2 + 1 - \delta_2^2}{2} - \mu, \quad y_o = \delta_1\delta_2\sqrt{b}, \quad z_o = 0$$

$$Px_o = -\delta_1\delta_2\sqrt{b}, \quad Py_o = \frac{\delta_1^{2+1} - \delta_2^2}{2} - \mu, \quad Px_o = 0$$

$$\alpha = \delta_1^3, \qquad \beta = \delta_2^3,$$

$$b = 1 - \frac{(\delta_1^2 + \delta_2^2 - 1)^2}{4\delta_1^2\delta_2^2}$$

If we make the change of variables

$$x = x_o + q_1, \quad y = y_o + q_2, \quad z = z_o + q_3$$

$$p_x = p_{x_o} + p_1, \; p_y = p_{y_o} + p_2, \; p_z = p_{z_o} + p_3 ,$$

the motion in question will correspond to the equilibrium position $q_i = p_i = 0$ $(i = 1,2,3)$.

The Hamiltonian function (i) may be expanded in the neighbourhood of the equilibrium position as

$$H = H_2 + H_3 + H_4 + \ldots + H_m + \ldots \tag{2}$$

where H_m is a homogeneous function of degree m with respect to q_i and p_i. In particular,

$$\begin{aligned} H_2 = {} & (1/2)(p_1^2 + p_2^2 + p_3^2) + p_1q_2 - p_2q_1 + q_1^2[1-(3/4)\{(1-\mu) \times \\ & \xi^2\delta_1^{-2} + \mu\eta^2\delta_2^{-2}\}]/2 - (3/2)q_1q_2\sqrt{b}/\delta_1\delta_2\{\delta_2^2(1-\mu)\xi + \delta_1^2\mu\eta\} \\ & -(1/2)q_2^2[3b\{(1-\mu)\delta_2^2 + \mu\delta_1^2\} - 1] + (1/2)q_3^2 \end{aligned} \tag{3}$$

$$\begin{aligned} H_3 = {} & (1/16)q_1^3[(1-\mu)\xi\delta_1^{-4}(5\xi^2 - 12\delta_1^2) + \mu\eta\delta_2^{-4}(5\eta^2 - 12\delta_2^2)] \\ & + (3/8)q_1^2q_2\sqrt{b}[(1-\mu)\delta_2\delta_1^{-3}(5\xi^2 - 4\delta_1^2) + \mu\delta_1\delta_2^{-3}(5\eta^2 - 4\delta_2^2)] \\ & + (3/4)q_1q_2^2[\xi\delta_1^{-2}(1-\mu)(5b\delta_2^2 - 1) + \eta\delta_2^{-2}\mu(5b\delta_1^2 - 1)] + \\ & + q_2^3\sqrt{b}[(1-\mu)\delta_2\delta_1^{-1}(5\delta_2^2 b - 3) + \mu\delta_1\delta_2^{-1}(5\delta_1^2 b - 3)]/2 \\ & -(3/4)q_1q_3^2[(1-\mu)\xi\delta_1^{-2} + \mu\eta\delta_2^{-2} - (3/2)q_2q_3^{-2}\sqrt{b}[(1-\mu)\delta_2\delta_1^{-1} \\ & + \mu\delta_1\delta_2^{-1}] \end{aligned} \tag{4}$$

$$\begin{aligned} H_4 = {} & -(1/8)q_1^4[(1-\mu)\delta_1^{-6}\{3\delta_1^4 - (15/2)\xi^2\delta_1^2 + (35/16)\xi^4\} \\ & + \mu\delta_2^{-6}\{3\delta_2^4 - (15/2)\eta^2\delta_2^2 + (35/16)\eta^4\}] \\ & + (5/4)q_1^3q_2\sqrt{b}[(1-\mu)\delta_2\delta_1^{-3}\xi\{3 - (7/4)\xi^2\delta_1^{-2}\} \end{aligned}$$

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$$+ \delta_1\delta_2^{-3}\eta\{3-(7/4)\eta^2\delta_2^{-2}\}]$$

$$+ (5/4)q_1q_2^3\sqrt{b}[(1-\mu)\xi\delta_2\delta_1^{-3}(3-7\delta_2^2 b) + \mu\eta\delta_1\delta_2^{-3}(3-7\delta_1^2 b)] +$$

$$+ (3/4)q_1^2q_1^2[(1-\mu)\delta_1^{-2}(-1+5\delta_2^2 b + (5/4)\xi^2\delta_1^{-2}$$

$$- (35/4)b\xi^2\delta_2^2\delta^{-2}) + \mu\delta_2^{-2}\{-1+5\delta_1^2 b + (5/4)\eta^2\delta_2^{-2}$$

$$- (35/4)\eta^2\delta_1^2 b\delta_2^{-2}\}] - (1/8)q_2^4[(1-\mu)\delta_1^{-2}(3-30b\delta_2^2 + 35\delta_2^4 b^2)$$

$$+ \mu\delta_2^{-2}(3-30b\delta_1^2 + 35\delta_1^4 b^2)]$$

$$+ (3/16)q_1^2q_3^2[-4(1-\mu)\delta_1^{-2} - 4\mu\delta_2^{-2} + 5(1-\mu)\xi^2\delta_1^{-5} +$$

$$+ 5\mu\eta^2\delta_2^{-5}] - (3/4)q_2^2q_3^2[(1-\mu)\delta_1^{-2}(1-5\delta_2^2 b) +$$

$$+ \mu\delta_2^{-2}(1-5\delta_1^2 b)$$

$$+ (15/4)\sqrt{b}q_1 1_2 q_3^2[(1-\mu)\xi\delta_2\delta_1^{-3} + \mu\eta\delta_1\delta_2^{-3}]$$

$$- (3/8)q_3^4[(1-\mu)\delta_1^{-2} + \mu\delta_2^{-2}] \qquad (5)$$

where $\xi = \delta_1^2 + 1-\delta_2^2$, $\eta = \delta_1^2 - 1-\delta_2^2$.

3. CHARACTERISTIC ROOTS AND THE FIRST ORDER STABILITY OF THE TRIANGULAR LIBERATION POINTS

Restricting to H_2 alone, we may write down the characteristic equation in the form

$$[\lambda^4 + \lambda^2 + 9\mu(1-\mu)b](\lambda^2+1) = 0 \qquad (6)$$

As in the planar case (1987) the value $\mu = 0.285954 = \mu_o$ (say) for b = 1 corresponds to a critical case which needs special consideration and we shall not take it up here. We shall investigate the stability for all admissible values of b for $\mu < \mu_o$, where μ_o is given by Table-I (1987).

If ω_1, ω_2 and ω_3 be the frequencies, then

TABLE

	I	II	III	IV	V	VI
	$\mu_0 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{36b}}$	$\mu_1 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{4}{225b}}$	$\mu_2 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{100b}}$	$\mu_3 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{144b}}$	$\mu_4 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{32}{721b}}$	$\mu_5 = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{64}{625b}}$
b =	$\omega_1 - \omega_2 = 0$	$\omega_1 - 2\omega_2 = 0$	$\omega_1 - 3\omega_2 = 0$	$2\omega_2 - \omega_3 = 0$	$3\omega_2 - \omega_3 = 0$	$2\omega_1 - \omega_2 - 2\omega_3 = 0$
0.00	not applicable	N.A	N.A	N.A	N.A	N.A
0.05	Imaginary	Imaginary	0 2763932	0.166666	Imaginary	Imaginary
0.10	Imaginary	0.231258	0.1122701	0.075080	Imaginary	Imaginary
0.15	0.2454124	0.1373962	0.0718255	0.048664	Imaginary	Imaginary
0.20	0.1666666	0.0986135	0.0527864	0.036019	0.325290	Imaginary
0.25	0.127322	0.0770474	0.0417424	0.028595	0.227285	Imaginary
0.30	0.1032539	0.632612	0.0345253	0.023710	0.178004	Imaginary
0.35	0.6869202	0.0536746	0.0294380	0.020250	0.147036	Imaginary
0.40	0.0750817	0.466176	0.0256583	0.017673	0 125480	0.422540
0 45	0.0660972	0.0412039	0.0227392	0.015677	0.109540	0.350186
0.50	0.0590414	0.0369185	0.0204168	0.014080	0.097240	0.287397
0.55	0.0533514	0.0334415	0.0185249	0.012780	0.087450	0.247373
0.60	0.0486645	0.0305637	0.0169541	0.011710	0.079470	0.218320
0.65	0.044363	0.0281424	0.0156288	0.010800	0.072830	0.195910
0.70	0.0413961	0.0260768	0.0144958	0.010021	0.067220	0.177940
0.75	0.0385208	0.0242938	0.0135160	0.009346	0.062420	0.163140
0.80	0.0360198	0.227392	0.0126602	0.008757	0.058260	0.150710
0.85	0.0338237	0.0213717	0.0119064	0.008237	0.054620	0.140090
0.90	0.0318805	0.0207594	0.0112373	0.007776	0.051410	0.130910
0.95	0.0301487	0.090773	0.0106395	0.007364	0.048560	0.122890

$$\omega_1^2 = -\lambda_{1,2}^2 = \frac{-1+\sqrt{(1-36\mu(1-\mu)b)}}{2} = \frac{1+M}{2}$$

$$\omega_2^2 = -\lambda_{3,4}^2 = \frac{-1-\sqrt{1-36\mu(1-\mu)b}}{2} = \frac{1-M}{2}$$

$$\omega_3^2 = -\lambda_{5,6}^2 = 1 \tag{7}$$

where $M = \sqrt{1-36\mu(1-\mu)b}$.

The expressions (7) show that

$$1 > \omega_1 > 1/\sqrt{2} > \omega_2 > 0$$

and also it is clear that

$$\omega_1^2\omega_2^2 = 9\mu(1-\mu)b = 1/4$$

when $\mu = \mu_o$.

4. ARNOLD'S THEOREM ON THE STABILITY AND THE EXISTENCE OF RESONANCES

Since the characteristic equation of the linearised system has imaginary roots and the Hamiltonian function (2) will not have a definite sign, so it is not possible to assert that the motion will be stable or unstable when all the terms of the Hamiltonian function are taken into consideration. If the frequencies satisfy the condition

$$0 < |n_1| + |n_2| + |n_3| \leq 4 \tag{8}$$

then there exists (1927) a real canonical transformation $(q_1, p_1) \rightarrow (q_1', p_1')$, specified by power series convergent in the neighbourhood of the origin such that the Hamiltonian function (2) may be written as

$$H = H^o + H'(q', p') \tag{9}$$

in the new variables, where H^o has the normal form

$$H^o = \omega_1 r_1 - \omega_2 r_2 + \omega_3 r_3 + C_{200} r_1^2 + C_{110} r_1 r_2$$
$$+ C_{101} r_1 r_3 + C_{020} r_2^2 + C_{011} r_2 r_3 + C_{002} r_3^2 \tag{10}$$

$$(2r_1 = q_i'^2 + p_i'^2)$$

and H' is a convergent series in powers of q_1', p_1' beginning with terms not lower than the fifth one. We shall now aim to apply Arnold's theorem (1963) on the stability of the equilibrium position which is stated as follows:

Let the Hamiltonian function be such that

(a) the characteristic equation of the linearised system has purely imaginary roots,

(b) the condition (8) is satisfied,

(c) the coefficient of the normal form (10) satisfy the inequality

$$D = \begin{vmatrix} \dfrac{\partial^2 H^0}{\partial r_1 \partial r_3} & \dfrac{\partial H^0}{\partial r_1} \\ \dfrac{\partial H^0}{\partial r_3} & 0 \end{vmatrix} \neq 0 \qquad (11)$$

Then for most (in the sense of Lebesgue measure) initial conditions the equilibrium position

$$q_1 = p_1 = 0,$$

will be stable.

During the present investigation, we aim to apply the above stated Arnold's theorem concerning the condition (8), we shall come across six resonance cases given as

(i) $\omega_1 - \omega_2 = 0$, (ii) $\omega_1 - 2\omega_2 = 0$, (iii) $\omega_1 - 3\omega_2 = 0$,

(iv) $2\omega_2 - \omega_3 = 0$, (v) $3\omega_2 - \omega_3 = 0$, (vi) $2\omega_1 - \omega_2 - \omega_3 = 0$.

In the adjoining figure 1, we have plotted the values of μ corresponding to the different values of b varying from 0 to 1 for all the six types of resonances. It has already been seen by Kumar and Choudhry (1987) that within the range of stability given by the values of μ for $\omega_1 = \omega_2$, which will be denoted as graph (i), we come across the resonances (iv), (v) and (vi) we find that the graph corresponding to the resonance (iv) lie within the graph (i) and the graphs for the (v) & (vi) lie beyond. It shows that within the range of linear stability given by the graph (i), there is the possibility of having the resonances of the types (ii)-(iv).

Since H_3 and H_4 are even functions of q_3 and so after the normalisation ϕ_3 will enter the arguments of sines and cosines as $2\phi_3$ and $4\phi_3$ and so corresponding to the cases (iv), (v) and (vi) we shall not have any critical case. So these cases need no special investigation. It is thus seen that except for (ii) and (iii) the Hamiltonian function can be reduced to the form (10) suitable for the application of Arnold's theorem.

If the values of μ corresponding to $\omega_1 = \omega_2$ be denoted by μ_o, then from the graph (i), it is clear that these values of μ_o will differ according to the different values of b.

From the equation (6) it follows that

$$\omega_1^2\omega_2^2 = 1/4$$

for all b where $\omega_1 = \omega_2$.

As in (1987) we shall restrict our investigation for the range of values of μ restricted to $0 < \mu < \mu_o$ given by Table 1 where μ_o corresponds to $\omega_1 - \omega_2 = 0$ for the different values of b and for such $\mu = \mu_o$, $\omega_1^2\omega_2^2 = 1/4$. Under such a restriction although the resonance (i) is avoided but we may come across the other resonance cases. As examined in (1972) the resonance relations (iv)-(vi) will not lead to the appearance of non-vanishing denominators and they will not prevent the normal form (10) from being obtained. So except for the two cases (ii) and (iii) the Hamiltonian can be reduced to the normal form (10) required for the application of Arnold's theorem. We shall need special consideration for the cases (ii) and (iii).

5. NORMAL FORM OF THE HAMILTONIAN FUNCTION AND THE STABILITY EXCEPT FOR THE RESONANCE CASES (ii) & (iii) WHERE $\mu < \mu_o$

Here we shall aim to reduce the Hamiltonian function given by (2) to the form (10) for which we shall use Birkhoff's method of normalisation (1985). If in the form (10), $H_2^{(0)}$ is of positive definite form, then the equilibrium position is stable by virtue of Liapunov's theorem (1956) for all orders and all time. If H_2 is not a function of definite sign, then we shall need the application of Arnold's theorem referred above. In the present case H_2 is not of definite form. To put H_2 in the form (10), we shall introduce the transformation referred in (1985), where we may write

$$H = (1/2)(p_1'^2 + \omega_1^2 q_1'^2) - (1/2)(p_2'^2 + \omega_2^2 q_2'^2)$$

$$+ (1/2)(p_3'^2 + \omega_3^2 q_3'^2) + \sum_{\alpha+\beta=3}^{\infty} h_{\alpha\beta} q'^{\alpha} p'^{\beta} \qquad (12)$$

$$\alpha = \alpha_1 + \alpha_2 + \alpha_3$$
$$\beta = \beta_1 + \beta_2 + \beta_3$$

where for simplicity we shall mean

$$h_{\alpha\beta} = h_{\alpha_1\alpha_2\alpha_3\beta_1\beta_2\beta_3}$$

$$q'^{\alpha} = q'^{\alpha_1}_1 \, q_2'^{\alpha_2} \, q_3'^{\alpha_3}$$

$$p'^{\beta} = p'^{\beta_1}_1 \, p_2'^{\beta_2} \, p_3'^{\beta_3}$$

and in subsequent investigations we shall not write these ranges with the summation sign. Taking the notations as used in (1985) it may be mentioned that $h_{\alpha\beta}$ upto the third order and the fourth order terms not including q_3' and p_3' are given in (1985) which we shall not rewrite here. The terms depending on q_3' and p_3' are given here as follows:

$$h_{102000} = a_1 (H_{102000} + c_1 H_{012000}),$$

$$h_{012000} = a_2 (H_{102000} + c_2 H_{012000}),$$

$$h_{002100} = a_1 b_1 H_{012000},$$

$$h_{002010} = -a_2 b_2 H_{012000},$$

$$h_{202000} = a_1^2 (H_{202000} + c_1^2 H_{022000} + c_1 H_{112000}),$$

$$h_{022000} = a_2^2 (H_{202000} + c_2^2 H_{022000} + c_2 H_{112000}),$$

$$h_{112000} = a_1 a_2 (2H_{202000} + 2c_1 c_2 H_{022000} + (c_1 + c_2) H_{112000}),$$

$$h_{102100} = 2a_1^2 b_1 c_1 H_{022000} + a_1^2 b_1 H_{112000},$$

$$h_{102010} = -2a_1 a_2 b_2 c_1 H_{022000} - a_1 a_2 b_2 H_{112000},$$

$$h_{012100} = 2a_1 a_2 b_1 c_2 H_{022000} + a_1 a_2 b_1 H_{112000},$$

$$h_{012010} = -2a_2^2 b_2 c_2 H_{022000} - a_2^2 b_2 H_{112000},$$

$$h_{002110} = -2a_1 a_2 b_1 b_2 H_{022000},$$

$$h_{004000} = H_{004000} \qquad (13)$$

Next we perform the following canonic transformations

$$q_1' = (1/2)q_1'' + (i/\omega_1)p_1'', \quad p_1' = (1/2)i\omega_1 q_1'' + p_1'' ,$$

$$q_2' = -(1/2)iq_2'' + (1/\omega_2)p_2'', \quad p_2' = -(1/2)\omega_2 q_2'' + ip_2'' ,$$

$$q_3' = (1/2)q_3'' + ip_3'' , \qquad p_3' = (1/2)iq_3'' + p_3'' \qquad (14)$$

In the new variables the Hamiltonian (12) may be written in the form

$$H = i\omega_1 q_1'' p_1'' + i\omega_2 q_2'' p_2'' + i\omega_3 q_3'' p_3'' + \Sigma\, h'_{\alpha\beta}\, q''^{\alpha} p''^{\beta} \qquad (15)$$

If $h'_{\alpha\beta} = x_{\alpha\beta} + iy_{\alpha\beta}$, then the coefficients of terms not involving q_3' and p_3' are already given in (1987) which we shall not repeat here, except it may be noted that there we have to replace $h_{\alpha_1\alpha_2\beta_1\beta_2}$ by $h_{\alpha_1\alpha_2 0\beta_1\beta_2 0}$ and similarly $h'_{\alpha_1\alpha_2\beta_1\beta_2}$ by $h'_{\alpha_1\alpha_2 0\beta_1\beta_2 0}$.

The coefficients with the terms involving q_3' and p_3' may be given as follows:

$$x_{002100} = (1/4)\, h_{002100}, \quad y_{002100} = (1/4\omega_1) h_{102000},$$

$$x_{000102} = -h_{002100}, \quad y_{000102} = -(1/\omega_1) h_{102000},$$

$$x_{001101} = -(1/\omega_1) h_{102000}, \quad y_{001101} = h_{002100},$$

$$x_{002010} = (1/4\omega_2) h_{012000}, \quad y_{002010} = (1/4) h_{002010},$$

$$x_{000012} = -(1/\omega_2) h_{012000}, \quad y_{000012} = -h_{002010},$$

$$x_{001011} = -h_{002010}, \quad y_{001011} = (1/\omega_2) h_{012000},$$

(16)

the remaining coefficients of the third order terms involving q_3'' and p_3'' may be given by

$$h'_{\alpha\beta} = (y_{\alpha\beta} + ix_{\alpha\beta})(-\omega_1/2)(\beta_1-\alpha_1)(\omega_2/2)(\beta_2-\alpha_2)(-\omega_3/2)(\beta_3-\alpha_3) \tag{17}$$

we shall firstly assume that the resonances of the types of (i)-(iii) are not present. As in (1987) we shall use Birkhoff's transformation

$$(q_3'', p_3'') \longrightarrow (q_3''', p_3''') \tag{18}$$

and nullify the third order terms.

The new Hamiltonian exclusive of the third order and the fourth order terms giving rise to the resonance of type (iii) and also those terms not necessary for the form (10) may be given as

$$\begin{aligned} H' = {} & i\omega_1 q_1''' p_1''' + i\omega_2 q_2''' p_2''' + i\omega_3 q_3''' p_3''' \\ & - C_{200}(q_1''' p_1''')^2 + C_{110}(q_1''' p_1''')(q_2''' p_2''') \\ & - C_{002}(q_3''' p_3''')^2 + C_{101}(q_1''' p_1''')(q_3''' p_3''') \\ & + C_{011}(q_2''' p_2''')(q_3''' p_3''') - C_{020}(q_2''' p_2''')^2 \\ & + 0(q_1^2 + p_1^2)^{5/2} \end{aligned} \tag{19}$$

where

$$\begin{aligned} C_{200} = {} & -h'_{200200} - (3/8)\omega_1^2(x^2_{000300} + y^2_{000300}) \\ & -(3/2)(x^2_{100200} + y^2_{100200}) + (1/2)(x^2_{100110} \\ & + y^2_{100110}) - \frac{\omega_1^2}{2\omega_2(2\omega_1-\omega_2)}(x^2_{010200} + y^2_{010200}) \\ & + \frac{\omega_1^2\omega_2}{8(2\omega_1+\omega_2)}(x^2_{000210} + y^2_{000210}), \end{aligned}$$

$$C_{020} = -h'_{020020} + (3/8)\omega_2^2 (x^2_{000030} + y^2_{000030})$$

$$+ (6/\omega_2^2)(x^2_{020010} + y^2_{020010})$$

$$- \frac{\omega_2^2}{2\omega_1(\omega_1 - 2\omega_2)} (x^2_{100020} + y^2_{100020})$$

$$- (1/2)(x^2_{010110} + y^2_{010110})$$

$$- \frac{\omega_1\omega_2^2}{8(\omega_1 + 2\omega_2)} (x^2_{000120} + y^2_{000120}),$$

$$C_{002} = - h'_{002002} - \frac{2\omega_1}{\omega_1 - 2} (x^2_{002100} + y^2_{002100})$$

$$- \frac{1}{8(\omega_1 + 2)} (x^2_{000102} + y^2_{000102})$$

$$- (1/2)(x^2_{001101} + y^2_{001101}) + \frac{2\omega_2}{\omega_2 - 2}(X^2_{002010} + Y^2_{002010})$$

$$+ \frac{\omega_2}{8(\omega_2 + 2)} (x^2_{000012} + y^2_{000012})$$

$$+ (1/2)(x^2_{001011} + y^2_{001011}),$$

$$C_{110} = h'_{110110} - \frac{2\omega_2^2}{\omega_1(\omega_1 - 2\omega_2)} (x^2_{100020} + y^2_{100020})$$

$$+ \frac{\omega_1\omega_2^2}{2(\omega_1 + 2\omega_2)} (x^2_{000120} + y^2_{000120})$$

$$- \frac{\omega_1^2\omega_2}{2(2\omega_1 + \omega_2)} (x^2_{000210} + y^2_{000210})$$

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$$- \frac{2\omega_1^2}{(2\omega_1 - \omega_2)\omega_2} (x_{010200}^2 + y_{010200}^2)$$

$$+ 2 (x_{010110} x_{100200} + y_{010110} y_{100200})$$

$$- (4/\omega_2) (x_{020010} y_{100110} + x_{100110} y_{020010}),$$

$$C_{101} = h'_{101101} + 2 (x_{100200} x_{001101} + y_{100200} y_{001101})$$

$$- (x_{100110} x_{001011} + y_{100110} y_{001011})$$

$$- \frac{8\omega_1}{\omega_1 - 2} (x_{002100}^2 + y_{002100}^2)$$

$$+ \frac{\omega_1}{2(\omega_1 + 2)} (x_{000102}^2 + y_{000102}^2),$$

$$C_{011} = h'_{011011} - 2 (x_{011001} x_{020010} + y_{011001} y_{020010})$$

$$+ (x_{010110} x_{001101} + y_{010110} y_{001101})$$

$$- \frac{2}{2(\omega_2 + 2)} (x_{000012}^2 + y_{000012}^2)$$

$$+ \frac{8\omega_2}{(\omega_2 - 2)} (x_{002010}^2 + y_{002010}^2),$$

$$h'_{200200} = -(3/2)\omega_1^2 h_{000400} - (3/2\omega_1^2) h_{400000} - (1/2) h_{200200},$$

$$h'_{020020} = -(3/2)\omega_2^2 h_{000040} - (3/2\omega_2^2) h_{040000} - (1/2) h_{020020},$$

$$h'_{002002} = -(3/2) h_{00400}$$

$$h'_{110110} = \omega_1 \omega_2 h_{000220} + (1/\omega_1 \omega_2) h_{220000}$$

$$+ (\omega_1/\omega_2) h_{020200} + (\omega_2/\omega_1) h_{200020},$$

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$$h'_{101101} = -(1/\omega_1)h_{202000},$$

$$h'_{011011} = (1/\omega_2)h_{202000}. \qquad (20)$$

Passing now to the real variables introducing the transformation

$$(q_3''', p_3''') \to (\overline{q_3}, \overline{p_3})$$

given by

$$q_1''' = (1/\sqrt{\omega_1})(\overline{q_1} - i\overline{p_1}), \quad p_1''' = (\sqrt{\omega_1}/2)(-i\overline{q_1} + \overline{p_1}),$$

$$q_2''' = (1/\sqrt{\omega_2})(i\overline{q_2} - \overline{p_2}), \quad p_2''' = (\sqrt{\omega_2}/2)(\overline{q_2} - i\overline{p_2}),$$

$$q_3''' = (1/\sqrt{\omega_3})(\overline{q_3} - i\overline{p_3}), \quad p''' = (\sqrt{\omega_3}/2)(-i\overline{q_3} + \overline{p_3}) \qquad (21)$$

and then to polar co-ordinates

$$\overline{q_3} = \sqrt{2r_3}\sin\phi_3, \quad \overline{p_3} = \sqrt{2r_3}\cos\phi_3 \qquad (22)$$

we shall find that the Hamiltonian (19) reduces to

$$\overline{H} = r_1\omega_1 - r_2\omega_2 + r_3\omega_3 + (1/4)[C_{200}r_1^2 + C_{020}r_2^2 + C_{002}r_3^2$$

$$+ C_{110}r_1r_2 + C_{101}r_1r_3 + C_{011}r_2r_3] + O(r_3)^{5/2} \qquad (23)$$

Now to test regarding the stability of the equilibrium points under reference we shall examine the value of det D given by

$$\det \begin{vmatrix} \dfrac{\partial^2 \overline{H}}{\partial r_i \partial r_j} & \dfrac{\partial \overline{H}}{\partial r_i} \\ \dfrac{\partial \overline{H}}{\partial r_j} & 0 \end{vmatrix}_{r_1 = r_2 = r_3 = 0} \qquad (24)$$

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On its expansion, we shall find that

$$16D = \omega_1^2 (C_{011}^2 - 4C_{020}C_{002}) + \omega_2^2 (C_{101}^2 - 4C_{200}C_{002})$$

$$+ \omega_3^2 (C_{110}^2 - 4C_{200}C_{020}) - 2\omega_1\omega_2 (C_{101}C_{011} + 2C_{002}C_{110})$$

$$- 2\omega_1\omega_3 (C_{011}C_{110} + 2C_{020}C_{101})$$

$$- 2\omega_2\omega_3 (C_{110}C_{101} + 2C_{200}C_{011}) \qquad (25)$$

After making some computations we may find the coefficients given as

$$C_{200} = \frac{\omega_2^2 (124\omega_1^4 - 696\omega_1^2 + 81)}{144 (1-2\omega_1^2)^2 (1-5\omega_1^2)},$$

$$C_{110} = \frac{-\omega_1\omega_2 (64\omega_1^2\omega_2^2 + 43)}{6 (1-2\omega_1^2)(1-2\omega_2^2)(1-5\omega_1^2)(1-5\omega_2^2)},$$

$$C_{101} = \frac{-8\omega_1\omega_2^2}{3 (1-2\omega_1^2)(4-\omega_1^2)}$$

$$C_{020} = \frac{\omega_1^2 (124\omega_2^4 - 696\omega_2^2 + 81)}{144 (1-2\omega_2^2)^2 (1-5\omega_2^2)}$$

$$C_{011} = \frac{8\omega_2\omega_1^2}{3 (1-2\omega_2^2)(4-\omega_2^2)}$$

$$C_{002} = \frac{-\omega_1^2\omega_2^2}{3 (4-\omega_1^2)(4-\omega_2^2)} \qquad (26)$$

Which coincide with those of Markeev (1972). Putting $u = \omega_1^{-2}\,\omega_2^{-2}$, we find that (25), on substitutions of the values for the co-efficients C_{ijk} given by (26), may be written as

$$16D = \frac{f(u)}{5184\,(4-u)^2\,(25-4u)^2\,(1+12u)^2} \qquad (27)$$

where $f(u) = 73908288u^5 - 356526576u^4 + 2645643564u^3$

$$- 5787985485u^2 - 759408680u - 317395600$$

and for $u = 4$,

$$f,\ f',\ f'',\ f''',\ f^{iv},\ f^{v}$$

are all positive where dashes denote the differentiations. Hence by Newton's theorem (Burnside and Panton, 1979) on the superior limits of the roots it follows that there will be no root for $u > 4$, whence it follows that $D \neq 0$ for such a restriction.

In Markeev's case when $\mu = \mu_o$, $u = 4$, but in our case $u = 4$ for all $\mu = \mu_o$ corresponding to the case $\omega_1 - \omega_2 = 0$ plotted in our graph - (i). Thus we find that for each b, we shall have different μ . For example, when $b = 0.20$ the motion will be stable for all $\mu < 0.1666666$ and so we find that corresponding to the different values of the pair (δ_1, δ_2) our range of stability will go on differing.

Since $\mu_o = 0.0285454$ is less than all the values of μ_o in the Table-1, so the motion will be stable for all $\mu < \mu_o$, but it leaves many values of μ for which also Arnold's theorem will hold and consequently the stability will hold except for the two resonance cases whose corresponding values of μ are all less than μ_o for each b. So the investigation of the stability for the resonance cases cannot be escaped.

6. STABILITY FOR THE RESONANCE CASES (ii) AND (iii):

(a) The resonance case (ii) $\omega_1 - 2\omega_2 = 0$:

As in (Kumar & Choudhry, 1988), we shall introduce the transformation (18) to the Hamiltonian (15) but now we retain the terms giving rise to the resonance case and we shall finally have

$$H' = i\omega_1 q_1''' p_1''' + i\omega_2 q_2''' p_2''' + i\omega_3 q_3''' p_3''' + h'_{100020} q_1''' p'''^2 + h'_{020100} q_2'''^2 p_1''' + \qquad (28)$$

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Passing now to real variables by means of the transformation (21), our Hamiltonian (28) reduces to

$$H' = 2\omega_2 r_1 - \omega_2 r_2 + r_3 - \sqrt{\omega_2 (x^2_{100020} + y^2_{100020})}\, r_2 \sqrt{r_1}$$

$$\times \sin(\phi_1 + 2\phi_2) + \tilde{H}(r_j, \phi_j) \qquad (29)$$

where $\tilde{H}$ has the period 2π in ϕ_j and $\tilde{H} = 0(r_1 + r_2)^2$.

If $x^2_{100020} + y^2_{100020} \neq 0$ the equilibrium point will be unstable by Markeev's theorem (1978).

It has been examined in the paper (Kumar & Choudhry, 1988) that $x^2_{100020} + y^2_{100020}$ which is the same as (δ) in Table II of the said paper is not zero for the region under consideration and its shows that the motion will be unstable.

(b) The resonance case $\omega_1 = 3\omega_2$

In this case proceeding similar as in the paper (1988) which we shall not rewrite here, the Hamiltonian may be reduced to

$$H = 3\omega_2 r_1 - \omega_2 r_2 + r_3 + C_{200} r_1^2 + C_{110} r_1 r_2 + C_{101} r_1 r_3$$

$$+ C_{011} r_2 r_3 + C_{020} r_2^2 + C_{002} r_3^2$$

$$+ (1/3)\omega_2 \sqrt{3 (x^2_{100030} + y^2_{100030})} \times r_2 \sqrt{r_1 r_2} \cos(\phi_1 + 3\phi_2)$$

$$+ 0(r_1 + r_2)^{5/2}$$

Denoting by

$$a = C_{200} + 3C_{110} + 9C_{020}$$

$$d = 3\omega_2 \sqrt{x^2_{100030} + y^2_{100030}}$$

it is known by Markeev's theorem (1978) that if $|a| < d$, the equilibrium position is unstable and if $|a| > d$, the equilibrium position is stable.

If a = 0, the consideration of higher order terms becomes necessary.

The values of a,d and the corresponding nature of the motion have already been computed in Table III of the referred work (1988). So here even in the three-dimensional case the nature will continue to be the same and it will not be rewritten.

CONCLUSIONS

Thus we have shown that the triangular solution of the three-dimensional photo-gravitational circular restricted three-body problem is stable, for most sufficiently small initial departures from the given solution except for the two resonance cases $\omega_1 = 2\omega_2$ and $\omega_1 = 3\omega_2$ and the range $\mu < \mu_o$. Under the resonance case $\omega_1 = 2\omega_2$, the motion is seen to be unstable and for $\omega_1 = 3\omega_2$ for some sets of values of the pair (δ_1, δ_2) the motion is stable and for others it is unstable. These values are given in the referred paper (1988).

REFERENCES

[1] Arnold, V.I. (1963), USP.Matem.Nauk., Vol.18, pp.86-101.
[2] Birkhoff, G.D. (1927), Dynamical Systems, Am. Math.Soc.
[3] Burnside, W.S. and Panton, A.W. (1979), Theory of Equations, S.Chand & Company Ltd., New Delhi (13th edition).
[4] Kumar, V. and Choudhry, R.K. (1987), Cel. Mech.Vol.40, No.2, pp. 155-170.
[5] Kumar, V. and Choudhry, R.K. (1988), Cel. Mech.Vol. 41, No.2, pp. 161-173.
[6] Liapunov, A.M. (1956), A General Problem of Stability of Motion, Collected Works, Vol.2, Acad. Sc., USSR.
[7] Manju and Choudhry, R.K. (1985), Cel. Mech.Vol.36, No.2, pp. 165-190.
[8] Markeev, A.P. (1972), Soviet Astronomy, Vol. 15, No. 4, pp. 682-686.
[9] Markeev, A.P. (1978), Liberation Points in Celestial Mechanics, and Cosmo-dynamics, Moscow "Science".

https://doi.org/10.1017/S0252921100066197 Published online by Cambridge University Press