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Nonchaotic N-expansive homeomorphisms

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In this paper, we give necessary conditions for an N-expansive homeomorphism of a compact metric space to be nonchaotic in the Li–Yorke sense. As application we give a partial answer to a conjecture in [2].

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There are expansive homeomorphisms on compact metric spaces which are nonchaotic in the sense of Li and Yorke. More precisely, the ones obtained by the restriction of Denjoy maps to their minimal set. On the other hand, there is a generalization of expansive homeomorphisms introduced in [8]. These are the *N-expansive homeomorphisms* for a given positive integer. The case $N = 1$ corresponds to the expansive homeomorphisms. It was proved that for every $N \in \mathbb{N}$ there is a compact metric space exhibiting a 2^N-expansive homeomorphism which is not $(2^N - 1)$-expansive. This was extended in [2]. Indeed, for every $N \in \mathbb{N}$ there is a compact metric space exhibiting a $(N + 1)$-expansive homeomorphism which is not N-expansive. Moreover, these examples have the shadowing property and countably infinitely many chain recurrent classes. They motivated the study of N-expansive homeomorphisms with the shadowing property on compact metric spaces. The N-expansive homeomorphisms constitute together with the continuum-wise expansive ones [4] the so-called levels of generalized expansivity [7].

In light of these results, we can ask about what nonchaotic N-expansive homeomorphisms are. At first glance, we can say that these homeomorphisms exist on totally disconnected spaces only. This follows from Kato [4] and the fact that all N-expansive homeomorphisms are continuum-wise expansive.

In this paper, we will embed the nonchaotic N-expansive homeomorphisms into levels of asymptotic expansivity [5]. The latter is a recent notion of expansivity adapted to certain expansive-like examples (e.g. basin of attraction of hyperbolic attractors). Precisely, we will prove that every nonchaotic N-expansive is asymptotically N-expansive. As application, we will give a partial positive answer to a conjecture in [2]. Let us state our result in a precise way.

Consider a map of a metric space $f : X \to X$ and $x \in X$. Given $\epsilon > 0$ and $x \in X$, we define the *local stable set*

$$\Phi_\epsilon(x) = \{y \in X : \mathrm{d}(f^i(x), f^i(y)) \leqslant \epsilon, \quad \forall i \geqslant 0\}$$

and, if f is bijective, we also define

$$\Gamma_\epsilon(x) = \{y \in X : \mathrm{d}(f^i(x), f^i(y)) \leqslant \epsilon, \quad \forall i \in \mathbb{Z}\}.$$

A bijective map f is *expansive* [10] if there is $e > 0$ (called expansivity constant) such that $\Gamma_e(x) = \{x\}$ for every $x \in X$. The following definition splits this concept into levels of generalized expansivity [7]. Hereafter, we fix $N \in \mathbb{N}$.

DEFINITION 1 [8]. *A bijective map $f : X \to X$ is N-expansive if there is $e > 0$ (called N-expansivity constant) such that $\Gamma_e(x)$ has at most N elements for every $x \in X$.*

The case $N = 1$ in this definition corresponds to the expansive homeomorphism. Another generalization studied in [5] is as follows. A homeomorphism of a metric space $f : X \to X$ is *asymptotically expansive* if there is $e > 0$ (called asymptotic expansivity constant) such that if $x_1, x_2 \in X$ and $\mathrm{d}(f^i(x_1), f^i(x_2)) \leqslant \epsilon$ for every $i \geqslant 0$, then

$$\lim_{i\to\infty} \mathrm{d}(f^i(x_1), f^i(x_2)) = 0.$$

Now we split this definition into levels as the ones for the classical expansivity.

DEFINITION 2. *A map of a metric space $f : X \to X$ is asymptotically N-expansive if there is $e > 0$ (asymptotic N-expansivity constant) such that if $x_1, \cdots, x_{N+1} \in X$ and $\mathrm{d}(f^i(x_l), f^i(x_r)) \leqslant e$, $\forall i \geqslant 0$ and $l, r \in \{1, \cdots, N+1\}$, then*

$$\lim_{i\to\infty} \mathrm{d}(f^i(x_l), f^i(x_r)) = 0 \quad \textit{for some distinct } l, r \in \{1, \cdots, N+1\}.$$

Since 1-expansivity is equivalent to expansivity (which in turn implies asymptotic expansivity), we have that every 1-expansive homeomorphism is asymptotically 1-expansive. The question is if we can replace 1 by any positive integer N. More precisely, we have the question below:

QUESTION 1. Is every N-expansive homeomorphism of a compact metric space asymptotically N-expansive?

(Actually, this is a reformulation of conjecture 4.3 in [2], see the last section.)

We will give a partial answer based on the very classical concept of Li–Yorke chaos. Recall that a *Li–Yorke pair* of a map $f : X \to X$ is a pair $x, y \in X$ satisfying

$$\liminf_{i\to\infty} \mathrm{d}(f^i(x), f^i(y)) = 0 < \limsup_{i\to\infty} \mathrm{d}(f^i(x), f^i(y)).$$

We say that f is *chaotic* (in the sense of Li–Yorke) if it has a Li–Yorke pair and *nonchaotic* otherwise. The literature about this concept is very extensive. It was

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introduced by Li and Yorke in their nowadays classical paper [6]. Among the many results about this kind of chaos, we can mention Blanchard et al. [1] proving that it holds under positive topological entropy.

We can state our main result.

THEOREM. *Every nonchaotic N-expansive homeomorphism of a compact metric space is asymptotically N-expansive.*

We prove this result in the next section. We will give a short application related to [2] in the last section.

1. Proof of the theorem

First remind that a bijective map of a metric space $f : X \to X$ is *uniformly expansive* (Sears [9]) if there is $e > 0$ (called *uniform expansivity constant*) such that for every $\epsilon > 0$ there is $M \in \mathbb{N}$ such that if $x_1, x_2 \in X$ satisfy $d(f^i(x_1), f^i(x_2)) \leqslant e$, for all $-M \leqslant i \leqslant M$, then $d(x_1, x_2) \leqslant \epsilon$.

This concept splits into levels depending on positive integers $N \in \mathbb{N}$ as follows [3].

DEFINITION 3. *A bijective map of a metric space $f : X \to X$ is uniformly N-expansive if there is $e > 0$ (called uniform N-expansivity constant) such that for every $\epsilon_* > 0$ there is $M \in \mathbb{N}$ such that if $x_1^*, \cdots, x_{N+1}^* \in X$ satisfy*

$$d(f^i(x_l^*), f^i(x_s^*)) \leqslant e, \quad \forall -M \leqslant i \leqslant M, \ \forall l, s \in \{1, \cdots, N+1\},$$

then

$$d(x_l^*, x_s^*) \leqslant \epsilon_*$$

for some distinct $l, s \in \{1, \cdots, N+1\}$.

Every expansive homeomorphism of a compact metric space is uniformly expansive (lemma 2 in [11]). Likewise, we obtain the following lemma.

LEMMA 1. *Every N-expansive homeomorphism of a compact metric space is uniformly N-expansive.*

Proof. Let $f : X \to X$ be an N-expansive homeomorphism of a compact metric space. We shall prove that every N-expansivity constant e of f is a uniform N-expansivity constant of f.

Suppose not. Then, there is $\epsilon > 0$ such that no $M \in \mathbb{N}$ satisfies the conclusion in definition 3. From this, we obtain $N + 1$ sequences

$$(x_1^M)_{M\in\mathbb{N}}, \cdots, (x_{N+1}^M)_{M\in\mathbb{N}}$$

in X such that

$$d(f^i(x_l^M), f^i(x_s^M)) \leqslant e, \forall -M \leqslant i \leqslant M, \forall l, s \in \{1, \cdots, N+1\} \tag{1.1}$$

but

$$d(x_l^M, x_s^M) > \epsilon, \quad \forall M \in \mathbb{N}, \forall \text{ distinct } l, s \in \{1, \cdots, N+1\}. \tag{1.2}$$

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Since X is compact, we can assume that $\forall l \in \{1, \cdots, N+1\}$ $\exists x_l \in X$ such that $x_l^M \to x_l$ as $M \to \infty$, $\forall l \in \{1, \cdots, N+1\}$.

Since f is continuous, we can obtain the inequalities below by fixing $i \in \mathbb{Z}$ and letting $M \to \infty$ in (1.1):

$$\mathrm{d}(f^i(x_l), f^i(x_r)) \leqslant e, \quad \forall i \in \mathbb{Z}, \forall l, s \in \{1, \cdots, N+1\}.$$

This proves

$$\{x_1, \cdots, x_{N+1}\} \subset \Gamma_e(x_1).$$

But e is an N-expansivity constant so $\Gamma_e(x_1)$ has at most N elements thus the above inclusion implies

$$x_l = x_s, \quad \text{for some distinct } l, s \in \{1, \cdots, N+1\}.$$

However, by letting $M \to \infty$ in (1.2), one gets $\mathrm{d}(x_l, x_s) \geqslant \epsilon$ hence

$$x_l \neq x_s, \quad \forall \text{ distinct } l, s \in \{1, \cdots, N+1\}.$$

This is a contradiction which completes the proof. □

We will need the auxiliary definition below.

DEFINITION 4. *A map of a metric space $f : X \to X$ is weak asymptotically N-expansive if there is $e > 0$ (weak asymptotic N-expansivity constant) such that if $x_1, \cdots, x_{N+1} \in X$ and $\mathrm{d}(f^i(x_l), f^i(x_r)) \leqslant e$, $\forall i \geqslant 0$ and $l, r \in \{1, \cdots, N+1\}$, then*

$$\lim_{j\to\infty} \inf\{\mathrm{d}(f^j(x_l), f^j(x_s)) : l, s \in \{1, \cdots N+1\} \textit{ are distinct}\} = 0.$$

The difference between this definition and that of asymptotically N-expansive bijective maps is the limit in the corresponding conclusions. In particular, every asymptotically N-expansive homeomorphism is weak asymptotically N-expansive. These concepts coincide with asymptotic expansivity for $N = 1$.

Now we prove a lemma closely related to proposition 11 in [**5**]. Its proof is based on proposition 1 in Sears [**9**].

LEMMA 2. *Every uniformly N-expansive bijective map of a metric space is weak asymptotically N-expansive.*

Proof. It suffices to prove that every uniformly N-expansive constant e of a bijective map of a metric space $f : X \to X$ is a weak asymptotic N-expansivity constant.

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Suppose not. More precisely, that e is not a weak asymptotic N-expansivity constant. Then, by definition 4, there are $x_1, \cdots, x_{N+1} \in X$ satisfying

$$d(f^j(x_l), f^j(x_s)) \leqslant e, \quad \forall j \geqslant 0, \ l, s \in \{1, \cdots, N+1\}, \tag{1.3}$$

but

$$\lim_{i\to\infty} \inf\{d(f^i(x_l), f^i(x_s)) : l, s \in \{1, \cdots N+1\} \text{ are distinct}\} \neq 0.$$

From this limit, we get $\epsilon > 0$ and a sequence

$$(i_k)_{k\in\mathbb{N}} \to \infty \quad (\text{as } k \to \infty) \tag{1.4}$$

such that

$$\inf\{d(f^{i_k}(x_l), f^{i_k}(x_s)) : l, s \in \{1, \cdots N+1\} \text{ are distinct}\} > \epsilon, \quad \forall k \in \mathbb{N}.$$

Then,

$$d(f^{i_k}(x_l) f^{i_k}(x_s)) > \epsilon, \tag{1.5}$$

for all $k \in \mathbb{N}$ and all distinct $l, s \in \{1, \cdots, N+1\}$.

Now, recall that e is a uniformly N-expansivity constant of f so we can apply definition 3. Then, by taking

$$\epsilon_* = \frac{\epsilon}{2}$$

in this definition we obtain $M \in \mathbb{N}$ such that if $x_1^*, \cdots, x_{N+1}^* \in X$ and

$$d(x_l^*, x_s^*) > \epsilon_*, \quad \forall \text{ distinct } l, s \in \{1, \cdots, N+1\}, \tag{1.6}$$

then

$$d(f^i(x_l^*), f^i(x_s^*)) > e,$$

for some $-M \leqslant i \leqslant M$ and some $l, s \in \{1, \cdots, N+1\}$.

By (1.4) we can fix $k \in \mathbb{N}$ such that

$$i_k \geqslant M$$

and, by (1.5),

$$d(f^{i_k}(x_l), f^{i_k}(x_s)) > \epsilon_*, \quad \forall \text{ distinct } l, s \in \{1, \cdots, N+1\}.$$

Then, defining $x_l^* = f^{i_k}(x_l)$ for $1 \leqslant l \leqslant N+1$ we obtain

$$x_1^*, \cdots, x_{N+1}^* \in X$$

satisfying (1.6). So, the choice of M implies that there are

$$-M \leqslant i \leqslant M$$

and two (necessarily distinct) indexes $l, s \in \{1, \cdots, N+1\}$ such that

$$d(f^i(f^{i_k}(x_l)), f^i(f^{i_k}(x_s))) > e.$$

Therefore,

$$d(f^j(x_l), f^j(x_s)) > e \quad \text{where} \quad j = i_k + i.$$

Since $j = i_k + i \geqslant i_k - M \geqslant 0$, we contradict (1.3) completing the proof. □

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Lemmas 1 and 2 reduce question 1 to the following one:

QUESTION 2. Is every weak asymptotically N-expansive homeomorphism of a compact metric space asymptotically N-expansive?

We can give positive answer for nonchaotic homeomorphisms.

LEMMA 3. *Every nonchaotic weak asymptotically N-expansive map of a metric space is asymptotically N-expansive.*

Proof. Consider a nonchaotic weak asymptotically N-expansive map of a metric space $f : X \to X$. Let e be a weak asymptotic N-expansivity constant of f. Suppose that $\frac{e}{2}$ is not an asymptotic N-expansive constant of f. Then, there are $x_1, \cdots, x_{N+1} \in X$ such that $d(f^i(x_l), f^i(x_r)) \leqslant \frac{e}{2}$ for all $i \geqslant 0$, $l, r \in \{1, \cdots, N+1\}$ but

$$\lim_{i\to\infty} \mathrm{d}(f^i(x_l), f^i(x_r)) \neq 0, \quad \forall \text{ distinct } l, r \in \{1, \cdots, N+1\}.$$

So, there is $\epsilon > 0$ such that $\forall$ distinct $l, r \in \{1, \cdots, N+1\}$ there is a sequence

$$(i_k(l,r))_{k\in\mathbb{N}} \to \infty \quad (\text{as } k \to \infty) \tag{1.7}$$

such that

$$\mathrm{d}(f^{i_k(l,r)}(x_l), f^{i_k(l,r)}(x_r)) > \epsilon, \tag{1.8}$$

for all $k \in \mathbb{N}$ and all distinct $l, r \in \{1, \cdots, N+1\}$.

On the other hand,

$$\mathrm{d}(f^j(x_l), f^j(x_r)) \leqslant e, \quad \forall j \geqslant 0, \forall l, r \in \{1, \cdots, N+1\}.$$

Since e is an asymptotic N-expansivity constant,

$$\lim_{j\to\infty} \inf\{d(f^j(x_l), f^j(x_r)) : l, r \in \{1, \cdots, N+1\} \text{ are distinct}\} = 0.$$

Then, there are sequences $l_j \neq r_j \in \{1, \cdots, N+1\}$ such that

$$\lim_{j\to\infty} \mathrm{d}(f^j(x_{l_j}), f^j(x_{r_j})) = 0.$$

By the Pigeon Principle, there is a sequence $(j_q)_{q\in\mathbb{N}} \to \infty$ such that $l_{j_q} = l$ and $r_{j_q} = r$ are constant. Then,

$$\lim_{q\to\infty} \mathrm{d}(f^{j_q}(x_l), f^{j_q}(x_r)) = 0.$$

This implies

$$\liminf_{j\to\infty} \mathrm{d}(f^j(x_l), f^j(x_r)) = 0.$$

Moreover, (1.8) implies

$$\limsup_{j\to\infty} \mathrm{d}(f^j(x_l), f^j(x_r)) \geqslant \epsilon > 0.$$

Then, (x_l, x_r) is a Li–Yorke pair of f which is a contradiction. This completes the proof. □

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Before proving our theorem, we observe that the converse of the above lemma is false. More precisely, though every asymptotically N-expansive homeomorphism is weak asymptotically N-expansive, there are asymptotically N-expansive homeomorphisms which are chaotic (e.g. any expansive homeomorphism with positive entropy). Besides, examples of nonchaotic asymptotically N-expansive homeomorphisms are the restriction of the Denjoy circle maps to its minimal set.

Finally, we prove our result.

Proof of the theorem. Let $f : X \to X$ be a nonchaotic N-expansive homeomorphism of a compact metric space. Since X is compact, lemma 1 implies that f is uniformly N-expansive and so weak asymptotically N-expansive by lemma 2. Since f is nonchaotic, f is asymptotically N-expansive by lemma 3. □

2. An application

The following definition for maps on metric spaces $f : X \to X$ is definition 4.2 in [**2**].

Definition 5. *Given $x \in X$, the number of different stable sets of f in $\Phi_\epsilon(x)$ is defined as the integer $n(x, \epsilon)$ satisfying the two properties below:*

- *There exists $E = E(x, \epsilon) \subset \Phi_\epsilon(x)$ with $n(x, \epsilon)$ elements such that*
$$y \notin W^s(z), \quad \forall \text{ distinct } y, z \in E(x, \epsilon).$$
- *If $y_1, \cdots, y_{n(x,\epsilon)+1}$ are $n(x, \epsilon) + 1$ distinct points of $\Phi_\epsilon(x)$, then there are distinct $l, r \in \{1, \cdots, n(x, \epsilon) + 1\}$ such that*
$$y_l \in W^s(y_r).$$

The following question is conjecture 4.3 in [**2**]:

Question 3. If f is an N-expansive homeomorphism defined in a compact metric space X, then there exists $\epsilon > 0$ such that
$$n(x, \epsilon) \leqslant N, \quad \forall x \in X. \tag{2.1}$$

(Actually, the original question was if $\exists \epsilon > 0$ such that $n(x, \bar{\epsilon}) \leqslant N$ for all $x \in X$, $0 < \bar{\epsilon} \leqslant \epsilon$. However, the statement above is equivalent to this original one.)

The answer looks to be positive. In fact, if we take ϵ as the N-expansivity constant of f, we note that $\Gamma_\epsilon(x)$ has at most N elements making then possible that $n(x, \epsilon) \leqslant N$ ($\forall x \in X$). But $\Phi_\epsilon(x)$ is usually bigger than $\Gamma_\epsilon(x)$ so $n(x, \epsilon)$ may have more than N elements for some x.

Proposition 4.4 in [**2**] gives a positive answer when f has the shadowing property. Here we use our theorem to give a positive answer when f is nonchaotic. We need some elementary facts.

Let $f : X \to X$ be a map of a metric space.

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Definition 6. *The number of stable sets of f in $A \subset X$ is defined by*

$$n(A) = \sup\{n \in \mathbb{N} : \exists E \subset A \text{ with } n \text{ elements such that } x \notin W^s(y) \text{ for all distinct } x, y \in E\}.$$

The link with $n(x, \epsilon)$ is given by

$$n(x, \epsilon) = n(\Phi_\epsilon(x)), \quad \forall x \in X,\, \epsilon > 0. \tag{2.2}$$

We obtain the following elementary lemma.

Lemma 4. *A bijective map of a metric space $f : X \to X$ is asymptotically N-expansive if and only if there is $\epsilon > 0$ such that*

$$n(\Phi_\epsilon(x)) \leqslant N \quad \forall x \in X.$$

Proof. Suppose that there is $\epsilon > 0$ satisfying the conclusion of the lemma. Let $x_1, \cdots, x_{N+1} \in X$ such that

$$\mathrm{d}(f^i(x_l), f^i(x_r)) \leqslant \frac{\epsilon}{2} \quad \forall i \geqslant 0,\, l, r \in \{1, \cdots, N+1\}.$$

Then, the triangle inequality implies $x_1, \cdots, x_{N+1} \in \Phi_\epsilon(x_1)$. It follows from the definition of $n(\Phi_\epsilon(x_1))$ that $x_l \in W^s(x_r)$ for some distinct $l, r \in \{1, \cdots, N+1\}$. Therefore,

$$\lim_{i\to\infty} d(f^i(x_l), f^i(x_r)) = 0 \quad \text{ for some distinct } l, r \in \{1, \cdots, N+1\}$$

and so $e = \frac{\epsilon}{2}$ is an asymptotic N-expansivity constant of f.

Conversely, let e be an asymptotic N-expansivity constant of f. Suppose that $x \in X$ and $x_1, \cdots, x_{N+1} \in \Phi_{\frac{e}{2}}(x)$. Then, $d(f^i(x_l), f^i(x_r)) \leqslant e$ for all $i \geqslant 0$, $l, r \in \{1, \cdots, N+1\}$ so $\lim_{i\to\infty} \mathrm{d}(f^i(x_l), f^i(x_r)) = 0$ (i.e. $x_l \in W^s(x_r)$) for some distinct $l, s \in \{1, \ldots, N+1\}$, hence $n(\Phi_\epsilon(x_1)) \leqslant N$ proving the result. □

Combining (2.2) and lemma 4, we obtain the following corollary.

Corollary 1. *A bijective map of a metric space is asymptotically N-expansive if and only if there is $\epsilon > 0$ satisfying (2.1).*

Finally, we give a partial positive answer for question 3.

Proposition 1. *For every nonchaotic N-expansive homeomorphism of a compact metric space $f : X \to X$, there is $\epsilon > 0$ satisfying (2.1).*

Proof. Our theorem implies that f is asymptotically N-expansive and so the desired ϵ exists by corollary 1. □

We conclude with the following remark.

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Remark 1. Every N-expansive homeomorphism with the shadowing property of a compact metric space is asymptotically N-expansive. Indeed, for all such homeomorphisms, there is $\epsilon > 0$ satisfying (2.1) (proposition 4.4 in [**2**]) and so they are asymptotically N-expansive by lemma 4. It is also worth to mention that question 1 is a mere reformulation of question 3. The answer for both questions would be positive if the answer for question 2 were positive too.

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