THREE-DIMENSIONAL COORDINATION NUMBER FROM TWO-DIMENSIONAL MEASUREMENTS: A NEW METHOD

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ABSTRACT. The average three-dimensional coordination number, n_3, is an important measure of firn structure. The value of n_3 can be estimated from n_2, the average measured two-dimensional coordination number, and from a function, Γ, that depends only on the ratio of average bond radius to grain radius in the sample. This method is easy to apply and does not require the use of unknown shape factors or tunable parameters.

RÉSUMÉ. *Nombre coordonné à trois dimensions à partir de mesures bi-dimentionnelles une nouvelle méthode.* La moyenne des coordonnées à trois dimensions, n_3, est une mesure importante de la structure du névé. La valeur de n_3 peut être estimée à partir de n_2, nombre moyen des mesures à deux dimensions et à partir d'une fonction Γ, qui dépend uniquement du rapport entre le rayon moyen de voisinage et de celui du grain de l'échantillon. Cette méthode est d'un emploi facile et ne nécessite pas l'utilisation de facteurs de forme inconnus ni de paramètres ajustables.

ZUSAMMENFASSUNG. *Dreidimensionale Zuordnungszahl aus zweidimensionalen Messungen: Ein neues Verfahren.* Die mittlere dreidimensionale Zuordnungszahl n_3 ist ein wichtiges Mass für die Firnstruktur. Der Wert von n_3 kann aus n_2, der mittleren gemessenen zweidimensionalen Zuordnungszahl, und aus einer Funktion Γ, die nur vom Verhältnis des mittleren Bindungsradius zum Kornradius abhängt, abgeschätzt werden. Dieses Verfahren ist leicht anzuwenden und erfordert keine Kenntnis von Formfaktoren oder Abstimmungsparametern.

INTRODUCTION

Some physical properties of any granular material depend on the average coordination number of grains in the material (Gubler, 1978) but coordination number has proven a difficult quantity to measure. (Here, coordination number is defined as the number of grains in direct contact with a given grain; the surface of contact between grains is a grain bond.) The most accurate way to measure coordination number is by careful examination of closely spaced serial sections of a material. This method is so time-consuming, however, that it is generally impractical. It is faster but less accurate to estimate coordination numbers from measurements taken on a single section plane. Several methods have been proposed for doing this but all suffer from serious flaws. Here, we briefly review these previously published methods and then present a new method for estimating three-dimensional coordination number from measurements on a single plane of section.

SYMBOLS USED

a	Average bond area
A	Average cross-sectional area of grains
A_m	Average cross-sectional area of grains on plane of section
C'	Shape factor
E	Harmonic mean of bond lengths on plane of section
F	Fraction of bonds intersected by a random cut of a grain
F_1	Value of F for cut near center of grain
F_2	Value of F for cut near edge of grain
i	Tunable parameter in model of Gubler (1978)
n_2	Average coordination number on plane of section
$f_2(n_{2j})$	Distribution of coordination numbers on plane of section
n_3	Average three-dimensional coordination number
$f_3(n_{3j})$	Distribution of three-dimensional coordination numbers
n_3'	Value of n_3 calculated following Alley and others (1982)
n_3''	Value of n_3 calculated following Kry (1975)
N_A	Number of grains per unit area on plane of section
N_{Ab}	Number of bonds per unit area on plane of section
N_{Lb}	Number of bonds per unit length of test line on plane of section
N_{Lf}	Number of free surfaces (ice–air contacts) per unit length of test line on plane of section
N_v	Number of grains per unit volume
N_{vb}	Number of bonds per unit volume
p	Probability of a cut intersecting a bond in model of Gubler (1978)
r	Average radius of grain bonds section
R	Average radius of grains
R_m	Average of radii of grains seen on a plane of section

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r' Average radius of grain bonds calculated from average bond area

R' Average radius of grains calculated from average grain area

z Position of section cut on a grain

z' Lower limit of bond-center positions intersected by a cut at z

z'' Upper limit of bond-center positions intersected by a cut at z

α Ratio of bond radius to grain radius, $= r/R$

α' Relative bond size for tangential bonds

β Fraction of grain surface occupied by bonds,

$$= \frac{2N_{Lb}}{2N_{Lb} + N_{Lf}}$$

Γ Best estimate of average fraction of bonds on a grain intersected by plane of section

Γ_{ins} Average fraction of inscribed bonds on a spherical grain intersected by plane of section

Γ_{tan} Average fraction of tangential bonds on a spherical grain intersected by plane of section

ϕ Angle defined in Figure 2

ψ Angle defined in Figure 2

PREVIOUS METHODS

All of the methods for estimating three-dimensional coordination number, n_3, from measurements on a planar cross-section involve the use of certain simple counting measurements and certain assumptions regarding the size, derived quantities used in such methods are given above. The counting measurements and simple derived quantities above (A_m, β, E, n_2, $f_2(n_{2j})$, N_A, N_{Ab}, N_{Lb}, N_{Lf}) require only that analysis be conducted on a random plane in an isotropic material or that averages be computed over all directions in an anisotropic material. All other quantities given above depend on assumptions regarding geometry in the material, and these assumptions generally cannot be tested rigorously. It is thus important that the method selected for calculating n_3 be insensitive to deviations from assumed geometry. Further discussion of both basic measurements and derived quantities can be found in a number of sources, including Underwood (1970), Kry (1975), and Alley and others (1982). The measurement of n_2 and its use as a qualitative indicator of firn densification has been discussed by Fuchs (1959) and Ebinuma and Maeno (1985). We will present only a sketchy development of this material here.

Calculation of n_3 for planar, circular grain bonds of uniform size between mono-sized spheres was considered by Underwood (1970, p. 102). He showed that

$$n_3 = \frac{2N_{vb}}{N_v}$$

$$N_{vb} = \frac{2(N_{Ab})^2}{\pi N_{Lb}}$$

(1)

where N_{Lb} is the number of intersections per unit length between randomly oriented test lines and grain bonds on the plane of section, N_{Ab} is the number of grain bonds per unit area intersected by the plane of section, N_{vb} is the number of grain bonds per unit volume, and N_v is the number of grains per unit volume. The quantity N_v is

strongly dependent on the grain shape. For spherical grains, Underwood (1970, p. 96) showed that

$$N_v = \frac{N_A}{2R}$$

(2)

where N_A is the number of grains per unit area intersected by the plane of section and R is the average grain radius. The variation of N_v with grain shape has been discussed by DeHoff and Rhines (1961), who showed that relatively small shape variations from sphericity can lead to errors in N_v in excess of 100% if Equation (2) is used. Although DeHoff and Rhines (1961) derived equations equivalent to Equation (2) for a variety of shapes including prolate and oblate spheroids of arbitrary axial ratio, they assumed that all grains in a material have the same shape. This requirement of shape constancy is not met in many real materials, so that estimates of N_v should be considered inaccurate. We thus follow Kry (1975) in considering that any estimate of n_3 based on N_v should not be used to draw quantitative conclusions.

The assumption of uniform bond size used in Equation (1) was relaxed by Fullman (1953) and Kry (1975), who showed that

$$n_3 = \frac{2N_{vb}}{N_v}$$

$$N_{vb} = \frac{8N_{Ab}E}{\pi^2}$$

(3)

where E is the harmonic mean of the lengths of intersections of bonds with the plane of section. Again, this method yields excellent results if and only if N_v can be evaluated accurately. Considering the uncertainty in N_v, Kry (1975) went so far as to suggest that this method may yield little better than order-of-magnitude estimates of n_3. Natural variation of n_3 is little more than an order of magnitude, so this is not too useful.

A slightly different approach was adopted by Alley and others (1982), who estimated that

$$n_3 = \frac{6C' A_m \beta}{a}$$

$$\beta = \frac{2N_{Lb}}{2N_{Lb} + N_{Lf}}$$

$$a = \frac{2N_{Lb}}{N_{vb}}$$

(4)

where C' is a shape factor (discussed below), A_m is the average area of grains observed on a plane of section, β is the fraction of grain surface occupied by bonds, N_{Lf} and N_{Lb} are respectively the number of intersections per unit length of randomly oriented test lines with ice-air surfaces and with grain bonds, a is the average area of bonds, and N_{vb} is given in Equation (3). (Note that the equation for β in Alley and others (1982, p. 9) is in error; the correct expression is given above.) The estimate of a is from Fullman (1953) and assumes circular, planar bonds; this assumption of bond shape was tested by Kry (1975) and Alley (paper in preparation), and seems to be an accurate approximation. The factor $(6C' A_m)$ in Equations (4) is the average surface area per grain, and the shape factor C' corrects for deviations of grains from spherical form and for grain-size distributions. The shape factor was taken to be identically 1 by Alley and others (1982) but it can vary significantly with grain shape (Underwood, 1970, p. 90-93; Table I). The shape factor is not known *a priori* for any real material and is thus the major source of error in this method. This method may be more accurate than those relying on N_v but less accurate than we would like.

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TABLE I. VALUES OF SHAPE FACTOR C' IN EQUATION (4), FROM UNDERWOOD (1970, p. 90-93)

Figure	C'
Oblate spheroid, axial ratio = 0.5	0.888
Sphere	1.000
Pentagonal dodecahedron	1.159
Truncated octahedron (tetrakaidecahedron)	1.184
Prolate spheroid, axial ratio = 0.5	1.269
Hemisphere	1.339
Rhombic dodecahedron	1.414
Cube	1.500

An interesting approach developed by Gubler (1978) for study of seasonal snow leads to an estimate of the distribution of coordination numbers $f_3(n_{3j})$. It was shown by Gubler (1978) that any assumed distribution $f_3(n_{3j})$ can be used to predict a distribution function of coordination numbers in a plane of section, $f_2(n_{2j}, p, i)$, that depends on a tunable parameter, i, and on the probability, p, that a random cut through a single grain with one bond will intersect that bond. The probability, p, depends on average grain radius, R, average bond radius, r, the number of grains per unit volume, N_v, an empirically evaluated constant, and a shape factor that must be estimated. To calculate three-dimensional coordination, p is calculated, trial values of $f_3(n_{3j})$ and i are selected, and the two-dimensional distribution function $f_2(n_{2j}, p, i)$ consistent with the trial values is calculated. Then $f_3(n_{3j})$ and i are adjusted until the predicted $f_2(n_{2j}, p, i)$ matches the observed two-dimensional distribution $f_2(n_{2j})$ as closely as possible. The trial distribution $f_3(n_{3j})$ that produces the closest match is then the best estimate of the actual three-dimensional distribution of coordination numbers. This method requires that $(r/R) \ll 1$, which is realized in the snow studied by Gubler (1978) but not in most firn (Alley and others, 1982; paper in preparation by R.B. Alley). Although this method may prove valuable in the study of seasonal snow, the use of a tunable parameter, a shape factor, an empirical constant, and the uncertain quantity N_v, and the requirement of small bonds, render it suspect for application in firn. The use of observed two-dimensional coordination numbers to estimate three-dimensional coordination is an excellent idea, however, and we adopt it in developing our new model.

MODEL

Stated briefly, we have developed a transfer function from the average two-dimensional coordination number on a plane of section, n_2, to the average three-dimensional coordination number in the material, n_3, based on the average probability that a plane of section will intersect a circular, planar bond on the surface of a spherical grain. No solution is available for the exact geometry of bonds on grains, so we construct limiting cases and choose their average value for our transfer function. The transfer function is

$$n_3 = \frac{n_2}{\Gamma(\alpha)}$$

$$\alpha = \frac{r}{R} \tag{5}$$

where Γ, the average fraction of bonds intersected by a plane of section through a sphere, depends only on α, the ratio of average bond radius, r, to average grain radius, R.

Consider a spherical grain of radius R with a planar, circular grain bond of radius r. Clearly, for $r > 0$ the grain
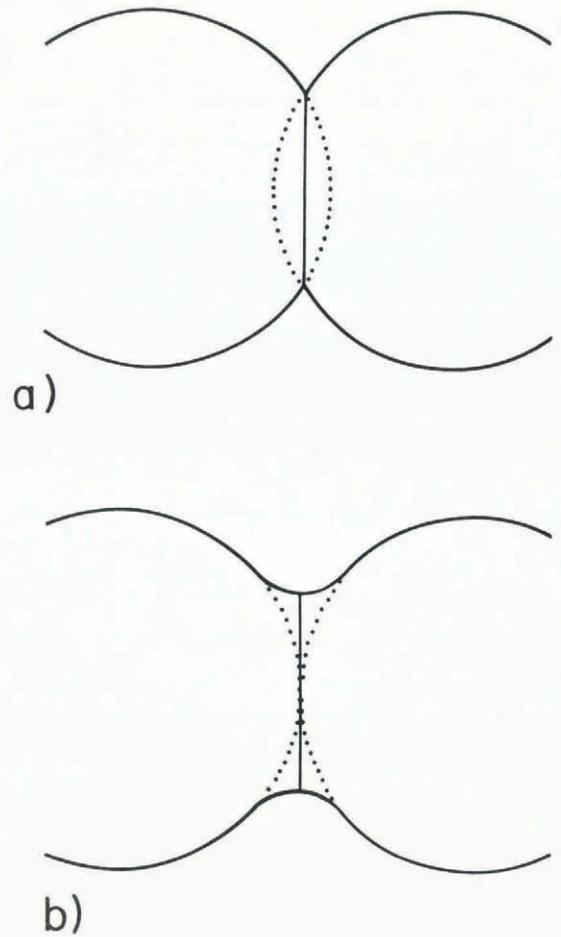

Fig. 1. Limiting geometries for grain bonds. (a) Inscribed bonds. (b) Tangential bonds.
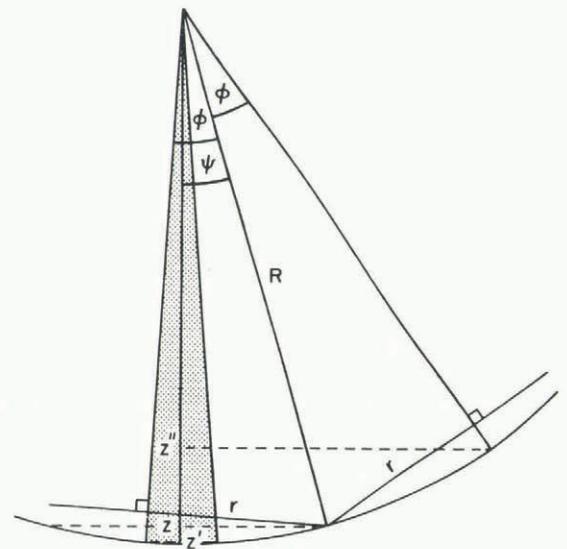

Fig. 2. Geometry for inscribed grain bonds, with plane of section near center of grain.

cannot be a true sphere. The actual geometry is limited by the cases of inscribed bonds (Fig. 1a) and tangential bonds (Fig. 1b). We model each case by calculating Γ_{ins} for inscribed bonds and Γ_{tan} for tangential bonds, and then take Γ to be the average of the two.

We begin with the inscribed case, which is shown in detail in Figure 2. Bonds are assumed to be circles of radius r distributed randomly over the grain. A section cut at position z will intersect all bonds with centers falling between z' and z'' (see Fig. 2). The region between z' and z'' contains surface area of the sphere $2\pi R(z' - z'')$ out of

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total surface area $4\pi R^2$; thus, the cut at z intersects fraction F_1 of the total bonds on the grain, where

$$F_1 = \frac{2\pi R(z' - z'')}{4\pi R^2}.$$ (6)

(The spatial distribution on a grain of bonds of non-zero size cannot be truly random; however, we require only that, on average, fraction F_1 of the surface area of a grain contains fraction F_1 of the bond centers on that grain, which is realized if bonds lack a preferred orientation.)

From the geometry of Figure 2

$$z' = R\cos(\psi - \phi),$$
$$z'' = R\cos(\psi + \phi).$$ (7)

Equations (7) can be re-written in terms of sines and cosines of ψ and ϕ using standard trigonometric identities, which can then be re-written in terms of R, α, and z. This leads to

$$z' = z\sqrt{1 - \alpha^2} + \alpha\sqrt{R^2 - z^2},$$
$$z'' = z\sqrt{1 - \alpha^2} - \alpha\sqrt{R^2 - z^2},$$
$$z' - z'' = 2\alpha\sqrt{R^2 - z^2}.$$ (8)

We have ignored one complication thus far, however. When $\psi < \phi$, a cut at z will not intersect the grain at all if there is a grain bond centered between $(\psi - \phi)$ and $(\phi - \psi)$, as shown in Figure 3. This region has area $2\pi R(R - z')$. If a cut with $\psi < \phi$ intersects the grain, then there
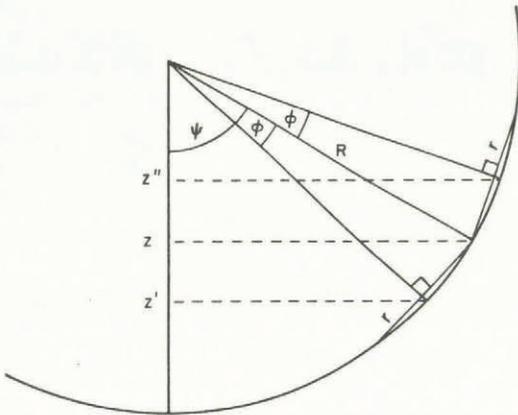

Fig. 3. Geometry for inscribed grain bonds, with plane of section near edge of grain. Shaded region lies between $(\phi - \psi)$ and $(\psi - \phi)$.

are no bonds centered in this region. In this case, the total area sampled by the cut is still given by $2\pi R(z' - z'')$, but the total area over which bonds can occur is $4\pi R^2 - 2\pi R(R - z')$. Thus, for $\psi < \phi$, a cut at z samples F_2 of the total bonds, where

$$F_2 = \frac{2\pi R(z' - z'')}{4\pi R^2 - 2\pi R(R - z')}.$$ (9)

By direct geometry, we can show that when $\phi = \psi$, $z = R\sqrt{1 - \alpha^2}$. This allows us to write the fraction of bonds, F, intersected by a cut at z, as

$$F = \begin{cases} F_2, & -R \leqslant z \leqslant -R\sqrt{1 - \alpha^2}, \\ F_1, & -R\sqrt{1 - \alpha^2} < z \leqslant R\sqrt{1 - \alpha^2}, \\ F_2, & R\sqrt{1 - \alpha^2} < z \leqslant R. \end{cases}$$ (10)

Because of the spherical symmetry of the problem, the average fraction of bonds sampled by a cut through a grain, Γ_{ins}, is simply the average of F over all z along any diameter of the spherical grain. Thus

$$\Gamma_{ins} = \frac{1}{2R} \int_{-R}^{R} F \, dz,$$ (11)

$$\Gamma_{ins} = \frac{1}{2R} \left\{ \int_{-R\sqrt{1-\alpha^2}}^{R\sqrt{1-\alpha^2}} \frac{\alpha}{R} \sqrt{R^2 - z^2} \, dz + 2\int_{R\sqrt{1-\alpha^2}}^{R} \frac{2\alpha\sqrt{R^2 - z^2}}{R + z\sqrt{1 - \alpha^2} + \alpha\sqrt{R^2 - z^2}} dz \right\}.$$ (12)

The first integral in Equation (12) can be evaluated directly, and Equation (12) becomes

$$\Gamma_{ins} = \alpha\sqrt{1 - \alpha^2} + \sin^{-1}\sqrt{1 - \alpha^2} + \frac{1}{R}\int_{R\sqrt{1-\alpha^2}}^{R} \frac{2\alpha\sqrt{R^2 - z^2}}{R + z\sqrt{1 - \alpha^2} + \alpha\sqrt{R^2 - z^2}} dz.$$ (13)

The remaining integral can be evaluated numerically without great difficulty and is a function of α only.

We should note here that the second integral in Equation (12) is a small correction term except at large α. Had we ignored this term and evaluated the first integral from $-R$ to R, the result would have been

$$\Gamma_{ins} \approx \frac{\pi\alpha}{4}.$$ (14)

Equations (13) and (14) differ by only 8% for $\alpha = 1.0$, by only 0.2% for $\alpha = 0.7$, and are identical to four significant figures for $\alpha < 0.5$; thus, Equation (14) could be used in place of Equation (13) in most cases.

Next, consider the tangential case shown in Figure 4. The tangential bond to a sphere of radius R is equivalent to an inscribed bond in a sphere of radius $R\sqrt{1 + \alpha^2}$. The tangential case with relative bond size α thus behaves like the inscribed case with bond size α', where

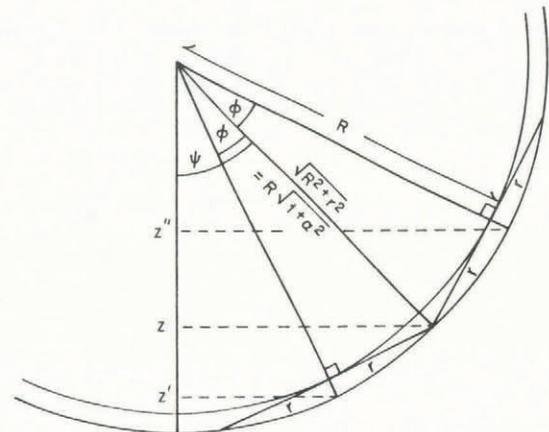$$\alpha' = \frac{\alpha}{\sqrt{1 + \alpha^2}}$$ (15)

Fig. 4. Geometry for tangential grain bonds.

and

$$\Gamma_{tan}(\alpha) = \Gamma_{ins}(\alpha'). \qquad (16)$$

The true Γ for a given grain lies between Γ_{ins} and Γ_{tan}, so we choose as the best estimate

$$\Gamma(\alpha) = \frac{\Gamma_{ins}(\alpha) + \Gamma_{tan}(\alpha)}{2} . \qquad (17)$$

Values of $\Gamma(\alpha)$ are listed in Table II, together with the relative difference between Γ and Γ_{ins} or Γ_{tan}. Based on experience, we do not expect natural values of α to exceed 0.7 commonly, so Table II shows that the maximum uncertainty introduced by choosing Γ rather than Γ_{ins} or Γ_{tan} is less than 10%. In most cases, the error introduced by using Γ should be very small.

Equation (17) and Table II represent our best estimate

TABLE II. Γ VERSUS α FROM EQUATION (17), AND MAXIMUM RELATIVE ERROR IN Γ FROM ASSUMED SHAPE

α	Γ	Maximum error %*
0.00	0.000	0.0
0.05	0.039	0.1
0.10	0.078	0.3
0.15	0.117	0.6
0.20	0.156	1.0
0.25	0.194	1.5
0.30	0.231	2.2
0.35	0.267	2.9
0.40	0.303	3.7
0.45	0.338	4.6
0.50	0.372	5.6
0.55	0.405	6.6
0.60	0.438	7.7
0.65	0.470	8.8
0.70	0.501	10.0
0.75	0.531	11.2
0.80	0.561	12.5
0.85	0.591	13.9
0.90	0.622	15.4
0.95	0.654	17.1
1.00	0.704	20.9

$$*error = \left[\frac{\Gamma - \Gamma_{tan}}{\Gamma}\right] \times 100\% = \left[\frac{\Gamma_{ins} - \Gamma}{\Gamma}\right] \times 100\%.$$

of Γ. If we use the approximation in Equation (14), then we can estimate Γ as

$$\Gamma \approx \frac{\pi\alpha}{4} \left[\frac{1 + (1+\alpha^2)^{-1/2}}{2}\right]. \qquad (18)$$

This is an excellent approximation of Equation (17) except at large α.

CALCULATIONS

This method requires that r and R, the average radii of bonds and grains, be known so that α can be calculated. For circular bonds that may exhibit a bond-size distribution, Fullman (1953) showed that

$$r = \frac{\pi}{4E} \qquad (19)$$

where E is the harmonic mean of the lengths of inter-

sections of bonds with the plane of section. For mono-sized spherical grains, it is not difficult to show that

$$R = \frac{4R_m}{\pi} \qquad (20)$$

where R_m is the average of individual radii of grains seen on the plane of section. This may be a good estimate for non-spherical grains of different sizes (Mendelson, 1969), although this is difficult to demonstrate rigorously. Methods for determining R from measured intercept lengths were also discussed by Mendelson (1969). (We recognize the imprecision introduced by not treating explicitly the effect on R and α of a distribution of grain shapes and sizes, but grain-size itself is not a well-defined quantity unless all grains have a specified shape (Underwood, 1970); we are continuing to investigate this problem.) Once r and R are known, then α is calculated from

$$\alpha = \frac{r}{R} . \qquad (21)$$

In many cases, it is easier to calculate α from r' and R', which are calculated from average areas. For circular bonds of different sizes, Fullman (1953) showed that the average bond area, a, is given by

$$a = \frac{\pi^2 N_{Lb}}{4N_{Ab}E} . \qquad (22)$$

The true average cross-sectional area of grains, A, is related in some fashion to the measured cross-sectional area on the plane of section, A_m. For mono-sized spherical grains, A is given exactly by

$$A = \frac{3}{2} A_m. \qquad (23)$$

This is a slight overestimate of A for a sample consisting of spherical grains of different sizes and a slight underestimate of A for mono-sized, non-spherical grains, and so should be a good estimate for non-spherical grains of different sizes (paper in preparation by R.B. Alley). Then

$$\alpha \approx \sqrt{\frac{a}{A}} = \frac{r'}{R'}. \qquad (24)$$

Empirically, we find that Equations (24) and (21) differ by less than 5% in most cases, so the investigator should choose the more convenient.

DISCUSSION

Because of the near-impossibility of learning n_3 exactly in a real sintered material, we cannot provide a rigorous test of our new method for calculating n_3. As discussed below, however, several factors recommend our method, including its insensitivity to deviations from assumptions, its independence from shape factors or tunable parameters, its accuracy, and its ease of computation.

No real system will match exactly the geometry assumed in our derivation. A grain can deviate from a spherical form toward some other convex form or toward a non-convex form. In either case, however, the deviation will affect in a similar manner both the total surface area of the grain and the average surface sampled by a plane of section; thus, Γ will vary more slowly than any single measure of grain shape. As an extreme example, the value of Γ for a thin disc of radius R is only about $(4/\pi)$ times Γ for a sphere of radius R, if r is held fixed.

As we discussed above, some other published methods of calculating n_3 rely on shape factors or tunable parameters that cannot be estimated well. Our new model does not require any of these and so is preferable. (Deviations from sphericity could be corrected by a shape factor but the slow variation of Γ with grain shape allows us to set this shape factor to 1.)

A major objection to other methods of calculating n_3 is

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their reliance on quantities, particularly N_v, that cannot be determined accurately. Our method depends on n_2 and on α; the latter does depend on a number of measured and calculated quantities. However, both experience and theory (Alley and others, 1982; paper in preparation by R.B. Alley) indicate that α (and thus Γ) varies by a factor of 2 or less after a bonded structure is developed. Most variations in n_3 thus result from variations in n_2. The value of n_2 can be determined with considerable accuracy. In typical firn of density 0.55 Mg m^{-3} from "Upstream B" on the Siple Coast of West Antarctica, a t-test on a count of 100 grains typically yields $n_2 = 2.5 \pm 0.17$ with 90% confidence. Counting more grains would narrow the confidence interval further. We believe that the total accuracy of our method is better than 20%, although we cannot demonstrate this rigorously. (For values of α less than 0.1, difficulty in recognizing a contact and a large standard deviation on n_2 will decrease the accuracy.)

Finally, our new method allows easy computation. Although r and R can be determined only after substantial effort, they are frequently of interest in their own right. Once r and R (or r' and R') are known, our model requires only that n_2 be measured and Γ determined from Table II or Equation (18), and these substituted into Equation (5). The total time required for measurement and computation after r and R are known is typically 10–15 min.

As an exercise, we compared n_3 calculated using our new method with n_3' calculated after Alley and others (1982) for 20 samples from "Upstream B" on the Siple Coast of West Antarctica. Values of n_3 are plotted in Figure 5. If we
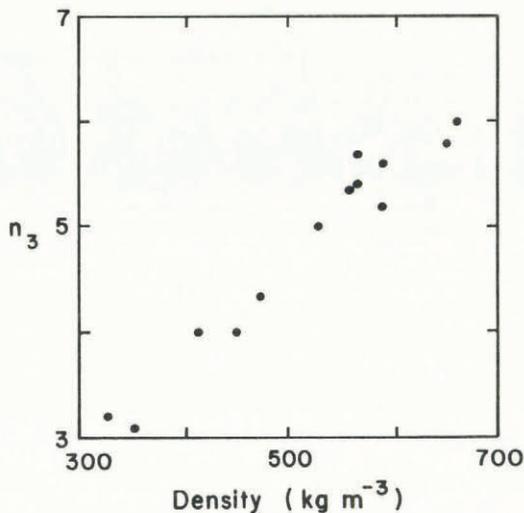
Fig. 5. Values of n_3 versus density for "Upstream B" on the Siple Coast of West Antarctica. Values of n_3 for anisotropic firn were obtained by averaging values for horizontal and vertical sections from the same sample.

take $C' = 1$ following Alley and others (1982), then $n_3' < n_3$ in every case considered. If we assume that n_3 from our new method is exact and calculate C' for the samples, we obtain a mean value of $C' = 1.29$ with a standard deviation of 0.15. Actual grains in shallow firn probably range from spheres to prolate spheroids and become more like truncated octahedra (tetrakaidecahedra) or dodecahedra with increasing depth; also, actual grains probably have some surface irregularities which would tend to increase C'. In the light of these considerations, Table I shows that $C' = 1.3$ is a reasonable value, which tends to lend credence to our model. We emphasize, however, that C' need not be the same in different samples and cannot be known *a priori*.

We also tested our new model against n_3^* from Fullman (1953) and Kry (1975), assuming spherical grains. Results show some variability but in general n_3^* is 10–20% less than n_3. Grains are not spherical in real firn but resemble prolate ellipsoids. Agreement between n_3^* and n_3 would be improved significantly if we assumed the grains to be prolate ellipsoids of axial ratio 0.9 (DeHoff and Rhines, 1961). This is a reasonable value based on observation but cannot be derived readily from measurements.

CONCLUSIONS

We have presented a new method for calculating n_3, the average three-dimensional coordination number in a granular material, from stereological measurements on a section plane. The model is computationally simple, accurate, insensitive to deviations from assumptions used in its derivation, and requires no shape factors or tunable parameters. Differences between results from our new model and previous models are explicable based on known weaknesses in the previous models. Thus, we believe that our model provides a useful way to estimate n_3. We now are using data on n_3 to study densification processes in firn (Alley and Bentley, in press).

REFERENCES

Alley, R.B., *and* Bentley, C.R. In press. Firn studies at Upstream B, West Antarctica. *Antarctic Journal of the United States.*

Alley, R.B., *and others.* 1982. Polar firn densification and grain growth, by R.B. Alley, J.F. Bolzan, and I.M. Whillans. *Annals of Glaciology*, Vol. 3, p. 7–11.

Dehoff, R.T., *and* Rhines, F.N. 1961. Determination of number of particles per unit volume from measurements made on random plane sections: the general cylinder and the ellipsoid. *Transactions of the Metallurgical Society of AIME*, Vol. 221, p. 975–82.

Ebinuma, T., *and* Maeno, N. 1985. Experimental studies on densification and pressure-sintering of ice. *Annals of Glaciology*, Vol. 6, p. 83–86.

Fuchs, A. 1959. Some structural properties of Greenland snow. *U.S. Snow, Ice, and Permafrost Research Establishment. Research Report 42.*

Fullman, R.L. 1953. Measurement of particle sizes in opaque bodies. *Transactions of the Metallurgical Society of AIME*, Vol. 197, p. 447–52.

Gubler, H. 1978. Determination of the mean number of bonds per snow grain and of the dependence of the tensile strength of snow on stereological parameters. *Journal of Glaciology*, Vol. 20, No. 83, p. 329–41.

Kry, P.R. 1975. Quantitative stereological analysis of grain bonds in snow. *Journal of Glaciology*, Vol. 14, No. 72, p. 467–77.

Mendelson, M.I. 1969. Average grain size in polycrystalline ceramics. *Journal of the American Ceramic Society*, Vol. 52, p. 443–46.

Underwood, E.E. 1970. *Quantitative stereology.* Reading, MA, Addison-Wesley Publishing.