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EMBEDDING AND TRACE RESULTS FOR VARIABLE EXPONENT SOBOLEV AND MAZ'YA SPACES ON NON-SMOOTH DOMAINS

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Abstract. We establish interior and trace embedding results for Sobolev functions on a class of bounded non-smooth domains. Also, we define the corresponding generalized Maz'ya spaces of variable exponent, and obtain embedding results similar as in the constant case. Some relations between the variable exponent Maz'ya spaces and the variable exponent Sobolev spaces are also achieved. At the end, we give an application of the previous results for the well-posedness of a class of quasi-linear equations with variable exponent.

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1. Introduction. Over the recent years, various mathematical problems with variable exponent have attracted the attention of many authors. Interest in variational problems and differential equations with non-standard growth conditions has grown, highly motivated by various applications, such as electrorheological fluids and image reconstruction (see [**1**, **6**, **8**, **27**], among others). For differential equations and boundary value problems, properties for Sobolev spaces such as embedding and trace results play an important role on the framework of these equations. The aim of this paper is to provide the corresponding interior and trace embedding for a class of non-smooth domains, and to define the ideal variable exponent function spaces needed in order to obtain the realization of a class of boundary value problems with variable exponent on general (non-smooth) domains.

The embedding results for Sobolev spaces with constant exponent into the Lebesgue spaces have been investigated by many authors. In fact, if $\Omega \subseteq \mathbb{R}^N$ is a $W^{1,p}$-extension domain (for some constant $p \in [1, N)$; see [**21**] for this definition), then it is well known that the interior embedding

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{Np}{N-p}}(\Omega, dx) \tag{1.1}$$

is bounded (e.g. [**20**, Theorem 5]). Also, in the variable exponent case, the boundedness of the embedding (1.1) has been investigated by various authors, where the most general case known was for Ω a bounded John domain (see [**15**]), although the statement for bounded $W^{1,p(\cdot)}$-extension domains follows in a straightforward manner (as we well see later on). On the other hand, concerning traces, we point out the results obtained by Biegert [**4**], where it was established that if $p \in (1, N)$ is constant and if Ω is a bounded

$W^{1,p}$-extension domain whose boundary is an upper d-set with respect to a finite Borel measure μ (see Section 2 for this definition), then the trace embedding

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{dp}{N-p}}(\partial\Omega, d\mu) \tag{1.2}$$

is bounded. Moreover, a boundary trace embedding result for variable exponent Sobolev spaces has been achieved by Fan [**17**] for Lipschitz domains.

Our first goal in this paper is to obtain similar continuous mappings as in (1.1) and (1.2) for the variable exponent case, assuming that Ω is a bounded $W^{1,p(\cdot)}$-extension domain (see Definition 2.3). In fact, we will prove that under these conditions, if $p \in C^{0,1}(\overline{\Omega})$ is such that $1 < p_* := \inf_{\overline{\Omega}} p(x) \le p^* := \sup_{\overline{\Omega}} p(x) < N$, then the interior and trace embedding maps

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx) \quad \text{and} \quad W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu), \tag{1.3}$$

are both bounded. The exponent $Np(\cdot)(N-p(\cdot))^{-1}$ in (1.3) is optimal. However, the value $q(\cdot) := dp(\cdot)/(N-p_*)$ may not be the optimal exponent (which we conjecture to be $dp(\cdot)/(N-p(\cdot))$), but in particular if μ is an upper $(N-1)$-Ahlfors measure, then the optimal exponent is achieved in this article.

In addition, we present the corresponding definitions of the variable exponent classical and extended Maz'ya spaces, and obtain the corresponding embedding results analogous as in the constant case. In particular, the classical Maz'ya inequality (e.g. [**23**, Corollary 2.11.2]) has been optimally generalized to the variable exponent case. To be more precise, given $p \in [1, N)$ and $1 \le r \le p(N-1)(N-p)^{-1}$, Maz'ya proved in [**23**, Corollary 2.11.2] that there exists a continuous embedding

$$W^1_{p,r}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{rN}{N-1}}(\Omega, dx), \tag{1.4}$$

where $W^1_{p,r}(\Omega, \partial\Omega)$ stands as the space introduced by Maz'ya in [**23**]. In this paper, we will show that the continuity of the embedding (1.4) remains valid if one replaces $p,\ r$ by corresponding functions $p,\ r \in C^{0,1}(\overline{\Omega})$ with $1 \le p_* \le p^* < N$ and $1 \le r_* \le r(x) \le (N-1)p(x)(N-p(x))^{-1}$. In our knowledge, variable exponent Maz'ya spaces have not been investigated, up to the present paper. Moreover, we comment that these results allowed us to investigate quasi-linear differential equations with variable exponent and with Robin boundary conditions, even on general bounded domains (see Section 6). For a treatment of these kind of boundary value problems on general domains in the constant case, we refer to [**5**, **10**].

The organization of the work is the following. In Section 2, we review the basic definitions of the variable exponent Lebesgue and Sobolev spaces, and in addition we present other important definitions and well-known results that will be applied in the subsequent sections. Section 3 is devoted to the interior trace embedding problem for non-smooth domains. The crucial result is the validity of the continuous embedding results in (1.3). Other consequences and compactness results are also achieved. In Section 4, we define the notion of the variable exponent Maz'ya space, and establish the corresponding embedding theorems related to these function spaces. In particular, we prove that in the case of bounded $W^{1,p(\cdot)}$-extension domains whose boundaries are upper d-set with respect to a measure μ, the extended variable exponent Maz'ya space coincide with the variable exponent Sobolev space, with equivalent norms. Section 5 presents briefly some concrete examples of $W^{1,p(\cdot)}$-extension domains. The construction of the extension operator is done directly from [**21**]. Finally, in section 6 we apply the results of the previous sections to obtain the realization of the $p(\cdot)$-Laplacian with Robin boundary conditions on bounded non-smooth domains.

2. Preliminaries and intermediate results. Let $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain whose boundary $\partial\Omega$ is finite with respect to a Borel regular measure μ. Given $E \subseteq \overline{\Omega}$ a positive measure space with respect to a finite Borel measure ν, we denote by $\mathcal{P}(E) := \{p : E \to [1, \infty] \text{ measurable}\}$, and set $E^p_\infty := \{x \in E \mid p(x) = \infty\}$. Throughout the rest of this article, we assume that $p \in \mathcal{P}(\overline{\Omega})$ is such that $1 \leq p_* := \operatorname{ess\,inf}_{\overline{\Omega}} p(x) \leq p^* := \operatorname{ess\,sup}_{\overline{\Omega}} p(x) < \infty$, and we also denote by $p'(\cdot)$ the conjugate of $p(\cdot)$ in the usual sense. In fact, in most cases, we will suppose that the function p lie on either the *log-Hölder continuous space* $\mathcal{P}^{\log}(\overline{\Omega})$, or the Lipschitz continuous space $C^{0,1}(\overline{\Omega})$. Here, we recall that $\mathcal{P}^{\log}(\overline{\Omega})$ denotes the set of functions $u \in \mathcal{P}(\overline{\Omega})$ such that the function $v := 1/u$ is globally log-Hölder continuous, that is, if there exist constants $c_1,\ c_2 > 0$ and a constant $\alpha \in \mathbb{R}$ such that

$$|v(x) - v(y)| \leq \frac{c_1}{\log(e + 1/|x - y|)} \qquad \text{and} \qquad |v(x) - \alpha| \leq \frac{c_2}{\log(e + |x|)}$$

for all $x,\ y \in \overline{\Omega}$. For properties of the space $\mathcal{P}^{\log}(\overline{\Omega})$, we refer to [**15**, Section 4.1].

Next, we define

$$L^{p(\cdot)}(E, d\nu) := \left\{u : E \to [-\infty, \infty] \text{ measurable} \mid \rho_{p,E}(u) < \infty\right\},$$

where

$$\rho_{p,E}(u) := \int_{E\setminus E^p_\infty} |u(x)|^{p(x)}\, d\nu + \|u\|_{L^\infty(E^p_\infty,\nu)}.$$

Because of our assumptions on the function p, it is easy to see that in our case $E^p_\infty = \emptyset$ and $E \setminus E^p_\infty = E$, so $L^{p(\cdot)}(E, d\nu)$ becomes the Musielak–Orlicz space $L^{\varphi_p}(E, d\nu)$ for $\varphi_p(x, u) := |u|^{p(x)}$, endowed with the Luxemburg norm

$$\|u\|_{p(\cdot),E} := \|u\|_{L^{p(\cdot)}(E,d\nu)} := \inf\left\{\lambda > 0 \mid \rho_{p,E}(u/\lambda) \leq 1\right\}.$$

(e.g. [**24**, Theorems 1.6 and 7.7], and [**13**]). The variable exponent L^p spaces of our interest will be $L^{p(\cdot)}(\Omega, dx)$ and $L^{p(\cdot)}(\partial\Omega, d\mu)$.

We will also consider the first-order Sobolev space with variable exponent, defined by

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega, dx) \mid \nabla u \in L^{p(\cdot)}(\Omega, dx)^N\right\},$$

and endowed with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(u/\lambda) + \rho_{p,\Omega}(|\nabla u|/\lambda) \leq 1\right\}.$$

For the classical properties of the variable exponent Lebesgue and Sobolev spaces, refer to [**9**, **15**, **18**, **22**, **24**], among others. Finally, for $r > 0$ and $x \in \mathbb{R}^N$, we will denote by $B_r(x)$ the ball of radius r and center x, and $u_{B_r(x)}$ will denote the average of u over $B_r(x)$.

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DEFINITION 2.1. For a function $u \in L^1_{loc}(\mathbb{R}^N, dx)$, the *precise representation* of u is defined by

$$\tilde{u}(x) := \begin{cases} \lim\limits_{r\to 0} u_{B_r(x)}, & \text{if this limit exists,} \\ 0, & \text{otherwise.} \end{cases}$$

For properties about the precise representation of a measurable function, refer to [**11**].

DEFINITION 2.2. Let $d \in (0, N)$ and μ a Borel measure supported on a compact set $F \subseteq \mathbb{R}^N$. Then, μ is said to be an *upper d-Ahlfors measure*, if there exist constants $M, R_0 > 0$ such that

$$\mu(B_r(x)) \leq Mr^d, \qquad \text{for all } 0 < r < R_0 \text{ and } x \in F. \tag{2.5}$$

If the condition (2.5) is fulfilled, then the set $F \subseteq \mathbb{R}^N$ is called an *upper d-(regular) set* with respect to the measure μ (cf. [**4**]). Moreover, the above condition can be reformulated as

$$\mu(B_r(x)) \leq M\frac{m_N(B_r(x))}{r^{N-d}}, \qquad 0 < r < R_0 \text{ and } x \in F, \tag{2.6}$$

where $m_N(\cdot)$ denotes the N-dimensional Lebesgue measure on $\mathbb{R}^N$.

DEFINITION 2.3. Let $p \in L^\infty(\mathbb{R}^N)$, $1 \leq p_* \leq p^* < \infty$. A domain $\Omega \subseteq \mathbb{R}^N$ is called a $W^{1,p(\cdot)}$*-extension domain*, if Ω has the $W^{1,p(\cdot)}$-extension property, that is, if there exists a bounded linear operator $P : W^{1,p(\cdot)}(\Omega) \to W^{1,p(\cdot)}(\mathbb{R}^N)$ such that $Pu = u$ a.e. on Ω. Thus, there exists a constant $C > 0$ such that

$$\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} := \|Pu\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)},$$

for every $u \in W^{1,p(\cdot)}(\Omega)$.

We conclude this section by stating some known results that will be applied in the subsequent sections.

THEOREM 2.4 *see [11]. Given a bounded set $D \subseteq \mathbb{R}^N$, there exist constants $\delta \geq 1$, $C > 0$ such that for any $x_0 \in D$, $0 < r \leq R_0$, and $u \in W^{1,1}(B_{\delta r}(x_0))$, one has*

$$\left|u(x) - u_{B_r(x_0)}\right| \leq C\int_{B_{\delta r}(x_0)} \frac{\theta(x, y)}{m_N(B_{\theta(x,y)}(x))}|\nabla u(y)|\, dy \tag{2.7}$$

(where $\theta(x, y) := |x - y|$ and $m_N(\cdot)$ denotes the usual N-dimensional Lebesgue measure on $\mathbb{R}^N$), whenever $x \in B_r(x_0)$ is such that $\lim_{r\to 0} u_{B_r(x)} = u(x)$. In particular, (2.6) holds for a.e. $x \in B_r(x_0)$.

THEOREM 2.5 *see [11]. Under the same notation as in Theorem 2.4, let $D \subseteq \mathbb{R}^N$ be a bounded set, let μ be an upper d-Ahlfors measure on D for $d \in [0, N)$, and let $\alpha > N - d$ be a fixed constant. Then for any $x \in D$ and $0 < r \leq R_0/2$, there exists a constant $C > 0$*

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such that

$$\int_{B_r(x)} \frac{\theta(x,y)^\alpha}{m_N(B_{\theta(x,y)}(x))}\, d\mu \;\le\; C r^{\alpha-N+d}. \tag{2.8}$$

If $\mu(\cdot) = m_N(\cdot)$, then the above conclusion holds for $d = 0$.

REMARK 2.6. If the measure μ in Theorem 2.5 is an upper $(N-1)$-Ahlfors measure, then by virtue of [**12**, formula (3.9)] it follow that there exists a constant $C > 0$ such that

$$\int_{B_r(x_0)} \left|u - u_{B_r(x_0)}\right| d\mu \;\le\; C \int_{B_{2r}(x_0)} |\nabla u|\, dx\,, \tag{2.9}$$

for each $x_0 \in D$ and $u \in W^{1,1}(B_{2r}(x_0))$.

REMARK 2.7. Let $p \in (1, N)$ be constant, let $d \in (N-p, N)$ and set $q := dp/(N-p)$. If μ is an upper d-Ahlfors measure on a compact set $F \subseteq \mathbb{R}^N$, then it follows from the proof of Theorem 1.9 in [**12**] that there exists a constant $C > 0$ such that

$$\left(\int_{B_r(x_0)} |u|^q\, d\mu\right)^{\frac{1}{q}} \;\le\; C\,\|u\|_{W^{1,p}(B_{2r}(x_0))}, \tag{2.10}$$

for each $x_0 \in F$ and $u \in W^{1,1}(B_{2r}(x_0))$.

3. Embedding results for Sobolev spaces on non-smooth domains. The main purpose of this section will be to establish the following two results.

THEOREM 3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a $W^{1,p(\cdot)}$-extension domain domain, where $p \in \mathcal{P}^{\log}(\overline{\Omega})$ fulfills $1 < p_* \le p^* < N$. Then there is a linear mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx)$ and a constant $C_1 > 0$ such that*

$$\|u\|_{\frac{Np(\cdot)}{N-p(\cdot)},\Omega} \;\le\; C_1\|u\|_{W^{1,p(\cdot)}(\Omega)}, \qquad \textit{for all } u \in W^{1,p(\cdot)}(\Omega). \tag{3.11}$$

Proof. The proof is done basically by following the approach in [**20**, proof of Theorem 2], and in [**14**, proof of Corollary 5.3]. Indeed, we begin with the important observation, that by [**15**, Proposition 4.1.7], $p \in \mathcal{P}^{\log}(\overline{\Omega})$ can be extended to a function $p \in \mathcal{P}^{\log}(\mathbb{R}^N)$. Having said this, under the above conditions, it follows from [**14**, Theorem 5.2] that the embedding $W^{1,p(\cdot)}(\mathbb{R}^N) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\mathbb{R}^N, dx)$ is continuous. Thus, it follows from this facts together with the $W^{1,p(\cdot)}$-extension property that

$$\|u\|_{\frac{Np(\cdot)}{N-p(\cdot)},\Omega} \;\le\; \|u\|_{\frac{Np(\cdot)}{N-p(\cdot)},\mathbb{R}^N} \;\le\; c\|u\|_{W^{1,p(\cdot)}(\mathbb{R}^N)} \;\le\; c'\|u\|_{W^{1,p(\cdot)}(\Omega)},$$

for some constants $c, c' > 0$ and for all $u \in W^{1,p(\cdot)}(\Omega)$, where we recall that we are writing $u := Pu$ (P the extension operator in Definition 2.3) on $\mathbb{R}^N$ for simplicity. This finishes the proof. $\square$

THEOREM 3.2. *Given $p \in C^{0,1}(\overline{\Omega})$ with $1 < p_* \le p^* < N$, let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain whose boundary is an upper d-set with respect to μ, for $d \in$*

$(N-p_*, N)$. Then there exists a linear mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$ and a constant $C_2 > 0$ such that

$$\|u\|_{\frac{dp(\cdot)}{N-p_*},\partial\Omega} \le C_2\|u\|_{W^{1,p(\cdot)}(\Omega)}, \qquad \text{for all } u \in W^{1,p(\cdot)}(\Omega). \tag{3.12}$$

Moreover, if $p_* \ge 1$ and $d \in [N-1, N)$, then there is a bounded linear operator $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{(N-1)p(\cdot)}{N-p(\cdot)}}(\partial\Omega, d\mu)$.

If $p \in (1, N)$ is a constant, then Theorem 3.2 has been by obtained by Biegert [**4**] (for $d \in (N-p, N)$), and for the variable exponent case, and Theorems 3.1 and 3.2 have been investigated in [**17**, **19**] if Ω is a bounded or unbounded Lipschitz domain. Because any Lipschitz domain is a $W^{1,p}$-extension domain whose boundary is an upper $(N-1)$-set with respect to the $(N-1)$-Hausdorff measure $\mu(\cdot) := \mathcal{H}^{N-1}(\cdot)$, the above result is a generalization of the result in [**17**] for a large class of domains that include among others, some examples of (ϵ, δ)-domains (see section 5 for the definition of such domain) and John domains. (refer to [**15**] for the definition and treatment on these spaces). In fact, the conclusion of Theorem 3.1 has been achieved for John domains (e.g. [**15**]). For $N = 2$, we stress out that $W^{1,p(\cdot)}$-domains are contained class of John domains (assuming Ω at least simply connected). Also is known that for bounded John domains, the classical Sobolev–Poincaré inequality holds (e.g. [**7**]). However, for $N \ge 3$, it remains an open problem to relate both class of domains, that is, it still not known if the class of $W^{1,p}$-extension domains is contained in the class of John domains.

To prove Theorem 3.2, we will employ some arguments in a similar way as in [**17**, **19**]. We start with the following estimate.

LEMMA 3.3. *Let $F \subseteq \mathbb{R}^N$ be a compact upper d-set with respect to μ, for $d \in [N-1, N)$. Then there exist $a \in \mathbb{R}^N$, and constants C_3, $\rho > 0$ such that*

$$\|u\|_{1,F} \le C_1\|u\|_{W^{1,1}(B_\rho(a))}, \qquad \text{for all } u \in W^{1,1}(\mathbb{R}^N). \tag{3.13}$$

Proof. We first deal with the case $d \in (N-1, N)$. For this part, it suffices to show (3.13) for the precise representation of $u \in W^{1,1}(\mathbb{R}^N)$, namely $\tilde{u} \in W^{1,1}(\mathbb{R}^N)$. Indeed, let F be as in the theorem, fix $x_0 \in F$, select a constant $\delta \ge 1$ such that the conclusion of Theorem 2.4 holds, and choose $r \in (0, R_0]$. Since $\tilde{u} \in W^{1,1}(B_{\delta r}(x_0))$, we apply the inequalities (2.6)–(2.8) to deduce that

$$\begin{aligned}
\int_{B_r(x_0)} |\tilde{u}(x)|\, d\mu_x &\le \int_{B_r(x_0)} \left|\tilde{u}_{B_r(x_0)}\right| d\mu_x + \int_{B_r(x_0)} \left|\tilde{u}(x) - \tilde{u}_{B_r(x_0)}\right| d\mu_x \\
&\le \mu(B_r(x_0)) \left|\tilde{u}_{B_r(x_0)}\right| + \int_{B_r(x_0)} \left(\int_{B_{\delta r}(x_0)} \frac{\theta(x,y)}{m_N(B_{\theta(x,y)}(x))} |\nabla \tilde{u}(y)|\, dy \right) d\mu_x \\
&\le Mr^{d-N}\|\tilde{u}\|_{1,B_r(x_0)} + \int_{B_{\delta r}(x_0)} \left(\int_{B_r(x_0)} \frac{\theta(x,y)}{m_N(B_{\theta(x,y)}(x))} d\mu_x \right) |\nabla \tilde{u}(y)|\, dy \\
&\le Mr^{d-N}\|\tilde{u}\|_{1,B_r(x_0)} + Cr^{1-N+d}\|\nabla\tilde{u}\|_{1,B_{\delta r}(x_0)} \le C_r\|\tilde{u}\|_{W^{1,1}(B_{\delta r}(x_0))},
\end{aligned}$$

where $C_r := \max\{Mr^{d-N}, Cr^{1-N+d}\}$. Now the compactness of F entails that there is a finite set $\{x_0, \ldots, x_m\} \subseteq F$ with $F \subseteq \bigcup_{j=0}^m B_r(x_j)$. This fact together with the above

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estimate imply the inequality (3.13) for $d \in (N-1, N)$. It remains to prove (3.13) for $d = N-1$. In fact, taking into account (2.6) and (2.9), one has

$$\begin{aligned}\int_{B_r(x_0)} |u|\, d\mu &\leq \int_{B_r(x_0)} \left|u - u_{B_r(x_0)}\right| d\mu + \int_{B_r(x_0)} \left|u_{B_r(x_0)}\right| d\mu \\ &\leq C \int_{B_{2r}(x_0)} |\nabla u|\, dx + Mr^{-1}\|u\|_{1,B_r(x_0)} \;\leq\; C_r'\|u\|_{W^{1,1}(B_{2r}(x_0))},\end{aligned}$$

for each $u \in W^{1,1}(B_{2r}(x_0))$, and where $C_r' := \max\{C, Mr^{-1}\}$. This gives the estimate (3.13) for $d = N-1$. Finally, since $\mathcal{B} := \bigcup_{j=0}^{m} B_r(x_j)$ is an open bounded set, we can find $a \in F$ and $\rho > 0$ large enough, such that $\mathcal{B} \subseteq B_\rho(a)$, as desired. □

Now let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain and let μ be an upper d-Ahlfors measure supported on $\partial\Omega$, for $d \in [N-1, N)$. Clearly, $W^{1,p(\cdot)}(\mathbb{R}^N) \subseteq W^{1,1}(\mathbb{R}^N)$, and thus the $W^{1,p(\cdot)}$-extension property together with Lemma 3.3 imply that for every $u \in W^{1,p(\cdot)}(\Omega)$ there already holds $u|_{\partial\Omega} \in L^1(\partial\Omega, d\mu)$. Therefore, the trace $u|_{\partial\Omega}$ of each function $u \in W^{1,p(\cdot)}(\Omega)$ has definite meaning.

The next result will be established using similar techniques as in [**17**], but we spell out the details here for completeness.

LEMMA 3.4. *Given $p \in C^{0,1}(\overline{\Omega})$ with $1 \leq p_* \leq p^* < N$, let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain and let μ be an upper d-Ahlfors measure supported on $\partial\Omega$, for $d \in [N-1, N)$. For each $u \in W^{1,p(\cdot)}(\Omega)$, if*

$$v := |u|^{\frac{(N-1)p(\cdot)}{N-p(\cdot)}},$$

then $v \in W^{1,1}(B_\rho(a))$, where $B_\rho(a)$ denotes the ball appearing in (3.13).

Proof. We begin by noticing from the fact $d \in [N-1, N)$, $q(\cdot) := (N-1)p(\cdot)/(N-p(\cdot)) < Np(\cdot)/(N-p(\cdot))$, and also we recall that $p \in C^{0,1}(\overline{\Omega})$ can be extended to a function $p \in C^{0,1}(\mathbb{R}^N)$. Hence, it follow from Theorem 3.1 and the $W^{1,p(\cdot)}$-extension property that $v \in L^1(\mathbb{R}^N, dx)$ (as the extension function); in particular, $v \in L^1(B_\rho(a), dx)$. It remains to prove that $\nabla v \in L^1(B_\rho(a), dx)$. Letting $L_q > 0$ denote the Lipschitz constant of $q(x)$ and applying Young's inequality, we have

$$\begin{aligned}|\nabla v| &\leq q^*|u|^{q(x)-1}|\nabla u| + L_q|u|^{q(x)}|\log(u)| \\ &\leq c_q|u|^{(q(x)-1)p'(x)} + c_0|\nabla u|^{p(x)} + L_q|u|^{q(x)}|\log|u|| \\ &= c_q|u|^{Np(x)/(N-p(x))} + c_0|\nabla u|^{p(x)} + L_q|u|^{q(x)}|\log|u||.\end{aligned}$$

It is clear that the first two terms in the last inequality are in $L^1(B_\rho(a), dx)$, so it remains to verify the last term. This is done recalling the well-known properties $\lim_{t\to 0^+} t^{\epsilon_1}|\log(t)| = 0$ and $\lim_{t\to\infty} t^{-\epsilon_2}|\log(t)| = 0$, which are valid for every $\epsilon_1,\ \epsilon_2 \in (0, 1]$. Thus there exist constants $c(\epsilon_1),\ c(\epsilon_2) > 0$ such that $\sup_{t\in(0,1]} t^{\epsilon_1}|\log(t)| \leq c(\epsilon_1)$ and $\sup_{t\geq 1} t^{-\epsilon_2}|\log(t)| \leq c(\epsilon_2)$. These properties entail that

$$|u|^{q(x)}|\log(u)| \;\leq\; c(1)|u|^{q(x)-1}, \qquad \text{for } \ |u| \leq 1,\ \ \epsilon_1 := 1,$$

and

$$|u|^{q(x)}|\log(u)| \;\leq\; |u|^{q(x)+\epsilon_2}|u|^{-\epsilon_2}|\log(u)| \;\leq\; c(\epsilon_2)|u|^{q(x)+\epsilon_2}, \qquad \text{for } \ |u| \geq 1,\ \ \epsilon_2 \in (0, k],$$

where $k := \min\{1, p_*(N-p_*)^{-1}\}$. These estimates show that $|u|^{q(x)}|\log(u)| \in L^1(B_\rho(a), dx)$, and completes the proof. $\square$

The next results follows immediately form the formula (2.10).

COROLLARY 3.5. *Let $q \in (1, N)$ be constant, let $d \in (N-q, N)$ and set $q_d := dq/(N-q)$. If μ is an upper d-Ahlfors measure on a compact set $F \subseteq \mathbb{R}^N$, then there exist $\tilde{a} \in \mathbb{R}^N$, and constants C_3, $\varrho > 0$, such that*

$$\|u\|_{q_d,F} \leq C_3\|u\|_{W^{1,q}(B_\varrho(\tilde{a}))}, \qquad \text{for all } u \in W^{1,q}(\mathbb{R}^N). \tag{3.14}$$

The following result is proved in a similar way as Lemma 3.4.

LEMMA 3.6. *Given $p \in C^{0,1}(\overline{\Omega})$ with $1 < p_* \leq p^* < N$, let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain and let μ be an upper d-Ahlfors measure supported on $\partial\Omega$, for $d \in (N-p_*, N)$. For each $u \in W^{1,p(\cdot)}(\Omega)$, if $w := |u|^{p(x)/p_*}$, then $w \in W^{1,p_*}(B_\varrho(\tilde{a}))$, for $B_\varrho(\tilde{a})$ the ball appearing in (3.14).*

Proof. By the $W^{1,p(\cdot)}$-extension property one only needs to show that $\nabla w \in L^{p_*}(B_\varrho(\tilde{a}), dx)$, given that $u \in W^{1,p(\cdot)}(\Omega)$. Using Young's inequality, we deduce that

$$\begin{aligned}|\nabla w| &\leq \frac{p^*}{p_*}|u|^{p(x)/p_*-1}|\nabla u| + \frac{L_p}{p_*}|u|^{p(x)/p_*}|\log(u)| \\ &\leq c_1|u|^{p(x)/p_*} + c_2|\nabla u|^{p(x)/p_*} + \frac{L_p}{p_*}|u|^{p(x)/p_*}|\log(u)|.\end{aligned}$$

From here, we proceed as in the proof of Lemma 3.4 to conclude that $\nabla w \in L^{p_*}(B_\varrho(\tilde{a}), dx)$, as required. $\square$

Proof of Theorem 3.2 Let us firs consider the case when μ be an upper d-Ahlfors measure supported on $\partial\Omega$, for $d \in (N-p_*, N)$. Given $u \in W^{1,p(\cdot)}(\Omega)$, let w be defined as in Lemma 3.6. Then by Corollary 3.5 and Lemma 3.6, one has $w \in W^{1,p_*}(B_\varrho(\tilde{a})) \hookrightarrow L^{\frac{dp_*}{N-p_*}}(\partial\Omega, d\mu)$, which means that $u \in L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$, and thus $W^{1,p(\cdot)}(\Omega) \subseteq L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$ (in the sense of traces). Now define a linear mapping $T : W^{1,p(\cdot)}(\Omega) \to L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$ by $Tu := u|_{\partial\Omega}$, and observe that the graph of T is closed in $W^{1,p(\cdot)}(\Omega) \times L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$, which implies the boundedness of T by virtue of the closed graph theorem. This proves (3.12). The case $d \in [N-1, N)$ follows similarly with the help of the $W^{1,p(\cdot)}$-extension property, Lemmas 3.3 and 3.4. The proof of Theorem 3.2 is complete. $\square$

COROLLARY 3.7. *Given $p \in C^{0,1}(\overline{\Omega})$ with $1 < p_* \leq p^* < N$, let $\Omega \subseteq \mathbb{R}^N$ be a $W^{1,p(\cdot)}$-extension domain whose boundary is an upper d-set with respect to μ, for $d \in (N-p_*, N)$. If $q \in \mathcal{P}(\partial\Omega)$ fulfills $1 \leq q(x) \leq dp(x)(N-p_*)^{-1}$ for $x \in \partial\Omega$, then there exists a linear continuous trace operator $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\partial\Omega, d\mu)$.*

REMARK 3.8. If μ is an upper d-Ahlfors measure on $\partial\Omega$, for $d \in (N-p_*, N)$, then it is still an open problem to establish the continuous trace $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{dp(\cdot)}{N-p(\cdot)}}(\partial\Omega, d\mu)$, provided that Ω is a $W^{1,p(\cdot)}$-extension domain. We conjecture that it is the optimal exponent, but no proof from this general case has been achieved. However, if $d = N-1$

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the optimal result has been obtained in Theorem 3.2. Moreover, if $p \in (1, N)$ then the conclusion of Theorem 3.2 agrees to the results obtained in [**4**, **11**].

We close this section with a compactness result. Recall that for the remaining of the article, for each $q \in C^{0,1}(\overline{\Omega})$, we will put $q_* := \inf_{\overline{\Omega}} q(x)$ and $q^* := \sup_{\overline{\Omega}} q(x)$.

THEOREM 3.9. *Let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \le p^* < N$, and let $\Omega \subseteq \mathbb{R}^N$ be a $W^{1,p(\cdot)}$-extension domain and a W^{1,p_*}-extension domain, with μ an upper d-set supported on $\partial\Omega$, for $d \in (N - p_*, N)$. If $q \in C^{0,1}(\overline{\Omega})$ is such that $(N - p_*)/d \le q_* < q(x) + \xi \le p(x)$ for $x \in \partial\Omega$ and for some constant $\xi > 0$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{dq(x)}{N-p_*}}(\partial\Omega, d\mu)$ is compact.*

Proof. Let $\alpha \in (0, \min\{p_*, q_*\})$ be chosen such that $w := |u|^{q(x)/\alpha}$ lie in $W^{1,p_*}(\Omega)$. Define a mapping $\Psi : W^{1,p(\cdot)}(\Omega) \to W^{1,p_*}(\Omega)$ by $\Psi(u) := w$. Proceeding as in the proof of Lemma 3.5 we see that Ψ is continuous and bounded. Take a sequence $\{u_n\} \subseteq W^{1,p(\cdot)}(\Omega)$ such that $u_n \to \bar{u}$ weakly on $W^{1,p(\cdot)}(\Omega)$. The compactness of the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^1(\Omega, dx)$ implies that $u_n \to \bar{u}$ a.e. on Ω. Because the sequence $\{\Psi(u_n)\}$ is bounded on the reflexive space $W^{1,p_*}(\Omega)$, we may assume (after taking a subsequence if necessary) that $\{\Psi(u_n)\} \to \bar{v}$ in $W^{1,p_*}(\Omega)$, with $\{\Psi(u_n)\} \to \bar{v}$ a.e. on Ω. But $\Psi(u_n) = |u_n|^{q(x)/\alpha} \to |\bar{u}|^{q(x)/\alpha} = \Psi(\bar{u})$ a.e on Ω, which shows that Ψ is weakly-weakly continuous. Since the trace $W^{1,p_*}(\Omega) \hookrightarrow L^{\frac{d\alpha}{N-p_*}}(\partial\Omega, d\mu)$ is compact (see [**4**, Corollary 7.4]), we deduce that $\Psi(u_n) \to \Psi(\bar{u})$ in $L^{\frac{d\alpha}{N-p_*}}(\partial\Omega, d\mu)$. Consequently, we may assume that $\Psi(u_n)|_{\partial\Omega} \to \Psi(\bar{u})|_{\partial\Omega}$, and moreover we have that

$$\lim_{n\to\infty} \int_{\partial\Omega} |u_n|^{\frac{dq(x)}{N-p_*}} \, d\mu = \int_{\partial\Omega} |\bar{u}|^{\frac{dq(x)}{N-p_*}} \, d\mu \,. \tag{3.15}$$

But (3.15) and the boundary convergence entail that $u_n \to \bar{u}$ in $L^{\frac{dq(x)}{N-p_*}}(\partial\Omega, d\mu)$, proving successfully the compactness of the trace $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{\frac{dq(x)}{N-p_*}}(\partial\Omega, d\mu)$. □

COROLLARY 3.10. *Let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 \le p_* \le p^* < N$, and let $\Omega \subseteq \mathbb{R}^N$ be a $W^{1,p(\cdot)}$-extension domain, with μ an upper d-set supported on $\partial\Omega$, for $d \in [N-1, N)$. If $q \in C^{0,1}(\overline{\Omega})$ is such that $1 \le q_* < q(x) + \xi \le (N-1)p(x)(N - p(x))^{-1}$ for $x \in \partial\Omega$ and for some constant $\xi > 0$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega, d\mu)$ is compact.*

The next result can be deduced following the same procedure given in [**19**, proof of Theorem 1.3] (see also the proof of Theorem 4.3 in the next section), so we omit its proof.

THEOREM 3.11. *Let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 < p_* \le p^* < N$, and let $\Omega \subseteq \mathbb{R}^N$ be a $W^{1,p(\cdot)}$-extension domain. If $q \in \mathcal{P}(\overline{\Omega})$ is such that $q(x) \ge p(x)$ for a.e. $x \in \overline{\Omega}$, and $\operatorname{ess\,inf}_{x\in\overline{\Omega}} \big(Np(x)(N - p(x))^{-1} - q(x)\big) > 0$, then the interior mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(x)}(\Omega, dx)$ is compact.*

REMARK 3.12. The results in this section can be obtained under weaker assumptions over the function $p(x)$. In fact, following the approach as in [**17**], one may be able to establish the majority of the result stated in this section under the assumption $p \in W^{1,\gamma}(\Omega)$, for $\gamma > N$. We do not go for details here.

4. The variable exponent Maz'ya space. In this section, we will give a precise definition of the generalized variable exponent Maz'ya spaces on a bounded domain

$\Omega \subseteq \mathbb{R}^N$. If $p,\ r \geq 1$ are constants, then the Maz'ya space $W^1_{p,r}(\Omega, \partial\Omega)$ was introduced by Maz'ya [**23**]. For the definition and properties of this function space, refer to [**23**]. We start right away with the definition of the Maz'ya space in the extended variable exponent case.

DEFINITION 4.1. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded set, and let μ be a finite Borel measure supported on $\partial\Omega$. Given $p,\ r \in C^{0,1}(\overline{\Omega})$ with $1 \leq p_* \leq p^* < \infty$ and $1 \leq r_* \leq r^* < \infty$, we define the *extended variable exponent Maz'ya space* $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$ to be the completion of the space

$$V^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\partial\Omega} \in L^{r(\cdot)}(\partial\Omega, d\mu)\}$$

with respect to the norm

$$\|u\|_{W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega,d\mu)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(|\nabla u|/\lambda) + \rho_{p,\Omega}(u/\lambda) + \rho_{r,\partial\Omega}(u/\lambda) \leq 1\right\}. \tag{4.16}$$

In addition, recalling that $\mathcal{H}^{N-1}(\cdot)$ denotes the $(N-1)$-dimensional Hausdorff measure over $\partial\Omega$, we define the *classical variable exponent Maz'ya space* $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega)$ as the completion of the space

$$V^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\partial\Omega} \in L^{r(\cdot)}(\partial\Omega, d\mathcal{H}^{N-1})\}$$

with respect to the norm

$$\|u\|_{W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(|\nabla u|/\lambda) + \rho_{r,\partial\Omega}(u/\lambda) \leq 1\right\}. \tag{4.17}$$

We point out that if $p,\ r \in [1, \infty)$ are constants, then the space $W^1_{p,r}(\Omega, \partial\Omega)$ coincides with the classical Maz'ya space defined by Maz'ya in [**23**], and the constant case of the extended Maz'ya space has been briefly investigated in [**5**]. Moreover, we will quote the following fundamental embedding result for the constant case.

THEOREM 4.2 *see [**23**]. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with finite measure, and let $p \in [1, N)$. Then for $1 \leq r \leq p(N-1)(N-p)^{-1}$, there exists a continuous embedding $W^1_{p,r}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{rN}{N-1}}(\Omega, dx)$. Moreover, if $1 \leq q < rN(N-1)^{-1}$, then the embedding $W^1_{p,r}(\Omega, \partial\Omega) \hookrightarrow L^q(\Omega, dx)$ is compact.*

From Theorem 4.2, we can derive the following important result.

THEOREM 4.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with finite measure, and let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 \leq p_* \leq p^* < N$. If $r \in C^{0,1}(\overline{\Omega})$ is such that $1 \leq r_* \leq r(x) \leq (N-1)p(x)(N-p(x))^{-1}$ for all $x \in \overline{\Omega}$, then there exists a linear mapping $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$ and a constant $C_1 > 0$ such that*

$$\|u\|_{\frac{Nr(\cdot)}{N-1},\Omega} \leq C_1\|u\|_{W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega)}, \qquad \textit{for all } u \in W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega). \tag{4.18}$$

Moreover, if $\tilde{r} \in \mathcal{P}(\overline{\Omega})$ is such that

$$r(x) \leq \tilde{r}(x) \ \textit{for a.e. } x \in \overline{\Omega} \qquad \textit{and} \qquad \operatorname*{ess\,inf}_{x\in\overline{\Omega}}\left\{\frac{(Nr(x)}{N-1} - \tilde{r}(x)\right\} > 0,$$

then the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow L^{\tilde{r}(\cdot)}(\Omega, dx)$ is compact.

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Proof. Let $p,\ r \in C^{0,1}(\overline{\Omega})$ be as in the theorem. We proceed in two steps, as follows:

- First, assume that $u \in W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \cap L^\infty(\Omega, dx)$. Then, we put

$$\lambda_r := \|u\|_{\frac{Nr(\cdot)}{N-1},\Omega}, \quad \text{and} \quad w := |u/\lambda_r|^{r(\cdot)}. \tag{4.19}$$

Clearly, $w|_{\partial\Omega} \in L^1(\partial\Omega, d\mathcal{H}^{N-1})$ and $w \in L^{\frac{N}{N-1}}(\Omega, dx)$, with $\|w\|_{\frac{N}{N-1},\Omega} = 1$. Then, since $\frac{(r(\cdot)-1)p(\cdot)}{p(\cdot)-1} \leq \frac{Nr(\cdot)}{N-1}$ if $r(x) \leq (N-1)p(\cdot)(N-p(\cdot)))^{-1}$, using Young's inequality, we have that

$$\int_\Omega |\nabla w|^{r(x)}\, dx \ \leq \ \epsilon \int_\Omega \left|\frac{u}{\lambda\, r}\right|^{\frac{(r(x)-1)p(x)}{p(x)-1}} dx + C_\epsilon \int_\Omega \left|\nabla \frac{|u|}{\lambda_r}\right|^{p(x)} dx + L_r \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} \left|\log\left(\frac{|u|}{\lambda_r}\right)\right| dx$$

$$\leq \ \frac{1}{8C_0} + C_p \int_\Omega \left|\nabla \frac{|u|}{\lambda_r}\right|^{p(x)} dx + L_r \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} \left|\log\left(\frac{|u|}{\lambda_r}\right)\right| dx,$$

for $\epsilon > 0$ (already selected suitably), for some constants $C_\epsilon = C_p > 0$, where $L_r > 0$ denotes the Lipschitz constant of $r(\cdot)$, and C_0 is the constant for the fulfillment of (4.18) for $p(\cdot) = q(\cdot) = 1$ (valid by Theorem 4.2). Now, from the well-known properties $\lim_{t\to 0^+} t^{\epsilon_1}|\log(t)| = 0$ and $\lim_{t\to\infty} t^{-\epsilon_2}|\log(t)| = 0$, which are valid for every $\epsilon_1,\ \epsilon_2 \in (0, 1]$. we see that there exist constants $c(\epsilon_1),\ \ c(\epsilon_2) > 0$ such that $\sup_{t\in(0,1]} t^{\epsilon_1}|\log(t)| \leq c(\epsilon_1)$ and $\sup_{t\geq 1} t^{-\epsilon_2}|\log(t)| \leq c(\epsilon_2)$. These properties, together with Young's inequality, entail that

$$L_r \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} \left|\log\left(\frac{|u|}{\lambda_r}\right)\right| dx \ \leq \ \frac{1}{8C_0} + c_1 \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} dx + c_2 \int_\Omega \left|\frac{u}{\lambda_r}\right|^{\frac{r(x)(2N-1)}{2(N-1)}} dx,$$

for some constants $c_1,\ c_2 > 0$. Thus, combining the previous two calculations, one sees that

$$\int_\Omega |\nabla w|^{r(x)}\, dx \ \leq \ C_r \left(\int_\Omega \left|\nabla \frac{|u|}{\lambda_r}\right|^{p(x)} dx + \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} dx + \int_\Omega \left|\frac{u}{\lambda_r}\right|^{\frac{r(x)(2N-1)}{2(N-1)}} dx \right) + \frac{1}{4C_0},$$

which implies that $\nabla w \in L^1(\Omega, dx)$. As the embedding $W^1_{1,1}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{N}{N-1}}(\Omega, dx)$ is continuous, this together with the above calculations and Young's inequality yield that

$$1 = \|w\|_{\frac{N}{N-1},\Omega} = \int_\Omega \left|\frac{u}{\lambda_r}\right|^{\frac{Nr(x)}{N-1}} dx$$

$$\leq \ C_r' \left(\int_\Omega \left|\nabla \frac{|u|}{\lambda_r}\right|^{p(x)} dx \int_{\partial\Omega} \left|\frac{u}{\lambda_r}\right|^{r(x)} d\mathcal{H}^{N-1} + \int_\Omega \left|\frac{u}{\lambda_r}\right|^{r(x)} dx + \int_\Omega \left|\frac{u}{\lambda_r}\right|^{\frac{r(x)(2N-1)}{2(N-1)}} dx \right) + \frac{1}{4}, \tag{4.20}$$

for some constant $C_r' > 0$. Now, since $\|w\|_{\frac{N}{N-1},\Omega} = \rho_{\frac{N}{N-1},\Omega}(w) = 1$, we may assume that $\rho_{\theta,\Omega}(w) \leq 1$ for all $\rho \in [1, \frac{N}{N-1}]$. Then, if $\lambda_r \leq 1$, then it is easy to conclude that $u \in L^{\frac{Nr(\cdot)}{N-1}}(\Omega)$

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by means of (4.20). Otherwise, when $\lambda_r > 1$, then by (4.20), Hölder's inequality, Young's inequality, and [**15**, Lemma 3.2.4], we have

$$\begin{aligned}
\lambda_r &\le C_r'\lambda_r\left(\int_\Omega \left|\nabla\frac{|u|}{\lambda_r}\right|^{p(x)} dx + \int_{\partial\Omega}\left|\frac{u}{\lambda_r}\right|^{r(x)} d\mathcal{H}^{N-1}\right.\\
&\quad\left. + \int_\Omega\left|\frac{u}{\lambda_r}\right|^{r(x)} dx + \int_\Omega\left|\frac{u}{\lambda_r}\right|^{\frac{r(x)(2N-1)}{2(N-1)}} dx\right) + \frac{\lambda_r}{4}\\
&\le C_r'\lambda_r\left(\int_\Omega \left|\nabla\frac{|u|}{\lambda_r}\right|^{p(x)} dx + \int_{\partial\Omega}\left|\frac{u}{\lambda_r}\right|^{r(x)} d\mathcal{H}^{N-1}\right.\\
&\quad\left. + \left\|\frac{u}{\lambda_r}\right\|_{r(\cdot),\Omega} + \left\|\frac{u}{\lambda_r}\right\|_{\frac{r(\cdot))(2N-1)}{2(N-1)},\Omega}\right) + \frac{\lambda_r}{4}\\
&\le C_r'\left(\lambda^{-(p_*-1)}\int_\Omega |\nabla u|^{p(x)}\,dx + \lambda^{-(r_*-1)}\int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1}\right.\\
&\quad\left. + \|u\|_{r(\cdot),\Omega} + \|u\|_{\frac{r(\cdot)(2N-1)}{2(N-1)},\Omega}\right) + \frac{\lambda_r}{4}\\
&\le C_r'\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1} + C_\Omega + \frac{\lambda_r}{4C_r'}\right) + \frac{\lambda_r}{4}\\
&= C_r'\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1} + C_\Omega\right) + \frac{\lambda_r}{2},
\end{aligned}$$

for some constant $C_\Omega > 0$. But the above calculation implies that

$$\lambda_r \le 2C_r'\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1} + C_\Omega\right), \tag{4.21}$$

which shows that $u \in L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$ for the case $\lambda_r > 1$. Moreover, combining the estimates for the separate cases $\lambda_r \le 1$, and $\lambda_r > 1$, we deduce that

$$\|u\|_{\frac{Nr(\cdot)}{N-1}} \le 2C_r'\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1} + C_\Omega'\right), \tag{4.22}$$

for all $u \in W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \cap L^\infty(\Omega, dx)$, and for some $C_\Omega' > 0$. It remains to show that $u \in L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$, whenever $u \in W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega)$ is arbitrary. In fact, let $u \in W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega)$. For each $n \in \mathbb{N}$, define

$$u_n := \begin{cases} u & \text{if } |u| \le n,\\ n\,\mathrm{sgn}(u) & \text{if } |u| > n.\end{cases} \tag{4.23}$$

Then, $\{u_n\} \subseteq W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \cap L^\infty(\Omega, dx)$, and moreover,

$$\int_{\partial\Omega}|u_n|^{r(x)}\,d\mathcal{H}^{N-1} \le \int_{\partial\Omega}|u|^{r(x)}\,d\mathcal{H}^{N-1} \tag{4.24}$$

and

$$\int_\Omega |\nabla u_n|^{p(x)}\,dx \le \int_\Omega |\nabla u|^{p(x)}\,dx, \tag{4.25}$$

for each $n \in \mathbb{N}$. Combining (4.22), (4.24), (4.25), and [**15**, Lemma 3.2.5], yield that

$$\begin{aligned}\rho_{\frac{Nr(\cdot)}{N-1},\Omega}(u_n) &\le \|u_n\|^{\hat r}_{\frac{Nr(\cdot)}{N-1},\Omega}\\ &\le C_{\hat r}\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx + \int_{\partial\Omega} |u_n|^{r(x)}\,d\mathcal{H}^{N-1} + C'_\Omega\right)^{\hat r}\\ &\le C_{\hat r}\left(\int_\Omega |\nabla u|^{p(x)}\,dx + \int_{\partial\Omega} |u|^{r(x)}\,d\mathcal{H}^{N-1} + C'_\Omega\right)^{\hat r},\end{aligned}$$

for some constant $C_{\hat r} > 1$, where

$$\hat r := \begin{cases} Nr^*(N-1)^{-1} & \text{if } \|u_n\|_{\frac{Nr(\cdot)}{N-1},\Omega} \ge 1,\\ Nr_*(N-1)^{-1} & \text{if } \|u_n\|_{\frac{Nr(\cdot)}{N-1},\Omega} < 1.\end{cases}$$

Hence, as $u_n \stackrel{n\to\infty}{\longrightarrow} u$ a.e. in Ω, and $\mathcal{H}^{N-1}$-a.e. on $\partial\Omega$, from Fatou's lemma we get that the inequality (4.22) remains valid for all $u \in W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega)$. Thus, $u \in L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$, which implies that $W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega) \subseteq L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$. Hence, letting $T : W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega) \to L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$ be defined by $Tu := u$, we observe that the graph of T is closed in $W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega) \times L^{\frac{Nr(\cdot)}{N-1}}(\Omega, dx)$, which implies the boundedness of T by virtue of the closed graph theorem. This gives (4.18). It remains to show the last statement. Indeed, let $\tilde r \in \mathcal{P}(\overline{\Omega})$ be such that

$$r(x) \le \tilde r(x) \ \text{ for a.e. } x \in \overline{\Omega} \qquad \text{and} \qquad \operatorname*{ess\,inf}_{x\in\overline{\Omega}}\left\{\frac{Nr(x)}{N-1} - \tilde r(x)\right\} > 0.$$

Choose $\epsilon > 0$ small enough, such that

$$\theta(x) := \tilde r(x)(1+\epsilon) < \frac{Nr(x)}{N-1}, \qquad \text{for all } x \in \overline{\Omega}.$$

Clearly, the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega) \hookrightarrow L^{\theta(\cdot)}(\Omega, dx)$ is bounded. Now let $F \subseteq W^1_{p(\cdot),r(\cdot)}(\Omega,\partial\Omega)$ be bounded. Then $F \subseteq L^{\theta(\cdot)}(\Omega, dx)$ is bounded, and thus there is a constant $M > 0$ such that

$$\rho_{\theta(\cdot),\Omega}(u) \le M, \qquad \text{for every } x \in F.$$

Put

$$\Sigma := \left\{ w := |u|^{\tilde r(\cdot)} \mid u \in F \right\},$$

and

$$\Phi(t) = t^\epsilon, \qquad \text{for all } t \in [0,\infty).$$

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Clearly, $\Phi : [0, \infty) \to [0, \infty)$ is increasing, with $\lim_{t\to\infty} \Phi(t) = +\infty$, and moreover,

$$\int_\Omega |w|\Phi(|w|)\, dx = \rho_{\theta(\cdot),\Omega}(u) \le M,$$

for each $w \in \Sigma$. Thus, it follows from [**25**, Theorem 7] (see also [**19**, Lemma 2.2]) that Σ possesses absolute equicontinuous integrals on Ω. As the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow W^1_{1,1}(\Omega, \partial\Omega)$ is continuous, and the embedding $W^1_{1,1}(\Omega, \partial\Omega) \hookrightarrow L^1(\Omega, dx)$ is compact, it follows that $F \subseteq L^1(\Omega, dx)$ is relatively compact. Thus, given a sequence $\{u_n\} \subseteq F$, it contains a subsequence (which we also denote by $\{u_n\}$), such that $u_n \overset{n\to\infty}{\longrightarrow} u$ in $L^1(\Omega, dx)$. It is easy to see that $u_n \overset{n\to\infty}{\longrightarrow} u$ a.e. on Ω. Moreover, because $\left\{|u_n|^{\tilde{r}(\cdot)}\right\} \subseteq \Sigma$ possesses absolutely equicontinuous integrals on Ω, it follows that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{\tilde{r}(x)}\, dx = \int_\Omega |u|^{\tilde{r}(x)}\, dx.$$

(cf. [**25**]). But this together with the obvious inequality $|u_n - u|^{\tilde{r}(\cdot)} \le 2^{\tilde{r}^*}\left(|u_n|^{\tilde{r}(\cdot)} + |u|^{\tilde{r}(\cdot)}\right)$ entail that the set $\{|u_n - u|^{\tilde{r}(\cdot)}\}$ contains absolutely equicontinuous integrals on Ω. Henceforth, $\lim_{n\to\infty} \rho_{\tilde{r}(\cdot),\Omega}(u_n - u) = 0$, which implies that F is a relative compact subset of $L^{\tilde{r}(\cdot)}(\Omega)$, and thus the embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow L^{\tilde{r}(\cdot)}(\Omega)$ is compact, completing the proof. □

Next, we turn our attention on the extended Maz'ya space $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$. Then, using the definition of $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$ together with the results of the previous sections, namely Theorems 3.1 and 3.2, we can establish directly the next results, and thus we will omit their proofs.

COROLLARY 4.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let μ be a finite Borel measure supported on $\partial\Omega$, and let $p,\ r \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \le p^* < N$, and $1 \le r_* \le r^* < \infty$. Then, there is a linear continuous embedding $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx)$.*

COROLLARY 4.5. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let μ be an upper d-Ahlfors measure supported on $\partial\Omega$ for $d \in (N - p_*, N)$, and let $p,\ r \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \le p^* < N$, and $1 \le r_* \le r^* < \infty$. Then, there is a linear continuous trace operator $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu) \hookrightarrow L^{\frac{dp(\cdot)}{N-p_*}}(\partial\Omega, d\mu)$.*

COROLLARY 4.6. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let μ be an upper d-Ahlfors measure supported on $\partial\Omega$ for $d \in (N - p_*, N)$, and let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \le p^* < N$. If $r \in C^{0,1}(\overline{\Omega})$ fulfils $1 \le r_* \le r(x) \le dp(x)(N - p_*)^{-1}$ for all $x \in \overline{\Omega}$, then the spaces $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega, d\mu)$ and $W^{1,p(\cdot)}(\Omega)$ coincide with equivalent norms.*

REMARK 4.7. If $\Omega \subseteq \mathbb{R}^N$ is a Lipschitz domain, then by virtue of Theorem 3.1, 3.2, and Theorem 4.3, if $p(x),\ r(x)$ are as in Corollary 4.4, it follows that the linear mappings $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{Np(\cdot)}{N-p(\cdot)}}(\Omega, dx)$ and $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega) \hookrightarrow L^{\frac{(N-1)p(\cdot)}{N-p(\cdot)}}(\partial\Omega, d\mathcal{H}^{N-1})$ are both bounded. Moreover, if in addition $1 \le r(x) \le (N - 1)p(x)(N - p(\cdot))^{-1}$ for all $x \in \overline{\Omega}$, then the spaces $W^1_{p(\cdot),r(\cdot)}(\Omega, \partial\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ coincide with equivalent norms.

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To conclude this section, we define the space $W_\mu(\Omega, \partial\Omega)$ as the completion of the space

$$V_\mu(\Omega, \partial\Omega) := \{u \in W^{1,p(\cdot)}(\Omega) \cap C_c(\overline{\Omega}) \mid u|_{\partial\Omega} \in L^{r(\cdot)}(\partial\Omega, d\mu)\}$$

with respect to the norm

$$\|u\|_{W_\mu(\Omega,\partial\Omega)} := \inf\left\{\lambda > 0 \mid \rho_{p,\Omega}(|\nabla u|/\lambda) + \rho_{r,\partial\Omega}(u/\lambda) \leq 1\right\}. \tag{4.26}$$

Then, we can deduce the following important result

THEOREM 4.8. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let μ be an upper d-Ahlfors measure supported on $\partial\Omega$ for $d \in (N - p_*, N)$, and let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < N$. If $r \in C^{0,1}(\overline{\Omega})$ fulfils $1 \leq r_* \leq r(x) \leq dp(x)(N - p_*)^{-1}$ for all $x \in \overline{\Omega}$, then the spaces $W_\mu(\Omega, \partial\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ coincide with equivalent norms.*

Proof. By virtue of Theorem 3.2, it suffices to show that $\|u\|_{p(\cdot),\Omega} \leq C\|u\|_{W_\mu(\Omega,\partial\Omega)}$ for all $u \in W^{1,p(\cdot)}(\Omega)$, and for some constant $C > 0$. To show this assertion, one just follow the same argument as in [**3**, Theorem 4.24], with the help of Theorem 3.11. □

5. Some examples of non-smooth domains. In this section, we present a class of domains where the previous results may be applied. We begin with the following definition, due to Jones [**21**].

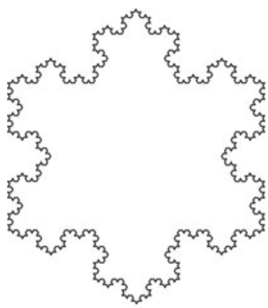
DEFINITION 5.1. A domain $\Omega \subseteq \mathbb{R}^N$ is called an (ϵ, δ)-*domain*, if there exist $\delta \in (0, +\infty]$ and $\epsilon \in (0, 1]$, such that for each $x, y \in \Omega$ with $|x - y| \leq \delta$, there exists a continuous rectifiable curve $\gamma : [0, t] \to \Omega$ such that $\gamma(0) = x$, $\gamma(t) = y$, $l(\{\gamma\}) \leq \frac{1}{\epsilon}|x - y|$, and $\text{dist}(z, \partial\Omega) \geq \epsilon \min\{|x - z|, |y - z|\}$ for all $z \in \{\gamma\}$.

Next, by virtue of [**15**, section 8.5], we have the following two results.

THEOREM 5.2 *see [**15**]. Let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 \leq p_* \leq p^* < \infty$. If $\Omega \subseteq \mathbb{R}^N$ is a bounded (ϵ, δ)-domain, then Ω is a $W^{1,p(\cdot)}$-extension domain*

THEOREM 5.3 *see [**15**]. Let $p \in \mathcal{P}^{\log}(\overline{\Omega})$ be such that $1 \leq p_* \leq p^* < \infty$, and let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain. Then, Ω is an (ϵ, δ)-domain if and only if Ω is a $W^{1,p(\cdot)}$-extension domain*

EXAMPLE 5.4. Let $\Omega \subseteq \mathbb{R}^2$ be the classical snowflake domain (see figure below).



By [**21**], it is an (ϵ, δ)-domain, and by [**30**] $\mathcal{H}^d$ is an upper d-Ahlfors measure supported on $\partial\Omega$, where $d := \log(4)/\log(3)$. Then, it follows that all the results of

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the previous sections are valid on this domain. Another example of a bounded (ϵ, δ)-domain whose boundary is an upper d-set with respect to the so called *self-similar measure* can be found in a beautiful paper by Achdou and Tchou [**2**].

6. An application to boundary value problems. The purpose of this section is to provide an application of the previous results to the solvability of a class of quasi-linear equations with variable exponent on non-smooth domains. To begin, we introduce the notion of $p(\cdot)$-generalized normal derivative, whose definition for the constant case is given in [**5**] (see also [**29**]).

DEFINITION 6.1. Let $p \in C^{0,1}(\overline{\Omega})$ be such that $1 < p_* \leq p^* < N$, let η be a Borel measure supported on $\partial\Omega$, and let $u \in W^{1,1}_{loc}(\Omega)$ be such that $|\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \in L^1(\Omega, dx)$ for all $v \in C^1(\overline{\Omega})$. If there exists a function $f \in L^1_{loc}(\mathbb{R}^N, dx)$ such that

$$\int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx = \int_{\Omega} f v \, dx + \int_{\partial\Omega} v \, d\eta,$$

for all $v \in C^1(\overline{\Omega})$, then we say that η is the $p(\cdot)$-*generalized normal derivative* of u, and we denote

$$|\nabla u|^{p(x)-2}\frac{\partial u}{\partial \nu_\eta} := \eta.$$

Having said that, given a bounded domain $\Omega \subseteq \mathbb{R}^N$ $(N \geq 2)$ and a finite Borel measure μ with support contained in $\partial\Omega$, we consider the generalized quasi-linear elliptic boundary value problem formally given by

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p(x)-2})\nabla u + \alpha(x)\,|u|^{p(x)-2}u = f & \text{in } \Omega\,, \\ |\nabla u|^{p(x)-2}\dfrac{\partial u}{\partial \nu_\mu} + \beta(x)\,|u|^{p(x)-2}u\,d\mu = g\,d\mu & \text{on } \partial\Omega\,. \end{cases} \tag{6.27}$$

Here, $p \in C^{0,1}(\overline{\Omega})$ satisfies $1 < p_* \leq p^* < \infty$, $\alpha \in L^\infty(\Omega, dx)$ and $\beta \in L^\infty(\partial\Omega, d\mu)$ fulfill $\inf_{x\in\Omega} \alpha(x) \geq \alpha_0$ and $\inf_{x\in\partial\Omega} \beta(x) \geq \beta_0$ for some constants α_0, $\beta_0 > 0$, and $f \in L^{q_1(\cdot)}(\Omega, dx)$, $g \in L^{q_2(\cdot)}(\partial\Omega, d\mu)$, for some measurable functions $q_1(x)$, $q_2(x)$ with $1 \leq q_1(x)$, $q_2(y) \leq \infty$ for each $x \in \Omega$, $y \in \partial\Omega$.

DEFINITION 6.2. Given $u, v \in W^1_{p(\cdot)}(\Omega, \partial\Omega, d\mu) := W^1_{p(\cdot),p(\cdot)}(\Omega, \partial\Omega, d\mu)$, set

$$\Lambda_p(u, v) := \int_{\Omega} |\nabla u|^{p(x)-2}\nabla u \nabla v \, dx + \int_{\Omega} \alpha |u|^{p(x)-2}uv \, dx + \int_{\partial\Omega} \beta |u|^{p(x)-2}uv \, d\mu. \tag{6.28}$$

A function $u \in W^1_{p(\cdot)}(\Omega, \partial\Omega, d\mu)$ is called a *weak solution of (6.27)*, if

$$\Lambda_p(u, v) = \int_{\Omega} f v \, dx + \int_{\partial\Omega} g v \, d\mu, \qquad \text{for all } v \in W^1_{p(\cdot)}(\Omega, \partial\Omega, d\mu) \tag{6.29}$$

Under the previous assumptions, we claim that the nonlinear form $\Lambda_p(u, v)$ is bounded, hemicontinuous, strictly monotone and coercive (see [**28**] for these definitions). Indeed, the strict monotonicity of Λ_p follows because it is known that $(|a|^{p(x)-2}a - |b|^{p(x)-2}b)(a-b) \geq c_p(|a|+|b|)^{p(x)-2}|a-b|^2$ for some $c_p > 0$, and for all $a, b \in \mathbb{R}^N$ (e.g. [**5**, Lemma 2.22]). Moreover, applying the generalized Hölder's inequality (see [**22**, Theorem 2.1]) we see that $|\Lambda_p(u,v)| \leq C\|u\|^{\bar{p}-1}_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)} \|v\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)}$ for every $u, v \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$, where $\bar{p}$ denotes either p^* or p_*, from where the boundedness follows. Furthermore, the hemicontinuity of Λ_p follows easily, once we recall the continuity of the norm function. To complete the proof of the claim, for each $u \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$, we put

$$\rho_\mu(u) := \rho_{p,\Omega}(|\nabla u|) + \rho_{p,\Omega}(u) + \rho_{p,\partial\Omega}(u).$$

If $\|u\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)} > 1$, then $\rho_\mu(u) > 1$ (e.g. [**18**, Theorem 1.3]). Since the map $t \mapsto \rho_\mu(u/t)$ is continuous and decreasing on $[1,\infty)$ (whenever $u \neq 0$ a.e. on $\overline{\Omega}$ and $\|u\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)} > 1$), for each $u \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$ there exists $\lambda > 1$ such that $\rho_\mu(u/\lambda) = 1$. Letting $\eta := \lambda^{1-p_*}$ one gets that

$$\eta\, \Lambda_p(u,u) \;\geq\; \lambda\, \rho_\mu(u/\lambda) \;=\; \lambda \;\geq\; \|u\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)}.$$

The above estimate shows that

$$\frac{\Lambda_p(u,u)}{\|u\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)}} \to \infty \quad \text{as} \quad \|u\|_{W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)} \to \infty,$$

that is, Λ_p is coercive. This completes the proof of the claim.

By the properties of the non-linear form Λ_p established above, for each $u \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$, there exists an operator $T_\mu : W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu) \to W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)^*$ such that $\Lambda_p(u,v) = \langle T_\mu(u), v\rangle$, for every $v \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$, where $\langle\cdot,\cdot\rangle$ denotes the duality between $W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$ and $W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)^*$. By the above estimates and properties fulfilled by Λ_p, we see that the operator T_μ is hemicontinuous, strictly monotone, coercive, and bounded, and by [**28**, Corollary 2.2] it follows that T_μ is surjective. Now let $f \in L^{q_1(\cdot)}(\Omega, dx)$ and $g \in L^{q_2(\cdot)}(\partial\Omega, d\mu)$, where q_1, q_2 are measurable functions fulfilling $q_1(x) \geq Np(x)(N(p(x)-1)+p(x))^{-1}$ for all $x \in \Omega$, and $q_2(x) \geq p(x)(p(x)-1)^{-1}$ for each $x \in \partial\Omega$. By Corollary 4.5, it follows that the functional $S : W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu) \to \mathbb{R}$ defined by

$$S(w) := \int_\Omega f w\, dx + \int_{\partial\Omega} g w\, d\mu$$

is continuous. Combining this with the all the above conclusions, taking into account Theorem 4.3, and applying Browder's theorem (e.g. [**16**, Theorem 5.3.22]), we immediately deduce the following result.

THEOREM 6.3. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, and let $f \in L^{q_1(\cdot)}(\Omega, dx)$ and $g \in L^{q_2(\cdot)}(\partial\Omega, d\mu)$, where q_1, q_2 are measurable functions fulfilling $q_1(x) \geq p(x)(p(x)-1)^{-1}$ for all $x \in \Omega$, and $q_2(x) \geq p(x)(p(x)-1)^{-1}$ for each $x \in \partial\Omega$. Then, the boundary value problem (6.27) admits an unique weak solution $u \in W^1_{p(\cdot)}(\Omega,\partial\Omega,d\mu)$. Moreover, in the case when $\mu = \sigma$, if $f \in L^{q_3(\cdot)}(\Omega, dx)$, where $q_3(x) \geq Np(x)(Np(x)-N+1)^{-1}$ for $x \in \Omega$, then the equation (6.27) has an unique weak solution $u \in W^1_{p(\cdot)}(\Omega,\partial\Omega)$.*

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Now set $p_{_N}(\cdot) := Np(\cdot)(N - p(\cdot))^{-1}$ and $p_{_d}(\cdot) := dp(\cdot)(N - p_*)^{-1}$. The next improved result follows as above, but with the help of Corollary 4.6 and Corollary 4.7.

COROLLARY 6.4. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded $W^{1,p(\cdot)}$-extension domain, let μ be an upper d-Ahlfors measure supported on $\partial\Omega$ for $d \in (N - p_*, N)$, and let $f \in L^{q_1(\cdot)}(\Omega, dx)$ and $g \in L^{q_2(\cdot)}(\partial\Omega, d\mu)$, where q_1, q_2 are measurable functions fulfilling $q_1(x) \geq p'_{_N}(x)$ for all $x \in \Omega$, and $q_2(x) \geq p'_{_d}(x)$ for each $x \in \partial\Omega$. Then, the boundary value problem (6.27) admits a unique weak solution $u \in W^{1,p(\cdot)}(\Omega)$.*

REMARK 6.5. By following the arguments given in the beautiful result by Biegert [**3**, Theorem 5.10], it may be possible to show that if we assume the conditions of Corollary 6.4, the (unique) weak solution of (6.27) is bounded on $\overline{\Omega}$. Since this is not the main purpose of this article, we do not go into further details here.

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