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ERDŐS–LIOUVILLE SETS

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Dedicated by the second author to George Szekeres

Abstract

In 1844, Joseph Liouville proved the existence of transcendental numbers. He introduced the set $\mathcal{L}$ of numbers, now known as Liouville numbers, and showed that they are all transcendental. It is known that $\mathcal{L}$ has cardinality $\mathfrak{c}$, the cardinality of the continuum, and is a dense G_δ subset of the set $\mathbb{R}$ of all real numbers. In 1962, Erdős proved that every real number is the sum of two Liouville numbers. In this paper, a set W of complex numbers is said to have the Erdős property if every real number is the sum of two numbers in W. The set W is said to be an Erdős–Liouville set if it is a dense subset of $\mathcal{L}$ and has the Erdős property. Each subset of $\mathbb{R}$ is assigned its subspace topology, where $\mathbb{R}$ has the euclidean topology. It is proved here that: (i) there exist $2^{\mathfrak{c}}$ Erdős–Liouville sets no two of which are homeomorphic; (ii) there exist $\mathfrak{c}$ Erdős–Liouville sets each of which is homeomorphic to $\mathcal{L}$ with its subspace topology and homeomorphic to the space of all irrational numbers; (iii) each Erdős–Liouville set L homeomorphic to $\mathcal{L}$ contains another Erdős–Liouville set L' homeomorphic to $\mathcal{L}$. Therefore, there is no minimal Erdős–Liouville set homeomorphic to $\mathcal{L}$.

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1. Introduction

It has been known for over 175 years that every Liouville number is transcendental and for 120 years that the set $\mathcal{L}$ of Liouville numbers is uncountable. Notwithstanding this, the set $\mathcal{L}$ is known to have Lebesgue measure zero. So in this sense, $\mathcal{L}$ is very small. Therefore, it is surprising that each real number equals the sum of two Liouville numbers. It is reasonable to ask if $\mathcal{L}$ is the smallest set, in some sense, with this property. In this paper, it is proved that there is an uncountable number of sets smaller

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than $\mathcal{L}$ which have this property. Indeed, there are $2^{\mathfrak{c}}$ such subsets of $\mathcal{L}$ no two of which are homeomorphic as subspaces of $\mathbb{R}$.

2. Preliminaries

REMARK 2.1. In 1844, Joseph Liouville proved the existence of transcendental numbers [2, 3]. He introduced the set $\mathcal{L}$ of real numbers, now known as Liouville numbers, and showed that they are all transcendental. A real number x is said to be a *Liouville number* if for every positive integer n, there exists a pair of integers (p, q) with $q > 1$ such that

$$0 < \left| x - \frac{p}{q} \right| < \frac{1}{q^n}.$$

This definition of a Liouville number can be reformulated as follows. For a given irrational x, let $p_k/q_k = p_k(x)/q_k(x)$, where $q_k(x) > 0$, denote the sequence of convergents of the continued fraction expansion of x; then for every $n \in \mathbb{N}$, there are infinitely many k such that $q_{k+1} > q_k^n$. A more restrictive class of Liouville numbers is obtained by requiring this inequality to hold for every $k > N = N(n) \in \mathbb{N}$. Such numbers are called *strong Liouville numbers*.

In 1962, Erdős [8] proved that every real number is the sum of two Liouville numbers (and also the product of two Liouville numbers). He gave two proofs. One was a constructive proof. The other proof used the fact that the set $\mathcal{L}$ of all Liouville numbers is a dense G_δ-set in $\mathbb{R}$ and showed that every dense G_δ-set in $\mathbb{R}$ has this property.

DEFINITION 2.2. A set W of complex numbers is said to have the *Erdős property* if every real number is a sum of two numbers in W.

REMARK 2.3. Recall that if A and B are subsets of the set $\mathbb{C}$ of all complex numbers, then the *sum-set* is defined to be $A + B = \{a + b : a \in A, b \in B\}$. So the subset W of $\mathbb{C}$ has the Erdős property if the sum-set $W + W$ contains the set $\mathbb{R}$. (See [4, 13].)

REMARK 2.4. By the theorem proved by Erdős mentioned above, the set $\mathcal{L}$ of all Liouville numbers has the Erdős property.

REMARK 2.5. If W is a set with the Erdős property, then every set containing W also has the Erdős property.

DEFINITION 2.6. A set W is said to be an *Erdős–Liouville set* if it has the Erdős property and is a dense subset of the set $\mathcal{L}$ of Liouville numbers.

REMARK 2.7. It is not immediately obvious that there exist any Erdős–Liouville sets other than the set $\mathcal{L}$ itself. It is known that some sets of positive Lebesgue measure have the Erdős property, but they are not subsets of $\mathcal{L}$ as the set $\mathcal{L}$ is known to have measure zero. (See, for example, [5].) According to Petruska, [12], Erdős asked if the set of strong Liouville numbers has the Erdős property. However, Petruska [12] proved that it does not. He did this by showing that the sum of two strong Liouville numbers is either a Liouville number or a rational number. Hence, the sum of two strong Liouville numbers cannot equal any irrational number other than a Liouville number. However, it is proved in [7], in the text following Corollary 1.4 and in Section 3, that there does exist another Erdős–Liouville set. In [10], the set of ultra-Liouville numbers is introduced and it is shown that this set is a dense G_δ-subset of $\mathcal{L}$ which is therefore an Erdős–Liouville set.

REMARK 2.8. In the literature, there are various strengthenings of the Erdős result on Liouville numbers. We mention explicitly [1, 14, 15]. The paper [9] shows that the set of Liouville numbers has a property stronger than the Erdős property. Though we do not study such properties, we record here that the $\mathfrak{c}$ Erdős–Liouville sets we produce in Theorem 4.6 also possess this stronger property, while Theorem 3.6 and the proof of Theorem 4.6 show that there are only $\mathfrak{c}$ dense G_δ subsets of $\mathbb{R}$. The relevant theorem from [9] describing this stronger property is the following result.

THEOREM 2.9. *Let G be a dense G_δ-subset of $\mathbb{R}$, I an interval in $\mathbb{R}$ with nonempty interior, and f a continuous function from I to $\mathbb{R}$ which is nowhere locally constant. (This means that f is not constant on any nonempty open subinterval of I.) Then there exists an $x \in G \cap I$ such that $f(x) \in G$. Indeed, there is an uncountable number of such x.*

If we put $f(x) = r - x$, for $r, x \in \mathbb{R}$ and $I = \mathbb{R}$, we see that f satisfies the conditions of the theorem and thus G has the Erdős property. However, as observed in [9], if we put $I = (0, \sqrt{r})$ and $f(x) = \sqrt{r - x^2}$, we see that for every Erdős–Liouville set G, every positive real number is the sum of two squares of numbers in G. Also, the argument in [9, pages 63–64] with $L^1 = \{\exp(\alpha) : \alpha \in \mathcal{L}\}$ leads to the observation that $L^1 \cap \mathcal{L}$ is an Erdős–Liouville set. Although it was not explicitly mentioned in [9], it follows by induction that if $L^n = L^{n-1} \cap \mathcal{L}$, for $n \in \mathbb{N}$, $n > 1$, then each L^n is an Erdős–Liouville set. However, we do not know if the sets L^n are distinct from each other and distinct from $\mathcal{L}$.

PROPOSITION 2.10. *Let S be a set of real numbers such that $W_1 \supset S \supset W_2$, where W_1 and W_2 are Erdős–Liouville sets. Then S is an Erdős–Liouville set.*

PROOF. As $S \supset W_2$, by Remark 2.5, it has the Erdős property. Also as W_2 is dense in $\mathbb{R}$, so too is S. Finally, as $S \subset W_1$, it is a subset of $\mathcal{L}$. Therefore, S is an Erdős–Liouville set. □

3. Some topology

Before proving the existence of an uncountable number of Erdős–Liouville sets, we need to record some topology, some of which was laid bare in [5, 6, 11].

DEFINITION 3.1. A topological space X is said to be *topologically complete* (or *completely metrisable*) if the topology of X is the same as the topology induced by a complete metric on X.

Of course, every complete metric space is topologically complete.

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We denote by $\mathbb{P}$ the set of all irrational real numbers with the topology it inherits as a subspace of the euclidean space $\mathbb{R}$.

A beautiful characterisation of the topological space $\mathbb{P}$ is given in [16, Theorem 1.9.8].

THEOREM 3.2. *The space of all irrational real numbers $\mathbb{P}$ is topologically the unique nonempty, separable, metrisable, topologically complete, nowhere locally compact, and zero-dimensional space.* □

This has a Corollary 3.3, [16, Corollary 1.9.9], which is often proved using continued fractions.

COROLLARY 3.3. *The space $\mathbb{P}$ is homeomorphic to the Tychonoff product $\mathbb{N}^{\aleph_0}$ of a countably infinite number of homeomorphic copies of the discrete space $\mathbb{N}$ of positive integers. Hence, $\mathbb{P} \times \mathbb{P}$ is homeomorphic to $\mathbb{P}$. Indeed, $\mathbb{P}$ is homeomorphic to $\mathbb{P}^{\aleph_0}$.*

REMARK 3.4. Recall that a subset X of a topological space Y is said to be a G_δ*-set* if it is a countable intersection of open sets in Y while X is said to be an F_σ*-set* if it is a countable union of closed sets in Y. Obviously, a subset X of a topological space Y is a G_δ-set if and only if its complement is an F_σ-set. We see immediately that in a metric space such as $\mathbb{R}$, the set $\mathcal{T}$ of all transcendental real numbers is a G_δ-set as its complement is the countably infinite set $\mathbb{A}$ of all real algebraic numbers.

Now we connect the notion of G_δ-set in $\mathbb{R}$ to the property of being topologically complete.

THEOREM 3.5 [16, Theorem A.63]. *A subset of a separable metric topologically complete space is a G_δ-set in that space if and only if it is topologically complete.*

Using Theorems 3.2, 3.5 and Corollary 3.3, we obtain the following result.

THEOREM 3.6. *Every G_δ subset of the set $\mathbb{P}$ of all irrational real numbers is homeomorphic to $\mathbb{P}$ and to $\mathbb{N}^{\aleph_0}$. In particular, the space $\mathcal{T}$ of all real transcendental numbers and the space $\mathcal{L}$ of all Liouville numbers, with their subspace topologies from $\mathbb{R}$, are both homeomorphic to $\mathbb{P}$ and to $\mathbb{N}^{\aleph_0}$.*

These results and a similar one [16, Theorem 1.9.6] characterising the space $\mathbb{Q}$ of all rational numbers with its euclidean topology, are used in [6, 11] to describe transcendental groups and topological transcendental fields.

4. The existence of $2^{\mathfrak{c}}$ Erdős–Liouville sets

THEOREM 4.1. *Let X be a topological space homeomorphic to $\mathbb{P}$. Then X has a dense G_δ-set Y which is homeomorphic to $\mathbb{P}$ such that the cardinality of the set $X \setminus Y$ is $\mathfrak{c}$, the cardinality of the continuum.*

PROOF. Consider the topological space $\mathcal{T}$ of all real transcendental numbers and the topological space $\mathcal{L}$ of all Liouville numbers. We saw in Corollary 2.6 and Remark 2.1

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that $\mathcal{L}$ is a dense G_δ-set, and $\mathcal{T}$ and $\mathcal{L}$ are homeomorphic to $\mathbb{P}$. Further, the cardinality of the set $\mathcal{T} \setminus \mathcal{L}$ is $\mathfrak{c}$. As the properties of being a dense G_δ-set and having cardinality $\mathfrak{c}$ are preserved by homeomorphisms, the theorem is proved. □

By Theorem 4.1 and Remark 2.1, we have the following corollary.

COROLLARY 4.2. *The space $\mathcal{L}$ of all Liouville numbers has a dense G_δ-set L_1 homeomorphic to $\mathcal{L}$. Further, L_1 is an Erdős–Liouville set.*

THEOREM 4.3. *If L is any Erdős–Liouville set homeomorphic to $\mathbb{P}$, then it has a proper subset L_1 which is an Erdős–Liouville set homeomorphic to $\mathbb{P}$. Therefore, there is no minimal Erdős–Liouville set homeomorphic to $\mathbb{P}$.* □

Our next theorem follows immediately from Corollary 4.2 and Theorem 4.1.

THEOREM 4.4. *There exist Erdős–Liouville sets $L_1, L_2, \ldots, L_n, \ldots$, for $n \in \mathbb{N}$, such that*

$$\mathcal{L} \supset L_1 \supset L_2 \supset \cdots \supset L_n \supset \cdots$$

with each $L_n \setminus L_{n+1}$ having cardinality $\mathfrak{c}$ and each L_{n+1} a G_δ-set in L_n which is homeomorphic to $\mathbb{P}$. □

THEOREM 4.5. *There exist $2^{\mathfrak{c}}$ Erdős–Liouville sets no two of which are homeomorphic.*

PROOF. First, we note that there are precisely $2^{\mathfrak{c}}$ subsets of the set $\mathcal{L}$ of all Liouville numbers as $\mathcal{L}$ has cardinality $\mathfrak{c}$. So the cardinality of the set of Erdős–Liouville sets is not greater than $2^{\mathfrak{c}}$.

Using the notation of Theorem 4.4, let W be any subset of $\mathcal{L} \setminus L_1$. As L_1 is an Erdős–Liouville set and $L_1 \subset \mathcal{L}$, Remark 2.5 implies that $L_1 \cup W$ is an Erdős–Liouville set. As there are $2^{\mathfrak{c}}$ subsets W of the set $\mathcal{L} \setminus L_1$, it follows that there are $2^{\mathfrak{c}}$ distinct Erdős–Liouville sets. So it remains to show only that amongst these, there are $2^{\mathfrak{c}}$ no two of which are homeomorphic.

By the Laverentieff theorem, [16, Theorem A8.5], there are at most $\mathfrak{c}$ subspaces of $\mathbb{R}$ which are homeomorphic. As there are $2^{\mathfrak{c}}$ distinct Erdős–Liouville sets, it follows that there are $2^{\mathfrak{c}}$ Erdős–Liouville sets no two of which are homeomorphic, as required. □

THEOREM 4.6. *There exist $\mathfrak{c}$ Erdős–Liouville sets each of which is homeomorphic to $\mathcal{L}$ with its subspace topology. So each is homeomorphic to $\mathbb{P}$.*

PROOF. Using the notation of Theorem 4.4, $\mathcal{L} \supset L_1$, and the set $\mathcal{L} \setminus L_1$ has cardinality $\mathfrak{c}$. Let $S = \{s_1, s_2, \ldots, s_n, \ldots\}$ be any countably infinite subset of $\mathcal{L} \setminus L_1$. As $\mathcal{L} \setminus L_1$ has cardinality $\mathfrak{c}$, there are $\mathfrak{c}$ distinct such subsets S. Then, $\mathcal{L} \setminus S = \bigcap_{i=1}^{\infty}(\mathcal{L} \setminus \{s_i\})$.

Observing that $\mathcal{L} \supset \mathcal{L} \setminus S \supset L_1$, Proposition 2.10 implies that each $\mathcal{L} \setminus S$ is an Erdős–Liouville set.

Noting that $\mathcal{L}$ is a G_δ-set in $\mathbb{R}$, and each $\mathcal{L} \setminus \{s_i\}$ is an open set in $\mathcal{L}$, it follows that $\mathcal{L} \setminus S$ is a G_δ-set. By Theorem 3.6, each of the $\mathfrak{c}$ sets $\mathcal{L} \setminus S$ is therefore homeomorphic to $\mathcal{L}$ and $\mathbb{P}$. □

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