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REMARKS ON BIANCHI SUMS AND PONTRJAGIN CLASSES

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Abstract

We use the exterior and composition products of double forms together with the alternating operator to reformulate Pontrjagin classes and all Pontrjagin numbers in terms of the Riemannian curvature. We show that the alternating operator is obtained by a succession of applications of the first Bianchi sum and we prove some useful identities relating the previous four operations on double forms. As an application, we prove that for a k-conformally flat manifold of dimension $n \geq 4k$, the Pontrjagin classes P_i vanish for any $i \geq k$. Finally, we study the equality case in an inequality of Thorpe between the Euler–Poincaré characteristic and the kth Pontrjagin number of a $4k$-dimensional Thorpe manifold.

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1. Introduction

Let (M, g) be a compact oriented Riemannian smooth manifold of dimension n and k a positive integer such that $4k \leq n$. We denote by R the Riemann curvature tensor of (M, g) seen as a $(2, 2)$ double form. It results from a theorem of Chern [4] that the object (which is written using double form operations)

$$P_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \operatorname{Alt}(R^k \circ R^k)$$

is a closed differential form of degree $4k$ that represents the kth Pontrjagin class of M. In particular, if $n = 4k$, the integral over M of $P_k(R)$ is a topological invariant, namely it is the kth Pontrjagin number of M.

In this paper, we first study in some detail the double form operations that were used to define the above differential form, namely the Alt operator, the exterior product of double forms that were used to define the kth power of R and finally the composition product $\circ$.

In particular, we show that the Alt operator is obtained by a succession of applications of the first Bianchi sum $\mathfrak{S}$ in the algebra of double forms; more precisely, we prove the following result.

Proposition 1.1. *Let ω be a (p,q) double form; then*

$$\mathrm{Alt}(\omega) = (-1)^{pq+(q(q-1)/2)} \frac{p!}{(p+q)!} \mathfrak{S}^q \omega,$$

where $\mathfrak{S}^q = \mathfrak{S} \circ \cdots \circ \mathfrak{S}$, q times. In particular, if ω satisfies the first Bianchi identity, then $\mathrm{Alt}(\omega) = 0$.

We also prove another useful identity.

Proposition 1.2. *Let $q, s \geq 1$ and $p \geq 0$. If ω_1 is a (p,q) double form and ω_2 is a $(q-1, s-1)$ double form, then*

$$\mathrm{Alt}(g\omega_2 \circ \omega_1) = (-1)^p \frac{p+1}{s} \mathrm{Alt}(\omega_2 \circ \mathfrak{S}(\omega_1)),$$

where g is the Riemannian metric. In particular, if ω_1 satisfies the first Bianchi identity, then $\mathrm{Alt}(g\omega_2 \circ \omega_1) = 0$.

As a direct consequence of the previous proposition, we show that Pontrjagin classes depend only on the Weyl part of the Riemann curvature tensor R. In particular, all the Pontrjagin classes of a conformally flat manifold vanish.

More generally, for $n \geq 4k$, following Kulkarni we shall say that a Riemannian n-manifold (M, g) is k-conformally flat if its Riemann curvature tensor R satisfies $R^k = gH$. In other words, the kth exterior power of R is divisible by g in the exterior algebra of double forms. We recover the usual conformally flat manifolds for $k = 1$.

Another consequence of the above formula is the following result.

Theorem 1.3. *If (M, g) is a k-conformally flat Riemannian manifold of dimension $n \geq 4k$, then the Pontrjagin classes P_i of M vanish for $i \geq k$.*

In Section 5, we study a generalization of four-dimensional Einstein manifolds, namely Thorpe manifolds. An oriented compact Riemannian manifold (M, g) of dimension $n = 4k$ is said to be a *Thorpe manifold* if $*R^k = R^k$, where R is the Riemann curvature tensor seen as a $(2,2)$ double form, R^k its exterior power and $*$ the double Hodge star operator acting on double forms. The usual Einstein 4-manifolds are obtained for $k = 1$.

The following theorem provides topological obstructions to the existence of a Thorpe metric.

Theorem 1.4. *Let (M, g) be a compact orientable $4k$-dimensional Thorpe manifold. Then*

$$\chi(M) \geq \frac{k!k!}{(2k)!} |p_k(M)|.$$

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Here $p_k(M)$ is the kth *Pontrjagin number of M and $\chi(M)$ is the Euler–Poincaré characteristic of M. Furthermore, if equality holds, then M is k-Ricci flat, that is,* $\mathrm{c}R^k = 0$.

The first part of the previous theorem is originally due to Thorpe [16]; however, the second part is a new result. Both parts of the theorem will be proved in the last section using the double form formalism.

In dimension four, the second part of the previous theorem tell us that a compact orientable Einstein manifold M that satisfies $2\chi(M) = |p_1(M)|$ must be Ricci flat. We note that Hitchin [7] showed that such a metric must then be either a flat or a Ricci-flat Kahler metric on the $K3$ surface (Calabi–Yau metric) or a quotient of it.

In dimension eight, Kim [9] showed if equality holds in the previous theorem (that is, $6\chi(M) = |p_2(M)|$) and if we assume also that the metric is Einstein, then the metric must be flat. We prove the following generalization of Kim's result.

Theorem 1.5. *A compact orientable manifold M of dimension $8k$ that is at the same time hyper $2k$-Einstein and Thorpe and satisfies $((4k)!/(2k)!(2k)!)\chi(M) = |p_{2k}(M)|$ must be k-flat, that is, $R^k = 0$.*

Recall that a metric is said to be hyper $2k$-Einstein if its Riemann curvature tensor satisfies $\mathrm{c}R^k = \lambda g^{2k-1}$ for some constant real number λ. We recover the usual Einstein metrics for $k = 1$.

In the final section, we prove a Stehney-type formula for all the (mixed) Pontrjagin numbers $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$. Recall that these numbers are important topological invariants for a manifold, they are oriented cobordism invariant and, together with Stiefel–Whitney numbers, they determine an oriented manifold's oriented cobordism class. Furthermore, the signature and the $\hat{A}$ genus can be expressed explicitly through linear combinations of Pontrjagin numbers.

Theorem 1.6. *Let (M, g) be a compact $4k$-dimensional Riemannian oriented manifold with Riemann curvature tensor R, seen as a $(2, 2)$ double form, and let $k_1, k_2, \ldots, k_m$ be a collection of natural numbers such that $k_1 + 2k_2 + \cdots + mk_m = k$. Then the Pontrjagin number $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ of M is given by the integral over M of the following $4k$-form:*

$$P_1^{k_1} P_2^{k_2} \cdots P_m^{k_m} = \frac{(4k)!}{[(2k)!]^2 (2\pi)^{2k}} \bigg(\prod_{i=1}^{m} \frac{[(2i)!]^2}{(i!)^{2k_i} (4i)!} \bigg) \mathrm{Alt}[(R \circ R)^{k_1} (R^2 \circ R^2)^{k_2} \cdots (R^m \circ R^m)^{k_m}],$$

where all the powers over double forms are taken with respect to the exterior product of double forms.

2. The exterior and composition algebras of double forms

2.1. The exterior algebra of double forms. Let (V, g) be a Euclidean real vector space of finite dimension n. Let $\Lambda V^* = \bigoplus_{p \geq 0} \Lambda^p V^*$ (respectively $\Lambda V = \bigoplus_{p \geq 0} \Lambda^p V$) denote the exterior algebra of the dual space V^* (respectively V). We define *the space of exterior double forms* of V (respectively *the space of exterior double vectors*) as

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$$\mathcal{D}(V^*) = \Lambda V^* \otimes \Lambda V^* = \bigoplus_{p,q\geq 0} \mathcal{D}^{p,q}(V^*)$$

$$\text{(respectively } \mathcal{D}(V) = \Lambda V \otimes \Lambda V = \bigoplus_{p,q\geq 0} \mathcal{D}^{p,q}(V)),$$

where $\mathcal{D}^{p,q}(V^*) = \Lambda^p V^* \otimes \Lambda^q V^*$ (respectively $\mathcal{D}^{p,q}(V) = \Lambda^p V \otimes \Lambda^q V$).

The space $\mathcal{D}(V^*)$ is naturally a bigraded associative algebra, called a *double exterior algebra of* V^*, where for simple elements $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}(V^*)$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r,s}(V^*)$, the multiplication is given by

$$\omega_1\omega_2 = (\theta_1 \otimes \theta_2)(\theta_3 \otimes \theta_4) = (\theta_1 \wedge \theta_3) \otimes (\theta_2 \wedge \theta_4) \in \mathcal{D}^{p+r,q+s}(V^*),$$

where $\wedge$ denotes the standard exterior product on the exterior algebra ΛV^*. A *double form of degree* (p, q) is by definition an element of the tensor product $\mathcal{D}^{p,q}(V^*) = \Lambda^p V^* \otimes \Lambda^q V^*$.

The above multiplication in $\mathcal{D}(V^*)$ is called the *exterior product of double forms.*

2.2. The composition algebra of double forms. We define a second product in the space of double exterior vectors $\mathcal{D}(V)$ (respectively in the space of double exterior forms $\mathcal{D}(V^*)$), which will be denoted by $\circ$ and will be called the *composition product*; see [5, 13]. Given $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r,s}$, two simple double exterior forms (or double exterior vectors), we set

$$\omega_1 \circ \omega_2 = (\theta_1 \otimes \theta_2) \circ (\theta_3 \otimes \theta_4) = \langle \theta_1, \theta_4 \rangle \theta_3 \otimes \theta_2 \in \mathcal{D}^{r,q}.$$

It is clear that $\omega_1 \circ \omega_2 = 0$ unless $p = s$. Then one can extend the definition to all double forms using linearity. We emphasize that, in contrast with the exterior product of double forms, the composition product clearly depends on the metric g.

It turns out that the space of double forms endowed with the composition product $\circ$ is an associative algebra.

For a double form (or vector) $\omega \in \mathcal{D}^{p,q}$, we denote by $\omega^t \in \mathcal{D}^{q,p}$ the transpose of ω. For a simple double form, it is defined by

$$(\theta_1 \otimes \theta_2)^t = \theta_2 \otimes \theta_1.$$

Using linearity, the previous definition can be extended to all double forms. Note that a double form ω is a symmetric double form if and only if $\omega^t = \omega$.

3. Basic maps in the space of double forms

Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V and $(e_1^*, \ldots, e_n^*)$ its dual basis of 1-forms.

Let $h = e_i^* \otimes e_j^* \in \mathcal{D}^{(1,1)}(V^*)$ be a simple double form. We define four basic linear maps $\mathcal{D}(V^*) \to \mathcal{D}(V^*)$ as follows. For $\omega = \theta_1 \otimes \theta_2 \in \mathcal{D}(V^*)$, we set

$$\begin{aligned}
L_h^{1,1}(\omega) &= e_i^* \wedge \theta_1 \otimes e_j^* \wedge \theta_2,\\
L_h^{-1,-1}(\omega) &= i_{e_i}\theta_1 \otimes i_{e_j}\theta_2,\\
L_h^{1,-1}(\omega) &= e_i^* \wedge \theta_1 \otimes i_{e_j}\theta_2,\\
L_h^{-1,1}(\omega) &= i_{e_i}\theta_1 \otimes e_j^* \wedge \theta_2,
\end{aligned}$$

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where i_{e_j} denotes the usual interior product. We then use linearity to extend first the previous maps $L_h^{a,b}$ to $\mathcal{D}(V^*)$ and secondly to define them for any $h =\in \mathcal{D}^{(1,1)}(V^*)$. For instance, for the inner product $g = \sum_{i=1}^n e_i^* \otimes e_i^* \in \mathcal{D}^{(1,1)}(V^*)$,

$$L_g^{a,b}(\omega) = \sum_{i=1}^n L^{a,b}_{e_i^*\otimes e_i^*}(\omega).$$

Recall that the exterior algebra ΛV^* inherits naturally an inner product, denoted $\langle . , . \rangle$, from the inner product g of V. This inner product can be canonically extended to an inner product, denoted also by $\langle . , . \rangle$, on the space of double forms $\mathcal{D}(V^*)$. More precisely, for simple double forms in $\mathcal{D}^{(p,q)}(V^*)$, we set

$$\langle \theta_1 \otimes \theta_2, \theta_3 \otimes \theta_4 \rangle = \langle \theta_1, \theta_3 \rangle \langle \theta_2, \theta_4 \rangle.$$

With respect to the previous canonical inner product, we have the following result.

Proposition 3.1. *For any $h \in \mathcal{D}^{(1,1)}(V^*)$:*

(1) *the map $L_h^{-1,-1}$ is the adjoint map of $L_h^{1,1}$;*
(2) *the map $L_h^{-1,1}$ is the adjoint map of $L_h^{1,-1}$.*

Proof. The proof is straightforward; it results directly from the fact that the interior product by a vector v is the adjoint of the exterior multiplication by v^*. □

The following proposition shows that the previous maps are the basic maps of the algebra of curvature structures as defined in [10].

Proposition 3.2. *Let (V, g) be a Euclidean vector space and $\omega \in \mathcal{D}(V^*)$ an arbitrary double form. Then:*

(1) *$L_g^{1,1}(\omega) = g\omega$ is the left multiplication map by g;*
(2) *$L_g^{-1,-1}(\omega) = c\,\omega$ is the contraction map of double forms;*
(3) *$L_g^{1,-1}(\omega) = \mathfrak{S}\omega$ is the first Bianchi sum;*
(4) *$L_g^{-1,1}(\omega) = \widetilde{\mathfrak{S}}\omega$ is the first adjoint Bianchi sum.*

Proof. The proof of (1) is straightforward; (2) results from the fact that the contraction map is the adjoint of the multiplication map by g; see [11]. To prove (3), we proceed as follows.

Let $\omega = e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \otimes e_{j_1}^* \wedge \cdots \wedge e_{j_q}^*$ be a simple (p, q) double form; then

$$\begin{aligned} L_g^{1,-1}(\omega) &= \sum_{i=1}^n e_i^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \otimes i_{e_i} e_{j_1}^* \wedge \cdots \wedge e_{j_q}^* \\ &= \sum_{i=1}^n e_i^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \otimes \sum_{k=1}^q (-1)^{q-1} \langle e_i^*, e_{j_k}^* \rangle e_{j_1}^* \wedge \cdots \widehat{e_{j_k}} \cdots \wedge e_{j_q}^* \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{i=1}^{n}\sum_{k=1}^{q}(-1)^{q-1}\langle e_i^*, e_{j_k}^*\rangle e_i^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \otimes e_{j_1}^* \wedge \cdots \widehat{e_{j_k}} \cdots \wedge e_{j_q}^* \\
&= \sum_{k=1}^{q}(-1)^{q-1} e_{j_k}^* \wedge e_{i_1}^* \wedge \cdots \wedge e_{i_p}^* \otimes e_{j_1}^* \wedge \cdots \widehat{e_{j_k}} \cdots \wedge e_{j_q}^* \\
&= \mathfrak{S}\omega.
\end{aligned}
$$

In the same way, one can prove that

$$
L_g^{-1,1}(\omega) = \sum_{k=1}^{q}(-1)^{q-1} e_{i_1}^* \wedge \cdots \widehat{e_{i_k}} \cdots \wedge e_{i_p}^* \otimes e_{i_k}^* \wedge e_{j_1}^* \wedge \cdots \wedge e_{j_p}^* = \widetilde{\mathfrak{S}}\omega.
$$

This completes the proof of the proposition. □

Remark 3.3. We remark that the first Bianchi sum and its adjoint are related by the formula (which can be easily checked)

$$
\widetilde{\mathfrak{S}}(\omega) = (\mathfrak{S}(\omega^t))^t.
$$

As a first direct application of the previous proposition, we show that the Bianchi sum is an antiderivation, as was observed in [10].

Proposition 3.4. *Let $\omega_1 \in \mathcal{D}^{(p,q)}(V^*)$ and $\omega_2 =\in \mathcal{D}^{(r,s)}(V^*)$; then*

$$
\mathfrak{S}(\omega_1\omega_2) = (\mathfrak{S}\omega_1)\omega_2 + (-1)^{p+q}\omega_1\mathfrak{S}\omega_2. \tag{3.1}
$$

Proof. Without loss of generality, we may assume that $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{(p,q)}(V^*)$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{(r,s)}(V^*)$. Then

$$
\begin{aligned}
\mathfrak{S}(\omega_1\omega_2) &= \mathfrak{S}(\theta_1 \wedge \theta_3 \otimes \theta_2 \wedge \theta_4) \\
&= \sum_{i=1}^{n} e_i \wedge \theta_1 \wedge \theta_3 \otimes i_{e_i}(\theta_2 \wedge \theta_4) \\
&= \sum_{i=1}^{n} e_i \wedge \theta_1 \wedge \theta_3 \otimes (i_{e_i}(\theta_2) \wedge \theta_4) + (-1)^q \theta_2 \wedge i_{e_i}(\theta_4) \\
&= \sum_{i=1}^{n}(e_i \wedge \theta_1 \otimes i_{e_i}\theta_2)(\theta_3 \otimes \theta_4) + \sum_{i=1}^{n}(-1)^p\theta_1 \wedge e_i \wedge \theta_3 \otimes (-1)^q\theta_2 \wedge i_{e_i}(\theta_4) \\
&= \sum_{i=1}^{n}(e_i \wedge \theta_1 \otimes i_{e_i}\theta_2)(\theta_3 \otimes \theta_4) + (\theta_1 \otimes \theta_2)(-1)^{p+q}\sum_{i=1}^{n} e_i \wedge \theta_3 \otimes i_{e_i}(\theta_4) \\
&= (\mathfrak{S}\omega_1)\omega_2 + (-1)^{p+q}\omega_1\mathfrak{S}\omega_2.
\end{aligned}
$$
□

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Recall that the alternating operator is a linear map $\mathcal{D}(V^*) \to \Lambda(V^*)$ defined as follows:

$$\mathrm{Alt}(\theta_1 \otimes \theta_2) = \frac{p!q!}{(p+q)!}\theta_1 \wedge \theta_2, \tag{3.2}$$

where $\theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}(V^*)$. It is clear that Alt is a surjective map. However, it is far from being injective. The next proposition shows in particular that the kernel of Alt contains all double forms satisfying the first Bianchi identity.

Proposition 3.5. *Let ω be a (p,q) double form. Then*

$$\mathrm{Alt}\omega = (-1)^{pq+(q(q-1)/2)}\frac{p!}{(p+q)!}\mathfrak{S}^q\omega,$$

where $\mathfrak{S}^q = \mathfrak{S} \circ \cdots \circ \mathfrak{S}$, q times.

Proof. By linearity, we may assume that $\omega = \theta_1 \otimes \theta_2$ is a decomposed (p,q) double form. Using Proposition 3.2,

$$\begin{aligned}
\mathfrak{S}^q\omega &= (L_g^{1,-1})^q(\omega) \\
&= \sum_{i_1,\dots,i_q=1}^{n} e_{i_1}^* \wedge \cdots \wedge e_{i_q}^* \wedge \theta_1 \otimes i_{e_{i_q}\wedge\cdots\wedge e_{i_1}}\theta_2 \\
&= \sum_{i_1,\dots,i_q=1}^{n} e_{i_1}^* \wedge \cdots \wedge e_{i_q}^* \wedge \theta_1 \otimes \langle e_{i_q}^* \wedge \cdots \wedge e_{i_1}^*, \theta_2\rangle \\
&= \sum_{i_1,\dots,i_q=1}^{n} \langle e_{i_q}^* \wedge \cdots \wedge e_{i_1}^*, \theta_2\rangle e_{i_1}^* \wedge \cdots \wedge e_{i_q}^* \wedge \theta_1 \\
&= (-1)^{(q(q-1)/2)} \sum_{i_1,\dots,i_q=1}^{n} \langle e_{i_1}^* \wedge \cdots \wedge e_{i_q}^*, \theta_2\rangle e_{i_1}^* \wedge \cdots \wedge e_{i_q}^* \wedge \theta_1 \\
&= q!(-1)^{(q(q-1)/2)}\theta_2 \wedge \theta_1 = q!(-1)^{pq+(q(q-1)/2)}\theta_1 \wedge \theta_2 \\
&= (-1)^{pq+(q(q-1)/2)}\frac{(p+q)!}{p!}\mathrm{Alt}(\omega).
\end{aligned}$$

□

The next proposition shows that the operator Alt defines an exterior algebra endomorphism.

Proposition 3.6. *Let $\omega_1 \in \mathcal{D}^{(p,q)}(V^*)$ and $\omega_2 \in \mathcal{D}^{(r,s)}(V^*)$; then $\omega_1\omega_2 \in \mathcal{D}^{(p+r,q+s)}(V^*)$ and*

$$\frac{(p+r+q+s)!}{(p+r)!(q+s)!}\mathrm{Alt}(\omega_1\omega_2) = (-1)^{qr}\Big(\frac{(p+q)!}{p!q!}\mathrm{Alt}\omega_1\Big) \wedge \Big(\frac{(r+s)!}{r!s!}\mathrm{Alt}\omega_2\Big).$$

In particular, if $\mathrm{Alt}(\omega_1) = 0$ for some double form ω_1, then $\mathrm{Alt}(\omega_1\omega_2) = 0$ for any double form ω_2.

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Proof. Without loss of generality, we may assume that $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{(p,q)}(V^*)$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{(r,s)}(V^*)$. Then

$$\begin{aligned}\mathrm{Alt}(\omega_1\omega_2) &= \mathrm{Alt}(\theta_1 \wedge \theta_3 \otimes \theta_2 \wedge \theta_4)\\ &= \frac{(p+r)!(q+s)!}{(p+r+q+s)!}\theta_1 \wedge \theta_3 \wedge \theta_2 \wedge \theta_4\\ &= \frac{(p+r)!(q+s)!}{(p+r+q+s)!}(-1)^{qr}\theta_1 \wedge \theta_2 \wedge \theta_3 \wedge \theta_4\\ &= (-1)^{qr}\frac{(p+r)!(q+s)!}{(p+r+q+s)!}\frac{(p+q)!(r+s)!}{p!q!r!s!}\mathrm{Alt}\omega_1 \wedge \mathrm{Alt}\omega_2. \qquad \square\end{aligned}$$

We shall say that a double form ω satisfies the first Bianchi identity if $\mathfrak{S}\omega = 0$. In the next proposition, we use the Bianchi first sum to reformulate the classical Plücker relations, also called the Grassmann quadratic p-relations.

Proposition 3.7. *A p-form $\alpha \in \Lambda V^*$ is decomposable if and only if the double form $\alpha \otimes \alpha \in \mathcal{D}^{p,p}(V^*)$ satisfies the first Bianchi identity.*

Recall that a p-form α is said to be decomposable if there exist p 1-forms α_i such that $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$.

Proof. Let $(e_1, \ldots, e_n)$ be an orthonormal basis of V. We look at double forms as multilinear forms on V and we use an alternative form of the first Bianchi sum as follows:

$$\begin{aligned}&\mathfrak{S}(\alpha \otimes \alpha)(e_{i_1}, \ldots, e_{i_{p+1}}; e_{j_1}, \ldots, e_{j_{p-1}})\\ &\quad= \sum_{k=1}^{p+1}(-1)^k(\alpha \otimes \alpha)(e_{i_1}, \ldots, \widehat{e_{i_k}}, \ldots, e_{i_{p+1}}; e_{i_k}, e_{j_1}, \ldots, e_{j_{p-1}})\\ &\quad= \sum_{k=1}^{p+1}(-1)^k\alpha(e_{i_1}, \ldots, \widehat{e_{i_k}}, \ldots, e_{i_{p+1}})\alpha(e_{i_k}, e_{j_1}, \ldots, e_{j_{p-1}}). \qquad \square\end{aligned}$$

The following lemma will be useful in proving some results of the next section.

Lemma 3.8. (1) *Let $r, p \geq 1$, $q \geq 0$ and $\omega_1 \in \mathcal{D}^{p,q}(V^*)$, $\omega_2 \in \mathcal{D}^{r-1,p-1}(V^*)$ be two double forms; then*

$$\mathrm{Alt}(\omega_1 \circ g\omega_2) = (-1)^{r-1}\frac{q+1}{r}\mathrm{Alt}(\widetilde{\mathfrak{S}}(\omega_1) \circ \omega_2).$$

In particular, if ω_1^t satisfies the first Bianchi identity, then $\mathrm{Alt}(\omega_1 \circ g\omega_2) = 0$.

(2) *Let $q, s \geq 1$, $p \geq 0$ and $\omega_1 \in \mathcal{D}^{p,q}(V^*)$, $\omega_2 \in \mathcal{D}^{q-1,s-1}(V^*)$ be two double forms; then*

$$\mathrm{Alt}(g\omega_2 \circ \omega_1) = (-1)^p\frac{p+1}{s}\mathrm{Alt}(\omega_2 \circ \mathfrak{S}(\omega_1)).$$

In particular, if ω_1 satisfies the first Bianchi identity, then $\mathrm{Alt}(g\omega_2 \circ \omega_1) = 0$.

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Proof. Let $\omega_1 = \theta_1 \otimes \theta_2 \in \mathcal{D}^{p,q}(V^*)$ and $\omega_2 = \theta_3 \otimes \theta_4 \in \mathcal{D}^{r-1,p-1}(V^*)$.

On one hand, using the definition of the composition product of double forms,

$$\omega_1 \circ g\omega_2 = (\theta_1 \otimes \theta_2) \circ \sum_{i=1}^{n} e_i^* \wedge \theta_3 \otimes e_i^* \wedge \theta_4 = \sum_{i=1}^{n} \langle \theta_1, e_i^* \wedge \theta_4 \rangle e_i^* \wedge \theta_3 \otimes \theta_2.$$

On the other hand, recall that the first adjoint Bianchi sum is given by

$$\widetilde{\mathfrak{S}}(\omega_1) = \sum_{i=1}^{n} i_{e_i}\theta_1 \otimes e_i^* \wedge \theta_2;$$

therefore,

$$\widetilde{\mathfrak{S}}(\omega_1) \circ \omega_2 = \sum_{i=1}^{n} \langle \theta_1, e_i^* \wedge \theta_4 \rangle \theta_3 \otimes e_i^* \wedge \theta_2.$$

To complete the proof, it is enough to check that

$$\mathrm{Alt}(\theta_3 \otimes e_i^* \wedge \theta_2) = (-1)^{r-1} \frac{q+1}{r} \mathrm{Alt}(e_i^* \wedge \theta_3 \otimes \theta_2).$$

The proof of the second statement is identical. □

4. Pontrjagin forms and scalars

4.1. Pontrjagin forms.

Definition 4.1. Let R be a $(2,2)$ double form on the Euclidean vector space (V, g). The kth *Pontrjagin form* of R, denoted $P_k(R)$, is the $4k$-form defined by

$$P_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \mathrm{Alt}(R^k \circ R^k) \in \Lambda^{4k}(V^*). \tag{4.1}$$

Remark 4.2. The previous definition is motivated by Riemannian geometry. If (M, g) is a Riemannian manifold of dimension n and k is a positive integer such that $4k \leq n$, let R be the Riemann curvature tensor seen as a $(2,2)$ double form. In Stehney [15], it is shown that the differential form $P_k(R)$ is a closed differential form of degree $4k$ that represents the kth Pontrjagin class of M. Let us note that the previous result of Stehney is a reformulation of a result originally due to Chern [4].

Proposition 4.3. *Let W denote the Weyl part of the $(2,2)$ double form R; then*

$$P_k(R) = P_k(W) = \frac{1}{(k!)^2(2\pi)^{2k}} \mathrm{Alt}(W^k \circ W^k).$$

In other words, the Pontrjagin forms depend only on the Weyl part of R.

This result was first proved by Avez [1], Bivens [2], Greub [6] and Branson and Gover [3].

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Proof. Recall that we have the decomposition $R = W + gh$, where h is the Schouten tensor. Then $R^k = W^k + gH$ for some double form H and therefore

$$R^k \circ R^k = (W^k + gH) \circ (W^k + gH) = W^k \circ W^k + W^k \circ gH + gH \circ W^k + gH \circ gH.$$

Since W^k and gH are both symmetric double forms and both satisfy the first Bianchi identity, Lemma 3.8 shows that

$$\mathrm{Alt}(W^k \circ gH) = \mathrm{Alt}(gH \circ W^k) = \mathrm{Alt}(gH \circ gH) = 0.$$

The proposition then follows immediately. □

As a direct consequence of the previous proposition, all the Pontrjagin classes of a conformally flat manifold vanish. In the next theorem, we are going to generalize this to more general q-conformally flat manifolds.

First, recall that for $n \geq 4k - 1$, the kth Gauss–Kronecker curvature R^k of a Riemannian n-manifold (M, g) admits a unique decomposition [10, 14]

$$R^k = \operatorname{con} R^k + gH,$$

where $\operatorname{con} R^k$ is a trace-free $(2k, 2k)$ double form and H is a $(2k-1, 2k-1)$ double form. This is somehow like a Euclidean division by the metric g. Note that for $k = 1$, $\operatorname{con} R$ is just the Weyl tensor of the Riemann curvature tensor R.

For $n \geq 4k$, following [10, 14], we say that a Riemannian n-manifold (M, g) is k-conformally flat if $\operatorname{con} R^k = 0$ or, equivalently, R^k is divisible by the metric g, that is, $R^k = gH$ for some $(2k-1, 2k-1)$ double form H.

We recover the usual conformally flat manifolds for $k = 1$. Using Lemma 3.8, one can easily prove the following result.

Theorem 4.4. *Let (M, g) be a k-conformally flat Riemannian manifold of dimension $n \geq 4k$. Then the Pontrjagin classes P_i of M vanish for $i \geq k$.*

Proof. Let $R^k = gH$ for some $(2k-1, 2k-1)$ double form H. Then, using Lemma 3.8,

$$\mathrm{Alt}(R^k \circ R^k) = \mathrm{Alt}(gH \circ R^k) = 0.$$

This completes the proof. □

Remark 4.5. Let us remark here that if $R^k = gH$ for some $(2k-1, 2k-1)$ double form H, then H automatically satisfies the first Bianchi identity, for Proposition 3.1 shows that

$$0 = \mathfrak{S}(R^k) = \mathfrak{S}(gH) = g\mathfrak{S}(H).$$

Recall that if $\mathfrak{S}(H)$ is a $(2k, 2k-2)$ double form, then, by [10, Proposition 2.5], we have $\mathfrak{S}(H) = 0$, as $2k + 2k - 2$ is less than n.

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4.2. Pontrjagin–Chern scalars of double forms.

4.2.1. The volume double form. Let (V, g) be a Euclidean vector space of even dimension $n = 4k$. Fix a volume form ω_g on V, that is, choose $\omega_g \in \Lambda^n(V^*)$ with norm 1. In other words, we choose an orientation on V. Using the inclusion $\Lambda^{4k}(V^*) \subset \mathcal{D}^{(2k,2k)}(V^*)$, one can look at ω_g as a $(2k, 2k)$ double form. The so-obtained double form, which we continue to denote by ω_g, shall be called the *volume double form of* (V, g).

It is easy to check that the operator $\Lambda^{2k}(V^*) \to \Lambda^{2k}(V^*)$ that is canonically associated to ω_g is nothing but the Hodge star operator $*$. In particular, for any two $2k$-vectors α, β, one has

$$\omega_g(\alpha, \beta) = \langle *\alpha, \beta\rangle = \langle \alpha, *\beta\rangle = *(\alpha \wedge \beta).$$

We list below some useful properties of this volume double form.

PROPOSITION 4.6. (1) *In the composition algebra of double forms $\mathcal{D}(V^*)$, the volume double form ω_g is a square root of unity, that is,*

$$\omega_g \circ \omega_g = \frac{g^{2k}}{(2k)!}.$$

(2) *For any two $(2k, 2k)$ double forms ω_1 and ω_2,*

$$\langle \omega_g \circ \omega_1, \omega_g \circ \omega_2\rangle = \langle \omega_1, \omega_2\rangle = \langle \omega_1 \circ \omega_g, \omega_2 \circ \omega_g\rangle.$$

(3) *If $*$ denotes the double Hodge star operating on double forms and ψ is an arbitrary $(2k, 2k)$ double form, then*

$$(*\psi) \circ \omega_g = \omega_g \circ \psi \quad \textit{and} \quad \psi \circ \omega_g = \omega_g \circ (*\psi).$$

In particular,

$$*\psi = \omega_g \circ \psi \circ \omega_g.$$

PROOF. Let (e_i) be a positive orthonormal basis of V; then it is easy to show that

$$\omega_g = \sum_I e_I^* \otimes *e_I^*,$$

where the index I runs over all multi-indices $i_1, \ldots, i_{2k}$ such that $1 \le i_1 < \cdots < i_{2k} \le n$ and the star power denotes duals as usual. Consequently,

$$\omega_g \circ \omega_g = \sum_{I,J} \langle e_I^*, *e_J^*\rangle e_J^* \otimes *e_I^* = \sum_I e_I^* \otimes e_I^* = \frac{g^{2k}}{(2k)!}.$$

This completes the proof of (1).

Next, note that

$$(\omega_g \circ \omega_1)^t \circ (\omega_g \circ \omega_2) = \omega_1^t \circ \omega_g \circ \omega_g \circ \omega_2 = \omega_1^t \circ \omega_2,$$

where we used the fact that ω_g is a symmetric double form. After taking full contractions of both sides, one gets the relation in (2).

To prove (3), without loss of generality let $\omega = e_I^* \otimes e_J^*$, where I and J are multi-indices as above. Then $*\omega = *e_I^* \otimes *e_J^*$ and

$$(*\omega) \circ \omega_g = \sum_K \langle *e_I^*, *e_K^*\rangle e_K^* \otimes *e_J^* = e_I^* \otimes *e_J^*.$$

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On the other hand,

$$\omega_g \circ \omega = \sum_K \langle e_K^*, e_J^* \rangle e_I^* \otimes *e_K^* = e_I^* \otimes *e_J^*.$$

This completes the proof of the proposition. □

The proof of the following properties is straightforward.

PROPOSITION 4.7. *If ω_g is the $(2k, 2k)$ volume double form, then*

$$*\omega_g = \omega_g, \quad c\omega_g = 0, \quad g\omega_g = 0 \quad \text{and} \quad \langle \omega_g, \omega_g \rangle = 1.$$

4.2.2. Pontrjagin–Chern scalars of double forms.

DEFINITION 4.8. Let R be a $(2, 2)$ double form on an oriented Euclidean vector space (V, g) of dimension $n = 4k$. The kth *Pontrjagin–Chern scalar* of R, denoted $p_k(R)$, is the real number defined by

$$p_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \langle R^k \circ R^k, \omega_g \rangle,$$

where ω_g is the volume double form of V.

It is easy to show that

$$p_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \langle \mathrm{Alt}(R^k \circ R^k), \omega_g \rangle = \langle P_k(R), \omega_g \rangle,$$

where the previous inner product is the one of $4k$-forms, ω_g is the volume form of (V, g) and $P_k(R)$ is the kth Pontrjagin form of R. In particular, since $\dim \Lambda^{4k} V^* = 1$,

$$P_k(R) = \langle P_k(R), \omega_g \rangle \omega_g = p_k(R) \omega_g.$$

The following proposition provides an alternative definition of the previous scalars.

PROPOSITION 4.9. *The kth* Pontrjagin–Chern scalar *$p_k(R)$ is given by a full contraction as follows:*

$$p_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}(2k)!} c^{2k}(R^k \circ R^k \circ \omega_g).$$

PROOF. It results from Proposition 4.6 that

$$\begin{aligned} p_k(R) &= \frac{1}{(k!)^2(2\pi)^{2k}} \langle R^k \circ R^k, \omega_g \rangle \\ &= \frac{1}{(k!)^2(2\pi)^{2k}} \langle R^k \circ R^k \circ \omega_g, \omega_g \circ \omega_g \rangle \\ &= \frac{1}{(k!)^2(2\pi)^{2k}} \left\langle R^k \circ R^k \circ \omega_g, \frac{g^{2k}}{(2k)!} \right\rangle \\ &= \frac{1}{(k!)^2(2\pi)^{2k}(2k)!} c^{2k}(R^k \circ R^k \circ \omega_g), \end{aligned}$$

where in the last step we used the fact that the contraction map c is the adjoint of the multiplication map by g; see [11]. □

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REMARK 4.10. We remark that if (e_i) is an orthonormal basis of (V, g) and R^k is seen as a bilinear form on Λ^{2k}, then

$$\mathrm{c}^{2k}(R^k \circ R^k \circ \omega_g) = \sum_{i_1,\dots,i_{2k}=1}^{n} R^k \circ R^k(e_{i_1} \wedge \cdots \wedge e_{i_{2k}}, *(e_{i_1} \wedge \cdots \wedge e_{i_{2k}})).$$

For a Riemannian oriented manifold of dimension $n = 4k$, let R be its Riemann curvature tensor seen as a $(2, 2)$ double form. The integral over M of the function $p_k(R)$ is a *topological invariant* called the kth Pontryagin number of M and is denoted by $p_k(M)$.

PROPOSITION 4.11. *Let R be a $(2, 2)$ double form on an oriented Euclidean vector space (V, g) of dimension $n = 4k$. Then*

$$|p_k(R)| \le \frac{1}{(k!)^2(2\pi)^{2k}} \|R^k\|^2.$$

Furthermore, if equality occurs, then the contraction $\mathrm{c}R^k = 0$.

PROOF. As the double form R^q is symmetric,

$$p_k(R) = \frac{1}{(k!)^2(2\pi)^{2k}} \langle R^k \circ R^k, \omega_g\rangle = \frac{1}{(k!)^2(2\pi)^{2k}} \langle R^k, R^k \circ \omega_g\rangle.$$

The Cauchy–Schwartz inequality shows that

$$(k!)^2(2\pi)^{2k}|p_k(R)| = |\langle R^k, R^k \circ \omega_g\rangle| \le \|R^k\|\,\|R^k \circ \omega_g\|.$$

Next, we apply Proposition 4.6 to show that $\|R^k \circ \omega_g\| = \|R^k\|$, as follows:

$$\|R^k \circ \omega_g\|^2 = \langle R^k \circ \omega_g, R^k \circ \omega_g\rangle = \langle R^k, R^k\rangle.$$

This completes the proof of the first part of the proposition. To prove the second part, first we use Lemma 4.12 below to show that

$$c(R^k \circ \omega_g) = \mathfrak{S}R^k \circ \omega_g = 0.$$

Finally, if equality holds, then R^k must be proportional to $R^k \circ \omega_g$ and therefore $cR^k = c(R^k \circ \omega_g) = 0$. □

LEMMA 4.12. *Let $n = 2p$ and ω be a (p, p) double form. Then*

$$\mathrm{c}(\omega \circ \omega_g) = \mathfrak{S}\omega \circ \omega_g,$$

where the last ω_g is the volume double form seen as a $(p - 1, p + 1)$ double form.

PROOF. Let $(e_1, \dots, e_n)$ be an orthonormal basis of V. By linearity, we may assume that $\omega = e_I^* \otimes e_J^*$, where I, J are two multi-indices of length p. Then

$$\omega \circ \omega_g = e_I^* \otimes *e_J^*.$$

Therefore,

$$\mathrm{c}(\omega \circ \omega_g) = \sum_{i=1}^{n} i_{e_i} e_I^* \otimes i_{e_i}(*e_J^*).$$

A straightforward computation shows that $i_{e_i}(*e_J^*) = *(e_i^* e_J^*)$; therefore,

$$\mathrm{c}(\omega \circ \omega_g) = \sum_{i=1}^{n} i_{e_i} e_I^* \otimes *(e_i^* e_J^*) = \sum_{i=1}^{n} (i_{e_i} e_I^* \otimes e_i^* e_J^*) \circ \omega_g = \mathfrak{S}\omega \circ \omega_g. \qquad \square$$

Remark 4.13. If W denotes the Weyl part of R, then in a similar way one can prove that

$$|p_k(R)| \leq \frac{1}{(k!)^2(2\pi)^{2k}} \|W^k\|^2.$$

5. Thorpe manifolds

An oriented compact Riemannian manifold (M, g) of dimension $n = 4k$ is said to be a *Thorpe manifold* if $*R^k = R^k$, where R is the Riemann curvature tensor seen as a $(2, 2)$ double form, R^{2k} its exterior power and $*$ the double Hodge star operator acting on double forms.

For $k = 1$, we recover four-dimensional Einstein manifolds. For $n > 4$, Thorpe manifolds are in general not Einstein and *vice versa*; see for instance [8].

Recall that the Gauss–Bonnet theorem asserts that

$$\int_M h_{4k} \omega_g = (2\pi)^{2k} (2k)! \chi(M),$$

where h_{4k} is the $4k$-Gauss–Bonnet curvature of (M, g), which is given by (see for instance [11])

$$h_{4k} = *R^{2k} = *(R^k R^k) = \langle *R^k, R^k \rangle.$$

In particular, for a Thorpe manifold,

$$h_{4k} = \|R^k\|^2 \geq 0.$$

Therefore, the Euler–Poincaré characteristic of a compact Thorpe manifold of dimension $n = 4k$ is always nonnegative, and it is zero if and only if the manifold is q-flat. This remark can be refined as follows.

Theorem 5.1. *Let (M, g) be a compact orientable $4k$-dimensional Thorpe manifold; then*

$$\chi(M) \geq \frac{k!k!}{(2k)!} |p_k(M)|,$$

where $p_k(M)$ is the kth *Pontrjagin number of M. Furthermore, if equality holds, then M is k-Ricci flat, that is,* $\mathrm{c}R^q = 0$.

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Proof. Using Proposition 4.11, we get

$$\begin{aligned}|p_k(M)| &\leq \int_M |p_k(R)|\omega_g \\ &\leq \frac{1}{(k!)^2(2\pi)^{2k}}\int_M \|R^k\|^2\omega_g \\ &= \frac{1}{(k!)^2(2\pi)^{2k}}\int_M h_{4k}\omega_g \\ &= \frac{(2k)!}{k!k!}\chi(M).\end{aligned}$$

Furthermore, if equality holds, then $|p_k(R)| = (1/(k!)^2(2\pi)^{2k})\|R^k\|^2$. Again, by Proposition 4.11, the metric must then be k-Ricci flat. This completes the proof of the theorem. □

We note that the first part of the theorem was first proved by Thorpe [16]. As a consequence of the second part of the theorem, we get the following main result of [9].

Corollary 5.2. *A compact orientable manifold M of dimension eight that is at the same time Einstein and Thorpe and satisfies $6\chi(M) = |p_2(M)|$ must be flat.*

Proof. On one hand, Theorem 5.1, once applied to the case $k = 2$, shows that $cR^2 = 0$. Consequently, the second Gauss–Bonnet curvature satisfies $h_4 = \frac{1}{4!}c^4R^2 = \frac{1}{4!}c^3(cR^2) = 0$.

On the other hand, an Einstein manifold with identically zero h_4 must be flat [11]. This completes the proof of the corollary. □

Next, we are going to generalize the above corollary to higher dimensions. First, we start with a definition.

Definition 5.3 [12]. Let $0 < 2q < n$. We shall say that a Riemannian n-manifold is *hyper* $(2q)$-*Einstein* if the first contraction of the tensor R^q is proportional to the metric g^{2q-1}, that is,

$$cR^q = \lambda g^{2q-1}.$$

We recover the usual Einstein metrics for $q = 1$. The following theorem generalizes a well-known result about four-dimensional Einstein manifolds.

Theorem 5.4 [11, 12]. *Let $k \geq 1$ and (M, g) be a hyper $(2k)$-Einstein manifold of dimension $n \geq 4k$. Then the Gauss–Bonnet curvature h_{4k} of (M, g) is nonnegative. Furthermore, $h_{4k} \equiv 0$ if and only if (M, g) is k-flat.*

Recall that k-flat means that the sectional curvature of R^k is identically zero.

Now we are ready to state and prove a generalization of Corollary 5.2.

THEOREM 5.5. *A compact orientable manifold M of dimension $8k$ that is at the same time hyper $2k$-Einstein and Thorpe and satisfies $((4k)!/(2k)!(2k)!)\chi(M) = |p_{2k}(M)|$ must be k-flat.*

We recover Corollary 5.2 for $k = 1$.

PROOF. Theorem 5.1 shows that $\mathrm{c}R^{2k} = 0$. Consequently, the $(4k)$th Gauss–Bonnet curvature satisfies $h_{4k} = (1/(4k)!)\mathrm{c}^{4k}R^{2k} = (1/(4k)!)\mathrm{c}^{4k-1}(\mathrm{c}R^{2k}) = 0$. Then Theorem 5.4 shows that the manifold must be k-flat. □

6. Final remarks

6.1. Mixed Pontrjagin numbers. Let (M, g) be a compact $4k$-dimensional Riemannian oriented manifold with Riemann curvature tensor R, seen as a $(2, 2)$ double form, and $k_1, k_2, \ldots, k_m$ a collection of natural numbers such that $k_1 + 2k_2 + \cdots + mk_m = k$.

The Pontrjagin number $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ of M is defined by the integral over M of the following $4k$-form:

$$P_1^{k_1} P_2^{k_2} \cdots P_m^{k_m} := \underbrace{(P_1 \wedge \cdots \wedge P_1)}_{k_1 \text{ times}} \wedge \underbrace{(P_2 \wedge \cdots \wedge P_2)}_{k_2 \text{ times}} \wedge \cdots \wedge \underbrace{(P_m \wedge \cdots \wedge P_m)}_{k_m \text{ times}},$$

where, for each i, $P_i = P_i(R)$ denotes the ith Pontrjagin form of R as defined in Section 4.1.

The above Pontrjagin numbers are important topological invariants for a manifold, they are oriented cobordism invariant and, together with Stiefel–Whitney numbers, they determine an oriented manifold's oriented cobordism class. Furthermore, the signature and the $\hat{A}$ genus can be expressed explicitly through linear combinations of the above Pontrjagin numbers.

We are now going to prove a Stehney-type formula for all these numbers.

THEOREM 6.1. *With the above notation,*

$$P_1^{k_1} P_2^{k_2} \cdots P_m^{k_m} = \frac{(4k)!}{[(2k)!]^2(2\pi)^{2k}}\Big(\prod_{i=1}^{m} \frac{[(2i)!]^2}{(i!)^{2k_i}(4i)!}\Big)\mathrm{Alt}[(R \circ R)^{k_1}(R^2 \circ R^2)^{k_2} \cdots (R^m \circ R^m)^{k_m}],$$

where all the powers over double forms are taken with respect to the exterior product of double forms. In particular, the Pontrjagin numbers are given by the integral

$$p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m} = \frac{(4k)!}{[(2k)!]^2(2\pi)^{2k}}\Big(\prod_{i=1}^{m} \frac{[(2i)!]^2}{(i!)^{2k_i}(4i)!}\Big) \times \int_M \frac{\mathrm{c}^{2k}}{(2k)!}[((R \circ R)^{k_1}(R^2 \circ R^2)^{k_2} \cdots (R^m \circ R^m)^{k_m}) \circ \omega_g]\omega_g,$$

where c *is the contraction map of double forms and ω_g is the volume form as above.*

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PROOF. Let ω_i, for $i = 1, 2, \ldots, k$, be a collection of arbitrary $(2p_i, 2p_i)$ double forms. Then successive applications of Proposition 3.6 show that

$$\frac{(\sum_{i=1}^k 4p_i)!}{[(\sum_{i=1}^k 2p_i)!]^2}\mathrm{Alt}(\omega_1\omega_2\cdots\omega_k) = \Big(\prod_{i=1}^k \frac{(4p_i)!}{[(2p_i)!]^2}\Big)\mathrm{Alt}(\omega_1)\wedge \mathrm{Alt}(\omega_2)\wedge\cdots\wedge\mathrm{Alt}(\omega_k).$$

In particular, if ω is an arbitrary $(2p, 2p)$ double form,

$$\frac{(4pk)!}{[(2pk)!]^2}\mathrm{Alt}(\omega^k) = \Big[\frac{(4p)!}{[(2p)!]^2}\Big]^k \underbrace{\mathrm{Alt}(\omega)\wedge\cdots\wedge\mathrm{Alt}(\omega)}_{k\text{ times}}.$$

Using the previous two formulas, one can directly and without difficulties complete the proof of the theorem. The second part can be easily proved by imitating the proof of Proposition 4.9. □

6.2. Stehney's formula for Pontrjagin forms. For the sake of completeness, we include here the derivation of Stehney's formula (4.1) from Chern's theorem [4].

Let (M, g) be a Riemannian n-manifold with Riemann curvature tensor R, $m \in M$ and (e_i) an orthonormal basis of the tangent space at m. Chern's theorem [4] shows that at m,

$$P_k(R) = \frac{[(2k)!]^2}{(4k)!(2\pi)^{2k}(2^k k!)^2}\sum_I \Omega_I \wedge \Omega_I,$$

where the sum runs over all multi-indices $I = (i_1, \ldots, i_{2k})$ such that $1 \le i_1 < \cdots < i_{2k} \le n$, and

$$\Omega_I(v_1, \ldots, v_{2k}) = 2^k R^k(v_1 \wedge \cdots \wedge v_{2k}; e_I).$$

We used the notation $e_I = e_{i_1} \wedge \cdots \wedge e_{i_{2k}}$. We emphasize that our convention for the wedge product is as defined by formula (3.2). Next, let J, K in the sums below run over all multi-indices of length $2k$, as was the case for I in the previous sum. Then

$$\begin{aligned}
\frac{1}{2^{2k}}\sum_I \Omega_I\wedge\Omega_I &= \frac{1}{2^{2k}}\sum_{I,J,K}\Omega_I(e_J)e_J^*\wedge\Omega_I(e_K)e_K^*\\
&= \sum_{I,J,K} R^k(e_J, e_I)R^k(e_K, e_I)e_J^*\wedge e_K^*\\
&= \sum_{J,K} R^k\circ R^k(e_J, e_K)e_J^*\wedge e_K^*\\
&= \frac{(4k)!}{(2k)!(2k)!}\mathrm{Alt}\Big(\sum_{J,K} R^k\circ R^k(e_J, e_K)e_J^*\otimes e_K^*\Big)\\
&= \frac{(4k)!}{(2k)!(2k)!}\mathrm{Alt}(R^k\circ R^k).
\end{aligned}$$

Finally,

$$P_k(R) = \frac{[(2k)!]^2}{(4k)!(2\pi)^{2k}(2^k k!)^2}\sum_I \Omega_I\wedge\Omega_I = \frac{2^{2k}\mathrm{Alt}(R^k\circ R^k)}{(2\pi)^{2k}(2^k k!)^2} = \frac{1}{(2\pi)^{2k}(k!)^2}\mathrm{Alt}(R^k\circ R^k).$$

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References

[1] A. Avez, 'Characteristic classes and Weyl tensor: applications to general relativity', *Proc. Natl. Acad. Sci. USA* **66**(2) (1970), 265–268.
[2] I. Bivens, 'Curvature operators and characteristic classes', *Trans. Amer. Math. Soc.* **269**(1) (1982), 301–310.
[3] T. Branson and A. R. Gover, 'Pontrjagin forms and invariant objects related to the Q-curvature', *Commun. Contemp. Math.* **9**(335) (2007), 335–358.
[4] S. S. Chern, 'On the curvature and characteristic classes of a Riemannian manifold', *Abh. Math. Semin. Univ. Hambg.* **20** (1956), 117–126.
[5] W. H. Greub, *Multilinear Algebra,* 2nd edn (Springer, New York, 1978).
[6] W. H. Greub, 'Pontrjagin classes and Weyl tensors', *C. R. Math. Rep. Acad. Sci. Can. III* **3** (1981), 177–183.
[7] N. Hitchin, 'On compact four-dimensional Einstein manifolds', *J. Differential Geom.* **9** (1974), 435–441.
[8] J. M. Kim, 'Einstein–Thorpe manifolds', PhD Thesis, SUNY at Stony Brook, 1998.
[9] J. M. Kim, '8-dimensional Einstein–Thorpe manifolds', *J. Aust. Math. Soc.* A **68** (2000), 278–284.
[10] R. S. Kulkarni, 'On the Bianchi identities', *Math. Ann.* **199** (1972), 175–204.
[11] M. L. Labbi, 'Double forms, curvature structures and the (p, q)-curvatures', *Trans. Amer. Math. Soc.* **357**(10) (2005), 3971–3992.
[12] M. L. Labbi, 'On generalized Einstein metrics', *Balkan J. Geom. Appl.* **15**(2) (2010), 61–69.
[13] M. L. Labbi, 'On some algebraic identities and the exterior product of double forms', *Arch. Math.* **49**(4) (2013), 241–271.
[14] T. Nasu, 'On conformal invariants of higher order', *Hiroshima Math. J.* **5** (1975), 43–60.
[15] A. Stehney, 'Courbure d'ordre p et les classes de Pontrjagin', *J. Differential Geom.* **8** (1973), 125–134.
[16] J. A. Thorpe, 'Some remarks on the Gauss–Bonnet integral', *J. Math. Mech.* **18**(8) (1969), 779–786.

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