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Smoothness of Quotients Associated With a Pair of Commuting Involutions

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Abstract. Let σ, θ be commuting involutions of the connected semisimple algebraic group G where σ, θ and G are defined over an algebraically closed field $\underline{k}$, char $\underline{k} = 0$. Let $H := G^\sigma$ and $K := G^\theta$ be the fixed point groups. We have an action $(H \times K) \times G \to G$, where $((h,k),g) \mapsto hgk^{-1}$, $h \in H$, $k \in K$, $g \in G$. Let $G/\!\!/(H \times K)$ denote the categorical quotient Spec $\mathcal{O}(G)^{H \times K}$. We determine when this quotient is smooth. Our results are a generalization of those of Steinberg [Ste75], Pittie [Pit72] and Richardson [Ric82] in the symmetric case where $\sigma = \theta$ and $H = K$.

1 Introduction

Let σ, θ be commuting involutions of the connected reductive algebraic group G where σ, θ and G are defined over an algebraically closed field $\underline{k}$, char $\underline{k} = 0$. Let $H := G^\sigma$ and $K := G^\theta$ be the fixed point groups. We have an action $(H \times K) \times G \to G$, where $((h,k),g) \mapsto hgk^{-1}$, $h \in H$, $k \in K$, $g \in G$. Let $G/\!\!/(H \times K)$ denote the categorical quotient Spec $\mathcal{O}(G)^{H \times K}$.

We want to determine when this quotient is smooth (resp., an affine space). Example 2 below shows that this is only a reasonable task when G is semisimple. If, in addition, G is simply connected, we have

Theorem 1 (See Corollary 2) *Suppose that G is semisimple and simply connected. Then $G/\!\!/(H \times K)$ is smooth.*

It is useful to first divide by the action of K. Let $\beta\colon G \to G$, $g \mapsto g\theta(g)^{-1}$. Then β induces an isomorphism $G/K \xrightarrow{\sim} P := \beta(G)$, $gK \mapsto \beta(g)$ [Ric82, 2.4]. The left action of G on G/K becomes the twisted action $g * x := gx\theta(g)^{-1}$, $g \in G$, $x \in P$. In particular, the $*$-action is conjugation when restricted to K. Instead of studying the quotient mapping $\pi\colon G \to G/\!\!/(H \times K)$ we study the quotient mapping $\pi_P\colon P \to P/\!\!/H$ where H acts via $*$.

Theorem 2 *Let G be semisimple, and let σ, H, etc. be as above. Then the following are equivalent*

(1) *$P/\!\!/H$ is smooth.*
(2) *$\mathcal{O}(P)^H$ is a polynomial algebra* (i.e., *$P/\!\!/H$ is an affine space).*
(3) *$\mathcal{O}(P)$ is a free $\mathcal{O}(P)^H$-module.*

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We establish Theorem 1 in §3 using slice theorem techniques from [HS01] and an argument along the lines of [Ric82, 14.3]. To establish Theorem 2 we have to use the fact that the quotient $P/\!\!/H$ can also be obtained in terms of a torus A divided by a "twisted" Weyl group $W_H^*(A)$ (see §2). We are able to reduce to the case that our twisted Weyl group $W_H^*(A)$ is of the form $W_H(A) \ltimes F_0$ where $W_H(A)$ is the usual Weyl group of A and F_0 is a subgroup of the elements of order two in A. In §4 we introduce our Main Theorem 7 which is a version of Theorem 2 in terms of A and $W_H^*(A)$. In §5 we obtain Theorem 2 as a consequence of our Main Theorem 7. We also reduce the proof of Theorem 7 to that of our Main Lemma 3 which we establish in §6. In §7 we consider the possible F_0 that can occur when G is adjoint.

2 Quotients of Tori

Let G, P, *etc.* be as in §1. A torus S in G is *θ-split* if $\theta(s) = s^{-1}$ for all $s \in S$ and S is *(σ, θ)-split* if it is θ-split and σ-split. Let A be a maximal (σ, θ)-split torus of G. From the twisted action $*$ we obtain a *twisted* Weyl group $W_H^*(A)$. Set $N_H^*(A) = \{h \in H \mid h * A = A\}$, $Z_H^*(A) = \{h \in H \mid h * a = a \text{ for every } a \in A\}$ and $W_H^*(A) = N_H^*(A)/Z_H^*(A)$.

Theorem 3 ([HS01, Theorem 6.5]) *Let H act on P by $*$. Then the inclusion $A \to P$ induces an isomorphism $A/W_H^*(A) \xrightarrow{\sim} P/\!\!/H$. In particular:*

(1) *The closed H-orbits are exactly those which intersect A.*
(2) *If $a \in A$, then $(H * a) \cap A = W_H^*(A) * a$.*

2.1 We now take a closer look at $W_H^*(A)$ and its action on A. If $h \in N_H^*(A)$, then $h * e = \beta(h) \in A$, hence $\sigma(\beta(h)) = \beta(h)^{-1}$. But σ fixes h, hence fixes $\beta(h)$. Thus $\beta(h) \in A^{(2)}$, the elements of A of order 2. If $a \in A$, then $h * a = hah^{-1}h\theta(h)^{-1} = hah^{-1}\beta(h)$, so that $N_H^*(A) = \{h \in N_H(A) \mid \beta(h) \in A^{(2)}\}$ and $Z_H^*(A) = \{h \in Z_H(A) \mid \beta(h) = e\}$. From the inclusions $N_H^*(A) \subset N_H(A)$ and $Z_H^*(A) \subset Z_H(A)$ we obtain a group homomorphism $\phi\colon W_H^*(A) \to W_H(A)$, and the mapping $h \mapsto \beta(h)$ induces a mapping (which we also call β) from $W_H^*(A)$ to $A^{(2)}$. The homomorphism ϕ has kernel $W_0 \simeq (Z_H(A) \cap N_H^*(A))/Z_H^*(A)$, and when restricted to W_0, β induces an isomorphism $W_0 \simeq F_0 := \beta(W_0)$. The subgroup F_0 is $W_H(A)$-stable.

Since $W_H(A)$ acts on A, we have a semidirect product $W_H(A) \ltimes A$, with multiplication $(w, a) * (w', a') = (ww', aw(a'))$, w, $w' \in W_H(A)$, a, $a' \in A$. We identify A with $\{(e, a) \mid a \in A\}$. The action of $W_H^*(A)$ on A factors through the injective homomorphism

$$\rho = (\phi, \beta)\colon W_H^*(A) \to W_H(A) \ltimes A^{(2)}.$$

2.2 Straightening the Action of $W_H^*(A)$

The group $W_H^*(A)$ (or rather, its embedding in $W_H(A) \ltimes A^{(2)}$) can be quite complicated. Fortunately, we can straighten things out, using quadratic elements. We say that $q \in A$ is *quadratic* if $q^2 \in Z(G)$. Let $Q(A)$ denote the set of quadratic elements

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in A. Given $q \in Q(A)$, let α_q denote the automorphism of A which is multiplication by q. If $(w, a) \in W_H(A) \ltimes A^{(2)}$ and $b \in A$, then

$$\alpha_q(w, a) * \alpha_q^{-1} b = qaw(q^{-1}b) = qw(q^{-1})aw(b) = (w, qw(q^{-1})a) * b.$$

Since $q^2 \in Z(G)$, $(qw(q^{-1}))^2 = q^2 w(q^{-2}) = e$, so that conjugation by α_q does indeed induce an automorphism, denoted η_q, of $W_H(A) \ltimes A^{(2)}$. Moreover, η_q acts as the identity on F_0.

Theorem 4 ([HS01, Theorem 9.3 and 9.13]) *There is a $q \in Q(A)$ such that*

$$\eta_q(\rho(W_H^*(A))) = W_H(A) \ltimes F_0.$$

Using Theorem 4 and our calculation above we may always reduce to the case that $\rho(W_H^*(A)) = W_H(A) \ltimes F_0$.

2.3 Determining F_0

Let $T \subset G$ be a torus. If T is invariant under an involution α, then we use T_+^α to denote $(T^\alpha)^0$ and T_-^α to denote the (unique) maximal α-split subtorus of T. Then $T = T_+^\alpha T_-^\alpha$.

If T is stable under our commuting involutions σ and θ, then we define $T_{++}^{\sigma,\theta}$ to be $(T_+^\sigma \cap T_+^\theta)^0$, and similarly for $T_{--}^{\sigma,\theta}$, $T_{+-}^{\sigma,\theta}$ and $T_{-+}^{\sigma,\theta}$. From [Hel88, 5.13] we know that there are (σ, θ)-stable maximal tori T of G such that

(1) $A = T_{--}^{\sigma,\theta}$.
(2) $AT_{+-}^{\sigma,\theta}$ is a maximal θ-split torus.
(3) $AT_{-+}^{\sigma,\theta}$ is a maximal σ-split torus.

We call such maximal tori *standard*. Now set $\tau := \sigma\theta$. We then have

Theorem 5 (See [HS01, Theorem 8.12]) *Let T be a standard maximal torus of G. Then $F_0 = T_-^\tau \cap A$.*

We use Theorem 5 to construct examples.

Example 1 One can easily find cases where $F_0 = A^{(2)}$: Let G_1 be a reductive group with maximal torus S and Weyl group W. Let α be an involution of G_1 such that S is α-split ([Hel88, 4.11]). Set $G := G_1 \times G_1$. For $(x, y) \in G$, set $\theta(x, y) = (\alpha(x), \alpha(y))$ and set $\sigma(x, y) = (y, x)$. Then σ and θ commute, and $A = \{(s, s^{-1}) \colon s \in S\}$ is a maximally (σ, θ)-split torus. Moreover, $S \times S$ is a standard torus T of G with $T_-^\tau = \{(s, s) \colon s \in S\}$. It follows that $F_0 = A \cap \{(s, s) \colon s \in S\} = A^{(2)}$.

Example 2 Here we show the complications that can occur if the group G is not semisimple. Let $G_1 = \mathrm{SL}_{2n}$, $n \geq 2$. Set $G_0 = G_1 \times G_1$, and let $\sigma = \theta$ send (x, y) to (y, x), $(x,y) \in G_1 \times G_1$. Then A can be taken to be $\{(s, s^{-1}) \mid s \in S\}$ where S is the standard diagonal maximal torus of SL_{2n}. The Weyl group $W_H^*(A)$ is just

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$W \simeq S_{2n}$ acting by $w(s, s^{-1}) = (w(s), w(s^{-1}))$, $s \in S$, $w \in W$, and the quotient A/W is smooth. Now set $G := (G_1 \times G_1 \times \underline{k}^*)$. Extend the actions of σ and θ to G such that $\theta(x, y, \lambda) = (y, x, \lambda)$ and $\sigma(x, y, \lambda) = (y, x, \lambda^{-1})$ for $(x, y, \lambda) \in G$. Then a maximal (σ, θ)-split torus A is $\{(s, s^{-1}, 1) \mid s \in S\}$. The torus T^{τ}_{-} is $\{(1, 1, \lambda) \mid \lambda \in \underline{k}^*\}$. Finally, divide G by the subgroup Z generated by $(z, z, -1) \in G_1 \times G_1 \times \underline{k}^*$ where z is minus the identity. In G/Z the intersection of A and T^{τ}_{-} has order 2 generated by the image of $(z, z, 1) \in G$. Thus the quotient space we get is $S/(S_{2n} \times \{e, z\})$, which is not smooth, as z does not act as a reflection on $S/S_{2n} \simeq \underline{k}^{2n-1}$. Thus adding a torus can change a smooth quotient to a nonsmooth one.

From now on we assume that G is semisimple.

3 G Simply Connected

We need to use properties of slice representations. Let A, $W^*_H(A)$, etc. be as above. Let $a \in A$, and let $\operatorname{Int}(a)$ denote conjugation by a. Set $\tilde\tau := \sigma\theta \operatorname{Int}(a)$, and let $\tilde G$ denote $G^{\tilde\tau}$. Then σ is an involution of $\tilde G$ and we define $\tilde H$ to be $\tilde G^{\sigma}$. One computes that $\tilde H = \{h \in H \mid h * a = a\} = H_a$. Then we have

Theorem 6 ([HS01, Theorem 5.11]) *Let $a \in A$, etc. be as above. Then there is an étale slice for the action of H, and this étale slice is isomorphic to an $\tilde H$-stable neighborhood of $e\tilde H$ in $\tilde G/\tilde H$ with the canonical action of $\tilde H$.*

Corollary 1 *Suppose that G is simply connected. Then the quotient $P/\!\!/H \simeq A/W^*_H(A)$ is smooth.*

Proof (Compare [Ric82, 14.3]) We continue with the notation of Theorem 6. Since $\tilde G$ is connected (as G is simply connected [Ste68]), it is known that the quotient of $\tilde G/\tilde H$ by $\tilde H$ is smooth near $e\tilde H$. In fact, up to étale morphisms, it is the quotient of the Lie algebra of a maximal σ-split torus of $\tilde G$ by the corresponding Weyl group (which is generated by reflections). Thus $P/\!\!/H$ is smooth near the image of $a \in A$ (for any a), and thus $P/\!\!/H$ is smooth. ■

3.1 The rest of this section is due to the referee. We continue to assume that G is simply connected.

Consider G as a $G \times G$-module via left and right multiplication. Then $\mathcal{O}(G) \simeq \bigoplus_\omega V(\omega) \otimes V(\omega)^*$ as $G \times G$-module, where ω runs over a system Ω of dominant integral weights and $V(\omega)$ denotes the irreducible module with highest weight $\omega \in \Omega$. By restriction we have the action of H and K on G and $\mathcal{O}(G)$ as in §1. Since H and K are spherical, for any simple module $V(\omega)$, $\omega \in \Omega$, the subspace of H (or K) fixed vectors is at most one-dimensional, and $V(\omega)$ has a fixed line if and only if $V(\omega)^*$ does. Set $\Omega_0 := \{\omega \in \Omega \mid \dim V(\omega)^H \neq 0 \neq V(\omega)^K\}$. Then $\mathcal{O}(G)^{H\times K}$ has a basis consisting of nonzero elements $\varphi_\omega \in V(\omega)^H \otimes (V(\omega)^*)^K$, $\omega \in \Omega_0$. By Theorem 2 and

Corollary 1, $\mathcal{O}(G)^{H\times K}$ is a polynomial algebra. This raises two natural questions:

Questions 1

(1) Is Ω_0 a free monoid on generators $\omega_1, \dots, \omega_l$, $l = \dim A$?
(2) If so, are the φ_{ω_i} generators for $\mathcal{O}(G)^{H\times K}$?

We are only able to answer these questions affirmatively when we are in the symmetric case ($\sigma = \theta$), which we assume until the end of this section.

3.2 Let A be a maximal θ-split torus in G and $T = T_0 A$ a θ-stable maximal torus of G, where θ is the identity on T_0. We have the character groups $X^*(T)$ and $X^*(A)$ and the root systems $\Phi(T)$ and $\Phi(A)$. We choose a system of positive roots $\Phi(T)_+$ which induces a system of positive roots $\Phi(A)_+$ for the (restricted) root system $\Phi(A)$. Since G is simply connected, the dominant integral weights Ω (resp., $\Omega(A)$) relative to $\Phi(T)_+$ (resp., $\Phi(A)_+$) correspond to characters of T (resp., A). As before, let Ω_0 denote the set of highest weights ω such that $V(\omega)^K \neq 0$. From [Vus74] or [Helg84, Ch. V, 4.2] we have:

Lemma 1 *Ω_0 consists of the highest weights ω which are trivial on T_0 and whose restrictions to A are in $2X^*(A)$. Conversely, any highest weight in $2X^*(A)$ extends to an element of Ω_0.*

Corollary 2 *Let $\omega_1', \dots, \omega_l'$ denote generators of the monoid of dominant integral weights relative to $\Phi(A)_+$ and let $\omega_1, \dots, \omega_l$ denote the extensions of $2\omega_1', \dots, 2\omega_l'$ to elements of Ω_0. Then $\omega_1, \dots, \omega_l$ freely generate Ω_0.*

3.3 Note that

$$\mathcal{O}(G/K) \simeq \mathcal{O}(G)^K \simeq \bigoplus_{\omega\in\Omega_0} V(\omega) \simeq (V(\omega) \otimes (V(\omega^*)^K),$$

and that for every $\omega \in \Omega_0$ we have our generator $\varphi_\omega \in V(\omega)^K$. Recall that for weights $w, w' \in \Omega$, we write $\omega \leq \omega'$ if the difference $\omega' - \omega$ is a sum of positive roots.

Lemma 2 *Let $\omega, \omega' \in \Omega_0$ and write $\varphi_\omega\varphi_{\omega'}$ as a sum $\sum_{\omega''\in\Omega_0} c(\omega'')\varphi_{\omega''}$. Then for every $c(\omega'') \neq 0$, $\omega'' \leq \omega + \omega'$, and $c(\omega + \omega') \neq 0$.*

Proof The product of $V(\omega)$ and $V(\omega')$ in $\mathcal{O}(G/K)$ is an image of the tensor product $V(\omega)\otimes V(\omega')$, hence has only factors $V(\omega'')$ where $w'' \leq \omega+\omega'$. In [Rui89, Theorem 3.2] it is shown that $c(\omega'') \neq 0$ if and only if $V(\omega'')$ actually occurs in the product $V(\omega)V(\omega')$. But clearly the product of the highest weight vectors in $V(\omega)$ and $V(\omega')$ generates a copy of $V(\omega + \omega')$. Thus $c(\omega + \omega') \neq 0$. ■

Proposition 1 *Let $\omega_1, \dots, \omega_l$ be the generators of Ω_0 of Corollary 2. Then $\mathcal{O}(G)^{K\times K}$ is the polynomial algebra on $\varphi_{\omega_1}, \dots, \varphi_{\omega_l}$.*

Proof Let $\omega \in \Omega_0$ and write $\omega = \sum_{i=1}^{l} n_i\omega_i$, $n_i \in \mathbb{N}$. By Lemma 2, $\prod_{i=1}^{l} \varphi_{\omega_i}^{n_i}$ is a sum $c\varphi_\omega + \sum d(\omega')\varphi_{\omega'}$ where $c \neq 0$ and $\omega' < \omega$ whenever $d(\omega') \neq 0$. By induction, $\sum d(\omega')\varphi_{\omega'}$ is in the span of monomials in the φ_{ω_i}, hence so is φ_ω. Thus we have $\mathbb{C}[\varphi_{\omega_1}, \dots, \varphi_{\omega_l}]$ mapping onto $\mathcal{O}(G)^{K\times K}$, a polynomial algebra in l-variables. It follows that the φ_{ω_i} are algebraically independent generators of $\mathcal{O}(G)^{K\times K}$. ■

4 The Main Theorem

Let A, $W_H^*(A)$, etc. be as in §2. Let V denote $X^*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. Then the roots of A in $\mathfrak{g}$ form a (not necessarily reduced) root system Φ in V, and $X^*(A)$ is contained in the associated weight lattice Λ [Hel88, Lemma 6.9]. The Weyl group $W_H(A)$ acts on Φ, $X^*(A)$ and Λ. We may split V into a direct sum of vector spaces V_j, $j = 1, \dots, r$, with irreducible root systems Φ_j, weight lattices Λ_j and Weyl groups W_j so that $\Phi = \cup_j \Phi_j$, $\Lambda = \oplus_j \Lambda_j$ and $W_H(A) = \prod_j W_j$. Let X_j denote $X^*(A) \cap V_j \supseteq \Phi_j$. Then $X^*(A)$ contains $\oplus_j X_j$, but it could be larger. By 2.2, Theorem 4, we may assume that $W_H^*(A) = W_H(A) \ltimes F_0$.

For each $j = 1, \dots, r$, let A_j be a torus with character group Λ_j. Let Z_j (the "center") be the kernel of all the roots in Φ_j considered as characters on A_j. If Φ_j is non-reduced, then $Z_j = \{e\}$. For reduced root systems Φ_j, the center Z_j corresponds to the center of the simply connected algebraic group corresponding to Φ_j. We have $A \simeq (\prod_j A_j)/Z'$ where $Z' \subset \prod_j Z_j$. Let $F' \subset \prod_j A_j$ denote the inverse image of $F_0 \subset A$. Then $Z' \subset F'$ and $A/W_H^*(A) \simeq (\prod_j A_j)/(W_H(A) \ltimes F')$. Set $F_j := F' \cap A_j$ and let W_j^* denote $W_j \ltimes F_j$. Then, in analogy to the symmetric case [Ric82], we have:

Theorem 7 **(Main Theorem)** *Let A and the A_j,* etc. *be as above. The following are equivalent:*

(1) *$A/W_H^*(A)$ is smooth.*
(2) *$\mathcal{O}(A)$ is a free $\mathcal{O}(A)^{W_H^*(A)}$-module.*
(3) *$A/W_H^*(A)$ is an affine space.*
(4) *Each A_j/W_j^* is smooth and $F' = \prod_j F_j$.*

Example 3 Suppose that we have $\Phi = \Phi_1 \cup \Phi_2$ where both Φ_1 and Φ_2 are reduced root systems of rank 1. Then $A_1 \simeq A_2 \simeq \underline{k}^*$ with Weyl group action $z \mapsto z^{-1}$, $z \in \underline{k}^*$. Suppose that $A = (A_1 \times A_2)/\pm I$ where $I = (1,1) \in A_1 \times A_2$. Then by Theorem 7(4), the quotient A/W is not smooth as we have $F_1 = F_2 = \{e\}$ and $F' = Z' \simeq \mathbb{Z}/2\mathbb{Z}$. One sees the nonsmoothness directly as follows. Let λ_1 and λ_2 be the usual coordinate functions on our two copies of $\underline{k}^*$. Then $\mathcal{O}(A_1 \times A_2)^W \simeq \mathbb{C}[\lambda_1 + \lambda_1^{-1}, \lambda_2 + \lambda_2^{-1}]$ and $\pm I$ acts by sending $\lambda_1 + \lambda_1^{-1}$ and $\lambda_2 + \lambda_2^{-1}$ to their negatives. Hence the quotient of A/W is not smooth.

The situation above arises when we consider $G_1 = \mathrm{SO}_4 \simeq (\mathrm{SL}_2 \times \mathrm{SL}_2)/(\mathbb{Z}/2\mathbb{Z})$ as a symmetric space, *i.e.*, one has $G = G_1 \times G_1$ and the involution θ sends (g_1, g_2) to (g_2, g_1), $g_1, g_2 \in G_1$. Then $K = G^\theta \simeq G_1$ acts on $P \simeq G_1$ by conjugation, and one gets the A and W considered above.

5 The Main Lemma

Let A_j, *etc.* be as in §4, $j = 1, \dots, r$. Let p_j be the projection of $\prod_{k=1}^r A_k$ to A_j, $j = 1, \dots, r$. To prove Theorem 7 we need the following

Lemma 3 **(Main Lemma)** *Let A_j,* etc. *be as above. Then*

(1) *Each $E_j := A_j/W_j^*$ is smooth if and only if it is an affine space.*
(2) *If E_j is an affine space, then it has coordinates such that $p_j(F')$ acts as linear transformations.*

We will establish Lemma 3 by examining the possibilities for the A_j, *etc.* case by case. The details are not hard, and the case of the Weyl group of type F_4 has a nice twist. In any case, we now use Lemma 3 to establish Theorem 7 and Theorem 2.

Proof of Theorem 7 Clearly (4) and Lemma 3 imply (3). If $A/W_H^*(A)$ is smooth, then the slice representations of $W_H^*(A)$ at all points of A must be linear representations of finite groups generated by reflections. But for such groups, the polynomials on the representation space are a free module over the invariants. This tells us that $\mathcal{O}(A)$ is locally a free $\mathcal{O}(A)^{W_H^*(A)}$-module over $A/W_H^*(A)$. If $A/W_H^*(A)$ is actually an affine space, then we get global freeness by the solution to the Serre Problem. Thus (3) implies (2). From commutative algebra [Mat80, Theorem 51], one sees that (2) implies (1). Thus we need only show that (1) implies (4).

Suppose that A_j/W_j^* is not smooth for some j, say $j = r$, the largest index. Note that, by construction, $F'' := F'/\prod_j F_j$ acts faithfully on $Q_r := \prod_{i<r} A_j/W_j^*$. Take a general point $x \in Q_r$ whose isotropy group is trivial for the action of F''. Let $y \in A_r/W_r^*$ be a non smooth point. Then the étale slice at (x, y) is a neighborhood of (x, y) in $Q_r \times A_r/W_r^*$. Clearly the slice is not smooth at (x, y), hence $A/W_H^*(A)$ is not smooth, and we have established necessity of the smoothness of each A_j/W_j^*.

It remains to show that if $F'' \neq \{e\}$ and each $E_j := A_j/W_j^*$ is smooth, then $A/W_H^*(A)$ is not smooth. By Lemma 3, each E_j is an affine space, and the action of F'' on each E_j is linear (and diagonalizable). Let $\alpha \in F''$ be nontrivial. Then, by construction of F'', α must act nontrivially on at least two of the spaces E_j. This shows that α is not a reflection, so that F'' contains no nontrivial reflections, hence the quotient $\prod E_j/F''$ is not smooth. ∎

Proof of Theorem 2 The equivalence of (1) and (2) follows from Theorem 7, and (3) implies (1) by [Mat80, Theorem 51]. Richardson (see [Ric81, Thm. B and Prop. 2.6] and [Ric82, §12]) gives criteria for (3) to hold, *i.e.,* for $\mathcal{O}(P)$ to be a free $\mathcal{O}(P)^H$-module. They are that

(a) $\mathcal{O}(P)^H$ is a polynomial algebra.
(b) All fibers of $\pi_P \colon P \to P/\!\!/H$ have the same dimension.
(c) There is a dense open subset $U \subset P$ consisting of H-orbits which are closed in P, and for every $x, y \in U$, the isotropy groups H_x and H_y are conjugate.

Now (b) above is true for symmetric varieties [Ric82, 9.11]. It then follows from the slice theorem 6 that (b) holds for π_P. From [HS01, 6.4] there is an open and dense subset A_{pr} of A such that $H * A_{\mathrm{pr}}$ is open in P and such that each $a \in A_{\mathrm{pr}}$ has isotropy group $Z^*_H(A)$. The H-orbits of points of A are closed by Theorem 3, hence we have (c). It follows that (a), *i.e.*, (2), implies (3). ■

6 Proof of The Main Lemma

We now have to establish Lemma 3. This involves some elementary combinatorics with root systems and Weyl groups. Let A_j, Z_j, p_j, *etc.* be as in §4 and §5, for $1 \leq j \leq r$. We will need to show that F_j and $p_j(F')$ are about the same size. To do this we use the following fact:

Remark 1 Let $f \in F'$ with $p_k(f) = f_k$, $k = 1, \dots, r$. Let $w \in W_j$ for some j, $1 \leq j \leq r$. Then $f_0 := f^{-1}w(f) = f_j^{-1}w(f_j)$ has trivial projection to A_k for $k \neq j$, hence $f_0 \in F_j$.

To lighten the notation, we will drop the index j and consider a torus A with root system Σ, Weyl group W, center Z, *etc.* We have a group F' and a homomorphism $p\colon F' \to A$ such that $F \subset p(F')$ are W-stable subgroups and such that $p(F')$ projects to a subgroup of $(A/Z)^{(2)}$. Remark 1 applies with F_j replaced by F. Let W^* denote $W \ltimes F$ and let E denote the quotient A/W^*. We say that the action of a group H on E is *linearizable* if E is an affine space which has coordinates such that the action of H is linear.

6.1 Type SL_n, $n \geq 2$

Let A denote the maximal torus of SL_n. Then A has character group generated by $\epsilon_1, \dots, \epsilon_n$ where (using additive notation) $\sum_j \epsilon_j = 0$. If $t = \mathrm{diag}(t_1, \dots, t_n) \in A$, then $\epsilon_j(t) = t_j$, $j = 1, \dots, n$. The roots are $\{\epsilon_j - \epsilon_k : j \neq k\}$ and the Weyl group is S_n, the symmetric group on n-letters, acting as usual. The center Z consists of the scalar matrices, *i.e.*, $\{f \in A : \epsilon_j(f) = \xi \text{ for all } j\}$ where $\xi^n = 1$.

Lemma 4 *Suppose that $p(F')$ contains an element not in $Z \simeq \mathbb{Z}/n\mathbb{Z}$. Then $A^{(2)} \subset F$.*

Proof Note that $p(F')$ is inside the pull back to A of $(A/Z)^{(2)}$ which consists of elements $f \in A$ such that $\epsilon_j(f) = \pm\xi$ for all j, where $\xi^{2n} = 1$. If $f \in p(F')$ is not in Z, then we must have that $\epsilon_k(f) = \xi$ and $\epsilon_l(f) = -\xi$ for some k and l, where $\xi^{2n} = 1$. Let $w \in S_n$ be the involution switching ϵ_k and ϵ_l. Then $f_0 := f^{-1}w(f)$ satisfies $\epsilon_k(f_0) = \epsilon_l(f_0) = -1$ and $\epsilon_j(f_0) = 1$ for $j \neq k, l$. Now $f_0 \in F$ by Remark 1, and f_0 and its images under the action of W generate $A^{(2)}$. ■

Corollary 3 *Let A,* etc. *be as above. Then the quotient E is smooth if and only if*

(1) $F = \{e\}$,
(2) $F = A^{(2)}$ *or*
(3) $n = 2$.

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Moreover, in each case where E is smooth, the $p(F')$-action on E is linearizable.

Proof First suppose that $p(F') \subset Z$. Then from [Ste75] we know that E is smooth if and only if $F = \{e\}$ or $n = 2$ and $F = Z = A^{(2)}$. In either case, the action of Z on E is linearizable, hence so is that of $p(F')$. If $p(F') \not\subset Z$, then we have that $A^{(2)} \subset F$. Now $A/A^{(2)}$ is essentially the same as A: the character group is generated by the weights $\pm 2\epsilon_j$ with the induced action of W. We replace F by $F/A^{(2)}$ and $p(F')$ by $p(F')/A^{(2)}$ and reduce to the case where $p(F') \subset Z$. Then we can repeat our argument above. ■

6.2 Type Sp_{2n}, $n \geq 2$

Let A be the standard n-torus, and let $\epsilon_1, \dots \epsilon_n$ be the standard characters on A. The root system of type C_n consists of the characters $\pm 2\epsilon_j$ and $\pm\epsilon_j \pm \epsilon_k$, $j < k$. The Weyl group W is generated by the reflections $\epsilon_j \mapsto -\epsilon_j$ and the symmetric group S_n permuting the ϵ_j. The torus A is the maximal torus of $\mathrm{Sp}(2n)$, and the character group (weight lattice) is generated by the ϵ_j. The center Z is of order 2 and is generated by the element z defined by the conditions $\epsilon_j(z) = -1$ for all j. Let $\frac{1}{2}A^{(2)}$ denote $\{f \in A^{(2)} \mid \epsilon_j(f) = -1$ for an even number of $j\}$. Note that $p(F')$ lies in the pullback $\tilde{A}^{(2)}$ of $(A/Z)^{(2)}$ to A, and $\tilde{A}^{(2)}$ consists of elements $f \in A$ with either $\epsilon_j(f) = \pm 1$ for all j or $\epsilon_j(f) = \pm i$ for all j.

Lemma 5 *Let A,* etc. *be as above, and suppose that $f \in p(F')$ with $\epsilon_j(f) = 1$ for some j and $\epsilon_k(f) = -1$ for some k. Then $F \supset \frac{1}{2}A^{(2)}$. If $f \in p(F')$ such that $\epsilon_j(f) = \pm i$ for some j, then $F \supset A^{(2)}$.*

Proof In the first case, let w denote the permutation of ϵ_j and ϵ_k. Then $f^{-1}w(f) \in F$, and $\epsilon_m(f) = 1$ for $m \neq j, k$ and $\epsilon_j(f) = \epsilon_k(f) = -1$. Clearly, then, F contains $\frac{1}{2}A^{(2)}$. In the second case with $\epsilon_j(f) = \pm i$ for some j, consider $f^{-1}w(f)$ where w is the reflection sending ϵ_j to $-\epsilon_j$. Then $\epsilon_k(f^{-1}w(f)) = 1$ for $k \neq j$, and $\epsilon_j(f^{-1}w(f)) = -1$. It follows that $F \supset A^{(2)}$. ■

Corollary 4 *Let A,* etc. *be as above. Then there are the following possibilities:*

(1) $\{e\} \subset F \subset p(F') \subset \{e, z\}$.
(2) $F = \frac{1}{2}A^{(2)} \subset p(F') \subset A^{(2)}$.
(3) $A^{(2)} \subset F \subset p(F') \subset \tilde{A}^{(2)}$.

Corollary 5 *Let A,* etc. *be as above. Then E is smooth if and only if*

(1) $F = \{e\}$,
(2) $F = \frac{1}{2}A^{(2)}$,
(3) $F = A^{(2)}$ *or*
(4) $n = 2$.

Moreover, whenever E is smooth, $p(F')$ acts linearizably on E.

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Proof We consider the cases in Corollary 4. In case (1) it is classical [Ste75] that E is smooth if $F = \{e\}$ and nonsmooth if $F = \{e, z\}$ and $n \geq 3$. Moreover, $p(F')$ acts linearizably on E when $F = \{e\}$. In case (2) the quotient of A by $\frac{1}{2}A^{(2)}$ has character group generated by the weights $\pm 2\epsilon_i$ and $\pm\epsilon_1 \pm \cdots \pm \epsilon_n$. This is the weight lattice for the maximal torus of $\mathrm{Spin}(2n+1)$, so we can apply our results in Corollary 6 below to $F/\frac{1}{2}A^{(2)}$ and $p(F')/\frac{1}{2}A^{(2)}$. They show that we have a linearizable action of $p(F')$ on E in case $F = \frac{1}{2}A^{(2)}$ or $F = A^{(2)}$.

In case (3), we may divide everything by $A^{(2)}$ as in the proof of Corollary 3 to reduce to case (1). Finally, when $n = 2$, we have $\{e, z\} = \frac{1}{2}A^{(2)}$, so that we have smoothness of the quotient E (and a linearizable action of $p(F')$ on E) in all of the cases (1)–(3). ∎

6.3 Type BC_n

Suppose that we have a root system of type BC_n. Then we can consider that we have the maximal torus A of Sp_{2n}, as above, where the root system has the characters $\pm 2\epsilon_j$ and $\pm\epsilon_j \pm \epsilon_k$, $j < k$, along with the characters $\pm\epsilon_j$. The Weyl group is that of Sp_{2n} and the center Z is trivial. Thus we have that $F \subset p(F') \subset A^{(2)}$. Let z be defined by $\epsilon_j(z) = -1$ for all j. Applying the arguments of Lemma 5, Corollary 4 and Corollary 5, we obtain

Proposition 2 *There are the following possibilities:*

(1) $\{e\} \subset F \subset p(F') \subset \{e, z\}$.
(2) $F = \frac{1}{2}A^{(2)} \subset p(F') \subset A^{(2)}$.
(3) $F = p(F') = A^{(2)}$.

Proposition 3 *Let A,* etc. *be as above. Then E is smooth if and only if*

(1) $F = \{e\}$,
(2) $F = \frac{1}{2}A^{(2)}$,
(3) $F = A^{(2)}$ *or*
(4) $n = 1$ *or* 2.

In all smooth cases, $p(F')$ acts linearizably on E.

6.4 Type Spin_{2n}, $n \geq 4$ and Spin_{2n+1}, $n \geq 2$

Let $\hat{A}$ be the double cover of the standard n-torus A. Then $\hat{A}$ has character group generated by the ϵ_j, $j = 1, \ldots, n$ and $\chi := \frac{1}{2}\sum_j \epsilon_j$. Let z_0 denote the element defined by: $\epsilon_j(z_0) = 1$ for all j and $\chi(z_0) = -1$. If n is even, let z_1 be defined by $\epsilon_j(z_1) = -1$ for all j, and $\chi(z_1) = 1$. If n is odd, define z_4 by $\epsilon_j(z_4) = -1$ for all j and $\chi(z_4) = i$. Then in the case of Spin_{2n}, the center Z is $\{e, z_4, z_4^2, z_4^3\}$ if n is odd, and is $\{e, z_0, z_1, z_0z_1\}$ if n is even. For Spin_{2n+1}, the center is always $\{e, z_0\}$. The Weyl group for the Spin_{2n+1} case is $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, where S_n permutes the roots as usual, and the generators of $(\mathbb{Z}/2\mathbb{Z})^n$ send ϵ_j to $-\epsilon_j$, $j = 1, \ldots, n$. The Weyl group for Spin_{2n} is isomorphic to $S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^{n-1}$ where S_n acts as usual and the generators of

$(\mathbb{Z}/2\mathbb{Z})^{n-1}$ send pairs of weights ϵ_j and ϵ_k to $-\epsilon_j$ and $-\epsilon_k$. The roots for Spin_{2n+1} are the $\pm\epsilon_j \pm \epsilon_k$, $j < k$ and the $\pm\epsilon_j$, while the roots of Spin_{2n} are the $\pm\epsilon_j \pm \epsilon_k$, $j < k$.

Lemma 6 *Let $\hat{A}$,* etc. *be as above. Then there are the following possibilities:*

(1) $\hat{A}^{(2)} \subset F$.
(2) $\{e\} \subset F \subset p(F') \subset Z$.
(3) *(n even,* Spin_{2n+1}*):* $\{e, z_0\} \subset F \subset p(F') \subset \{e, z_0, z_1, z_0z_1\}$.
(4) *(n odd,* Spin_{2n+1}*):* $\{e, z_0\} \subset F \subset p(F') \subset \{e, z_4, z_4^2, z_4^3\}$.

Proof Suppose that there is an $f \in p(F')$ with $\epsilon_j(f) = \pm i$ for some j. Then, from the description of the centers, we must be in the case of Spin_{2n} and every ϵ_k must be $\pm i$, $k = 1, \dots, n$. Let $w \in W$ send ϵ_1 and ϵ_2 to their negatives and leave all other ϵ_k fixed. Then $f_0 := f^{-1}w(f) \in F$, and $\epsilon_1(f_0) = \epsilon_2(f_0) = -1$ while $\epsilon_k(f_0) = 1$ for $k > 2$.

Now we suppose that there is an $f \in p(F')$ with $\epsilon_j(f) = 1$ and $\epsilon_k(f) = -1$ for some j and k. Let w be the generator of W which sends ϵ_j and ϵ_k to their negatives and leaves all other ϵ_m fixed. Note that $w(\epsilon_m)(f) = \epsilon_m(f)$ for all $m = 1, \dots, n$. Let λ be a weight of the form $\frac{1}{2}(\pm\epsilon_1 \pm \cdots \pm \epsilon_n)$. Then $w(\lambda) = \lambda \pm \epsilon_j \pm \epsilon_k$ for some choice of pluses and minuses. It follows that $w(\lambda)(f) = -\lambda(f)$, so that $w(f) = fz_0$ and $f^{-1}w(f) = z_0 \in F$. Let $w' \in W$ interchange j and k. Then $f_1 := f^{-1}w'(f))$ satisfies $\chi(f_1) = 1$, $\epsilon_m(f_1) = 1$ for $m \neq j, k$, and $\epsilon_j(f_1) = \epsilon_k(f_1) = -1$. Finally, z_0 and all the W translates of f_1 generate $\hat{A}^{(2)}$. Thus we are in case (1).

It remains to consider the cases where $p(F')$ consists of elements f with $\epsilon_j(f) = 1$ for all j or $\epsilon_j = -1$ for all j. In the case of type Spin_{2n}, one easily sees that we are always in case (2) (where Z has order 4). We need only now consider the case of Spin_{2n+1}, and to get (3) and (4) we only have to show that if $p(F')$ is of order 4, then $F \supset \{e, z_0\}$. In case (3), suppose that $z_1 \in p(F')$. Let $w \in W$ fix all ϵ_j except that it sends ϵ_1 to $-\epsilon_1$. Then one shows as above that $z_1^{-1}w(z_1)$ equals z_0. Similarly, if $z_4 \in p(F')$ (n odd), then $z_4^{-1}w(z_4) = z_0$. ■

Corollary 6 *Let $\hat{A}$, etc. be as above. Then E is smooth in precisely the following cases:*

(1) $F = \{e\}$ *or* $F = \hat{A}^{(2)}$.
(2) *F is of order 2 or is an order 2 extension of $\hat{A}^{(2)}$ and W is of type* Spin_{2n+1}.
(3) *W is of type* Spin_5.

Moreover, if E is smooth, then $p(F')$ acts linearizably on E.

Proof In the case of type Spin_{2n} there is nothing to show, since everything is classical (or becomes classical upon division by $\hat{A}^{(2)}$). In the case of type Spin_{2n+1} (assuming that we have already divided by $\hat{A}^{(2)}$ if need be) we end up in one of cases (2), (3) or (4) of Lemma 6. Case (2) is classical. In cases (3) and (4) we may consider the quotient of $\hat{A}$ by the action of z_0. The weights on the quotient of $\hat{A}$ by $\{e, z_0\}$ are generated by the ϵ_i. In other words, we are in the case of the maximal torus of Sp_{2n}, and we can apply Corollary 5. ■

6.5 Type G_2

Proposition 4 *Let A be a maximal torus of G_2. Then $p(F') = \{e\}$ or $F = p(F') = A^{(2)}$. In particular, the quotient E is an affine space with trivial action of $p(F')$.*

Proof Since Z is trivial in this case, F and $p(F')$ are subgroups of $A^{(2)}$. It follows from Lemma 4 (applied to the action of the copy of $S_3 \subset W(G_2)$) that if $p(F')$ is not trivial, then $F = A^{(2)}$. Thus we can always reduce to the classical case $F = \{e\}$ where $E \simeq \underline{k}^2$. ■

6.6 Type E_8

There is a homomorphism $\mathrm{SL}_9 \to E_8$ such that the adjoint representation of E_8 (which is faithful) decomposes as $\wedge^3(\underline{k}^9) \oplus \wedge^6(\underline{k}^9)$ plus the adjoint representation of SL_9 (see [Dyn52, Table 25]).

Proposition 5 *Let A be the maximal torus of E_8. Then $p(F') = \{e\}$ or $F = p(F') = A^{(2)}$. In particular, the quotient E is an affine space with trivial action of $p(F')$.*

Proof Let $\tilde{F}$ (resp., $\tilde{F}'$) be the inverse image of F (resp., $p(F')$) in the maximal torus $\tilde{A}$ of SL_9. If $\tilde{f} \in \tilde{F}'$ maps to $f \in p(F')$ and $\tilde{\omega} \in W(\mathrm{SL}_9)$ has image $\omega \in W(A)$, then $\tilde{f}^{-1}\tilde{\omega}(\tilde{f})$ has image $f^{-1}\omega(f) \in F$, so that $\tilde{f}^{-1}\tilde{\omega}(\tilde{f})$ lies in $\tilde{F}$. So we can calculate in $\tilde{A}$.

Since the kernel $\tilde{Z}$ of $\tilde{A} \to A$ has order 3, $\tilde{F}$ (resp., $\tilde{F}'$) is a product of a subgroup $\tilde{F}_0$ (resp., $\tilde{F}_0'$) of $\tilde{A}^{(2)}$ and $\tilde{Z}$. Lemma 4 shows that $\tilde{F}_0' = \{e\}$ or $\tilde{F}_0 = \tilde{A}^{(2)}$. Hence $p(F') = \{e\}$ or $F = p(F') = A^{(2)}$. Dividing by $A^{(2)}$ if necessary we arrive at the classical case $F = \{e\}$ where $E \simeq \underline{k}^8$. ■

6.7 Type E_6

Let A denote a maximal torus of E_6. From [Dyn52, Table 25] there is a homomorphism $(\mathrm{SL}_3)^3 \to E_6$ such that the fundamental (27-dimensional) representation V of E_6 restricts to the representation

$$V_1 \otimes V_2^* \oplus V_1^* \otimes V_3 \oplus V_2 \otimes V_3^*,$$

where V_j is the fundamental three-dimensional representation of the j-th copy of SL_3. Thus we have an injection $(\mathrm{SL}_3)^3/(\mathbb{Z}/3\mathbb{Z}) \to E_6$. There is also an injection $(\mathrm{SL}_6 \times \mathrm{SL}_2)/(\mathbb{Z}/2\mathbb{Z}) \to E_6$, where V decomposes as $\wedge^2(\underline{k}^6) \oplus (\underline{k}^6)^* \otimes \underline{k}^2$. Recall that the center Z of E_6 is cyclic of order 3.

Lemma 7 *Let A be as above, and let $\hat{F}$ be a W-stable subgroup of $A^{(2)}$. Then $\hat{F} = \{e\}$ or $\hat{F} = A^{(2)}$.*

Proof Since we have an injection of the maximal torus of $(\mathrm{SL}_3)^3/(\mathbb{Z}/3\mathbb{Z})$ into A, the order of $\hat{F}$ is 1, 4, 16 or 64. Since we have an injection of the maximal torus of $(\mathrm{SL}_6 \times \mathrm{SL}_2)/(\mathbb{Z}/2\mathbb{Z}) \to A$ where $(\mathbb{Z}/2\mathbb{Z})$ sits diagonally in $\mathrm{SL}_6 \times \mathrm{SL}_2$, $\hat{F}$ cannot have order 4 or 16. Thus the order of $\hat{F}$ is 1 or 64. ■

Lemma 8 *Let* A, etc. *be as above. Then* $F \subset p(F') \subset Z$, *or* $A^{(2)} \subset F$.

Proof Since Z has order 3, $p(F')$ splits as a direct sum of a subgroup of Z and a W-stable 2-group F''. By Lemma 7, if $F'' \neq \{e\}$, then $F'' = A^{(2)}$, and then clearly F also has to contain $A^{(2)}$. ∎

From [Ste75] we get

Corollary 7 *The quotient* E *is smooth if and only if* $F = \{e\}$ *or* $F = A^{(2)}$. *In either of these cases,* $p(F') = F$ *or* $p(F')$ *is an extension of* F *by the center* Z, *and* $p(F')$ *acts linearizably on* E.

6.8 Type E_7

Let A and W be the maximal torus and Weyl group of E_7. From [Dyn52, Table 25] there is a homomorphism $\mathrm{SL}_8 \to E_7$ such that the fundamental (56-dimensional) representation of E_7 restricts to $\wedge^2(\underline{k}^8) \oplus \wedge^6(\underline{k}^8)$. The adjoint representation of E_7 restricts to the representation $\wedge^4(\underline{k}^8)$ plus the adjoint representation of SL_8. There is also a homomorphism $\mathrm{SL}_6 \times \mathrm{SL}_3 \to E_7$ such that the fundamental representation of E_7 restricts to $\underline{k}^6 \otimes \underline{k}^3 \oplus \wedge^3(\underline{k}^6) \oplus (\underline{k}^6)^* \otimes (\underline{k}^3)^*$. Let A_{j-1} denote the standard maximal torus of SL_j for $j = 3, 6, 8$. Then the homomorphism $A_5 \times A_2 \to A$ has kernel a cyclic group of order 3 and the kernel of the homomorphism $A_7 \to A$ has order 2.

Let $\tilde{z} \in A_7$ be defined by $\epsilon_j(\tilde{z}) = i$ for $j = 1, \dots, 8$. If t is an eighth root of unity, let $tA_7^{(2)} \subset A_7$ denote $\{tf : f \in A_7^{(2)}\}$.

Lemma 9 *Let* $\tilde{F}'$ *(resp.,* $\tilde{F}$*) denote the inverse image of* $p(F')$ *(resp.,* F*) in* A_7. *Then*

(1) $\tilde{F}' \subset \{1, \tilde{z}, \tilde{z}^2, \tilde{z}^3\}$ *or*
(2) $\tilde{F} \supset A_7^{(2)} + iA_7^{(2)}$.

Proof Let t be a primitive eighth root of 1. Suppose that $\tilde{F}' \not\subset \{1, t, \dots, t^7\} \simeq \mathbb{Z}/8\mathbb{Z}$, the center of the maximal torus A_7 of SL_8. Lemma 4 then shows that $\tilde{F} \supset A_7^{(2)}$. If $\tilde{F} = A_7^{(2)}$, then the image of $\tilde{F}$ in A has order 2^6 and is W-stable. But this subgroup of $A^{(2)}$ must also be the image of an $(S_3 \times S_6)$-stable subgroup of $A_2^{(2)} \times A_5^{(2)}$, and there is no stable subgroup of order 2^6. Thus $\tilde{F}$ contains $A_7^{(2)}$ and $t^i A_7^{(2)}$ for some $i = 1, 2$, or 3. It follows that F contains $A_7^{(2)} + iA_7^{(2)}$ whose image in A is $A^{(2)}$.

We are left with showing that $\tilde{F}' = \{1, \dots, t^7\}$ is not possible. If this case occurs, then the image of $p(F')$ in A/Z is a W-stable subgroup of order 2. From our decomposition of $\mathrm{Ad}\, E_7$ as the SL_8-representation $\mathrm{Ad}\,\mathrm{SL}_8 \oplus \bigwedge^4 \underline{k}^8$, one sees that t acts as -1 (resp., 1) on $\bigwedge^4 \underline{k}^8$ (resp., $\mathrm{Ad}\,\mathrm{SL}_8$). Since W acts transitively on the roots of E_7, t does not generate a W-stable subgroup. ∎

Corollary 8 *We have*

(1) $F \subset p(F') \subset Z(E_7) \simeq \mathbb{Z}/2\mathbb{Z}$ *or*
(2) $F \supset A^{(2)}$.

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Then E is smooth if and only if $F = \{e\}$ or $F = A^{(2)}$, in which case $p(F')$ acts linearizably on E.

Proof Parts (1) and (2) are immediate from Lemma 9. The smoothness criterion is classical; in case $F \supset A^{(2)}$ one first needs to quotient by $A^{(2)}$. ■

6.9 Type F_4

Let $\hat{A}$ denote the double cover of the standard 4-torus. Then $\hat{A}$ has character group generated by the ϵ_j, $j = 1, \dots, 4$ and $\beta := \frac{1}{2}(\epsilon_1 + \cdots + \epsilon_4)$ or $\gamma := \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)$. The roots of F_4 are:

(1) the long roots $\pm\epsilon_j \pm \epsilon_k$, $j < k$,
(2) the short roots $\pm\epsilon_j$,
(3) the short roots $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ where the number of minus signs is even, and
(4) the short roots $\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$ where the number of minus signs is odd.

Note that these roots are the weights of the fundamental representations of the group D_4 where ω_2, the adjoint representation, has weights in the long roots (1), the representation ω_1 has the weights in (2), ω_3 has the weights in (3) and ω_4 has the weights in (4). The center Z is trivial. The Weyl group of F_4 is a semidirect product of a copy of S_3 and the Weyl group of D_4. The S_3 subgroup acts as permutations on the highest weights $\alpha := \epsilon_1$, β and γ and the subgroup permutes the long roots amongst themselves.

The center of D_4 is a copy of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and S_3 acts transitively on the complement of $\{e\}$. No nontrivial proper subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is S_3-stable. We have $F \subset p(F') \subset \hat{A}^{(2)}$.

Proposition 6 *There are the following possibilities:*

(1) *$\{e\} = F = p(F')$ or $\hat{A}^{(2)} = F = p(F')$.*
(2) *$\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} = F = p(F')$ or $F = p(F')$ is an extension of $\hat{A}^{(2)}$ by $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.*

In each case, E is an affine space and $p(F')$ acts trivially on E.

Proof From our results for the case of D_4, we know that $p(F') \not\subset \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ implies that $F \supset \hat{A}^{(2)}$. Dividing by $\hat{A}^{(2)}$ we can then reduce to the case that $p(F') \subset \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then, since no nontrivial subgroup of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is S_3-stable, we see that (1) or (2) has to hold. If $\{e\} = F = p(F')$, there is nothing to prove, since we know, classically, that $\hat{A}/W \simeq \underline{k}^4$. We only need to consider the case $F = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and show that the quotient is an affine space.

We calculate the invariants of $W \ltimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ by first finding the invariants of the normal subgroup $W(D_4)$. The invariants are:

(1) $f_2 := \sum_{w \in W(D_4)} w(\epsilon_1 + \epsilon_2)$,
(2) $f_x := \sum_{w \in W(D_4)} w(\epsilon_1)$,
(3) $f_y := \sum_{w \in W(D_4)} w(\beta)$, and

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(4) $f_z := \sum_{w \in W(D_4)} w(\gamma)$.

The action of S_3 permutes f_x, f_y and f_z while leaving f_2 fixed.

We now bring in the action of $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. The action on f_2 is trivial, while the action on the other variables has generators the first of which fixes f_x and sends f_y and f_z to their negatives, and the second of which fixes f_z and sends f_x and f_y to their negatives. The action of $S_3 \ltimes (\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z})$ on the span of f_x, f_y and f_z is just the standard reflection representation of $W(D_3)$, hence $E \simeq \underline{k}^4$. The generators of the invariants of the total Weyl group action are f_2, $f_x^2 + f_y^2 + f_z^2$, $f_x^2 f_y^2 + f_x^2 f_z^2 + f_y^2 f_z^2$ and $f_x f_y f_z$. ■

We have now completed the proof of the Main Lemma 3.

7 Classification of F_0, Adjoint Case

Throughout this section we assume that G is adjoint.

Remark 2 Let $T \supset A$ be standard. Since G is adjoint, $X^*(T)$ is the lattice generated by $\Phi(T)$. Moreover, $X^*(A)$ is the lattice generated by $\Phi(A)$. This easily follows from the fact that $X^*(A)$ is the set of restrictions of the elements of $X^*(T)$ to A and that $\Phi(A)$ is the set of roots obtained by restricting the elements of $\Phi(T)$ to A (see [Hel88, Lemma 5.6]).

7.1 To compute F_0, it suffices to reduce to the case that $\Phi(A)$ is irreducible. We will use the following notation. Let $T \supset A$ be standard and let $\mathfrak{g}(A, \lambda)$ denote the root space corresponding to $\lambda \in \Phi(A)$. Since $\sigma(\lambda) = \theta(\lambda) = -\lambda$, $\tau = \sigma\theta$ stabilizes $\mathfrak{g}(A, \lambda)$. Set

$$\mathfrak{g}(A, \lambda)^{\tau}_{\pm} = \{X \in \mathfrak{g}(A, \lambda) \mid \tau(X) = \pm X\},$$
$$m^{\pm}(\lambda, \tau) = \dim \mathfrak{g}(A, \lambda)^{\tau}_{\pm}.$$

For $\lambda \in \Phi(A)$ call $(m^+(\lambda, \tau), m^-(\lambda, \tau))$ the *signature* of λ. Let Δ be a basis of $\Phi(A)$. Following [Hel88, 6.11] we say that (σ, θ) is a *standard pair* if $m^+(\lambda, \tau) \geq m^-(\lambda, \tau)$ for any $\lambda \in \Delta$. One can always make a pair (σ, θ) standard (without changing F_0) by replacing θ by $\theta\,\mathrm{Int}(q)$ for some quadratic element q. Then $W_H^*(A) \simeq W_H(A) \ltimes F_0$ (see [HS01, Theorem 9.13] and §2.2, Theorem 4), so it suffices to determine F_0 in the case that (σ, θ) is a standard pair.

We use [HS01, Theorem 10.7] to classify F_0. In particular, $F_0 = \{e\}$ iff $m^+(\lambda, \tau) \neq m^-(\lambda, \tau)$ for all $\lambda \in \Delta$ and $F_0 = A^{(2)}$ iff $m^+(\lambda, \tau) = m^-(\lambda, \tau)$ for all $\lambda \in \Delta$. The classification of the pairs of commuting involutions in [Hel88, Tables II, III, IV and V] includes a classification of the restricted root systems $\Phi(A)$ and a classification of the signatures for the basis elements of $\Phi(A)$. So one can easily determine in which cases $F_0 = A^{(2)}$ or $F_0 = \{e\}$. We refer to both these cases as the trivial case.

Remark 3 From the classification of the signatures in [Hel88] it follows that for each type of irreducible root system $\Phi(A)$ each trivial case occurs for some triple

(G, σ, θ). The case $F_0 = \{e\}$ occurs when $\sigma = \theta$, since then $T_-^\tau = \{e\}$. The case $F_0 = A^{(2)}$ occurs for any type of (reduced or non reduced) irreducible root system $\Phi(A)$ for the following triples (G, σ, θ). Take $G = G_1 \times G_1$, $\sigma(x, y) = (y, x)$ and $\theta(x, y) = (\theta_1(x), \theta_1(y))$, $(x, y) \in G$ with θ_1 any involution of G_1. For the case that $\Phi(A)$ is reduced one can also use Example 1. For G simple and $\sigma \neq \theta$ the root system $\Phi(A)$ can only be of type A_n, B_n, C_n, BC_n or F_4 and for these not all trivial cases for F_0 occur.

All the results that follow heavily depend on the classification of the pairs of commuting involutions in [Hel88, Tables II, III, IV and V], [HS01, Theorem 10.7] and a case by case verification.

7.2 Classification of F_0 for G Adjoint and $\Phi(A)$ Irreducible

In the following we discuss which subgroups F_0 of $A^{(2)}$ occur in the case that G is adjoint and $\Phi(A)$ is irreducible. We also discuss smoothness of the corresponding quotient of A. Let $X_*(A)$ denote the group of rational one-parameter multiplicative subgroups of A. The group $X^*(A)$ can be put in duality with $X_*(A)$ by a pairing $\langle \cdot, \cdot \rangle$ defined as follows: if $\chi \in X^*(A)$, $\lambda \in X_*(A)$, then $\chi(\lambda(t)) = t^{<\chi, \lambda>}$ for all $t \in \underline{k}^*$.

Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be a basis of $\Phi(A)$ and let $\{\lambda_1, \ldots, \lambda_n\}$ be the one parameter subgroups dual to $\alpha_1, \ldots, \alpha_n$, *i.e.*, $\langle \alpha_i, \lambda_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$. The elements $q_i = \lambda_i(-1) \in A$ are quadratic elements (see 2.2), and since $\Phi(A)$ is irreducible, any quadratic element is $W_H(A)$-conjugate to one of the q_i, $i = 1, \ldots, n$ (see [BdS49] or [Hel88, Theorem 8.13]). Since G is adjoint, $Z(G) = \{e\}$, and hence $\{q_1, \ldots, q_n\} \subset A^{(2)}$.

7.2.1 $\Phi(A)$ of Type A_1

In this case F_0 is always trivial and by Corollary 3 the quotient is smooth.

7.2.2 $\Phi(A)$ of Type A_n, $n \geq 2$

In this case F_0 is always trivial. The group F in Corollary 3 is an extension of F_0 by the center $\mathbb{Z}/n\mathbb{Z}$, and Corollary 3 shows that the quotient is not smooth.

From now on let $\epsilon_1, \ldots \epsilon_n$ be the standard characters on the standard n-torus A_1 sitting inside GL_n, and let $\tilde{\epsilon}_i \in X_*(A_1)$ be the standard one-parameter subgroup of A_1 dual to ϵ_i, $1 \leq i \leq n$. We use additive notation for the one-parameter subgroups.

7.2.3 $\Phi(A)$ of Type B_n, $n \geq 2$

The roots are $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$, $\alpha_n = \epsilon_n$, and λ_j is just $\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_j$, $1 \leq j \leq n$. The only case with F_0 nontrivial occurs when $m^+(\alpha_i, \tau) \neq m^-(\alpha_i, \tau)$ for $i = 1, \ldots, n-1$ and $m^+(\alpha_n, \tau) = m^-(\alpha_n, \tau)$. Then F_0 is generated by $q_n = \lambda_n(-1) = (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_n)(-1)$, which is fixed under the Weyl group. So F_0 has order 2, which corresponds to the case where F has order 4 in Corollary 6. Hence

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the quotient is not smooth for $n \geq 3$ and smooth for $n = 2$. In all trivial cases the quotient is smooth.

7.2.4 $\Phi(A)$ of Type C_n, $n \geq 3$

The roots are $\alpha_1 = \epsilon_1 - \epsilon_2, \ldots, \alpha_{n-1} = \epsilon_{n-1} - \epsilon_n, \alpha_n = 2\epsilon_n$, which we consider as characters on the torus $A_2 = A_1/(\pm \mathrm{Id})$. Then $\lambda_i = \tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_i$, $1 \leq i \leq n-1$ which we consider as elements of $X_*(A_2)$ via projection from A_1. We have $\lambda_n = \frac{1}{2}(\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_n)$, *i.e.*, $\lambda_n(t)$ is the projection to A_2 of $\mathrm{diag}(\sqrt{t}, \ldots, \sqrt{t})$ (same choice of $\sqrt{t}$ in each slot). The only case with F_0 nontrivial occurs when $m^+(\alpha_i, \tau) = m^-(\alpha_i, \tau)$ for $i = 1, \ldots, n-1$ and $m^+(\alpha_n, \tau) \neq m^-(\alpha_n, \tau)$. By [HS01, Theorem 10.7(1)] F_0 consists of the W-orbits in $A_2^{(2)}$ represented by $q_i = (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_i)(-1)$, where i runs from 1 to $n-1$, and does not contain the W-orbit in $A_2^{(2)}$ represented by $q_n = \lambda_n(-1) = \frac{1}{2}(\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_n)(-1)$. Now $\lambda_n(-1)$ is the image in A_2 of $q = \mathrm{diag}(i, \ldots, i) \in A_1$. The Weyl group orbit of q consists of elements $\mathrm{diag}(\pm i, \ldots, \pm i)$. If we multiply all entries of such an element by -1, we get the same image in A_2. Thus the W-orbit of $\lambda_n(-1)$ has cardinality 2^{n-1}, hence F_0 has cardinality 2^{n-1}. Lifting F_0 to A_1 we get $A_1^{(2)}$, so by Corollary 5 the quotient is smooth. In both trivial cases the quotient is not smooth.

7.2.5 $\Phi(A)$ of Type BC_n, $n \geq 1$

The root system is the union of those for type B_n (7.2.3) and C_n (7.2.4). The α_i and λ_i are as in (7.2.3). The only case with F_0 nontrivial occurs when $m^+(\alpha_i, \tau) \neq m^-(\alpha_i, \tau)$ for $i = 1, \ldots, n-1$ and $m^+(\alpha_n, \tau) = m^-(\alpha_n, \tau)$. Then $F_0 = \{e, q_n\}$ where $q_n = \lambda_n(-1) = (\tilde{\epsilon}_1 + \cdots + \tilde{\epsilon}_n)(-1)$ is W-fixed. This corresponds to the case $F_0 = \{e, z\}$ in Proposition 3. The quotient is nonsmooth if $n \geq 3$ and smooth if $n = 1$ or 2. For both trivial cases the quotient is smooth.

7.2.6 $\Phi(A)$ of Type D_n, $n \geq 4$

In this case F_0 is always trivial and by Corollary 6 the quotient is never smooth.

7.2.7 $\Phi(A)$ of Type F_4

By Borel and de Siebenthal [BdS49] (see also [Hel88, Theorem 8.13]) there are two nontrivial W-orbits in $A^{(2)}$ with representatives q_1 and q_4. Since $m^+(\alpha_1, \tau) \neq m^-(\alpha_1, \tau)$ and $m^+(\alpha_4, \tau) = m^-(\alpha_4, \tau)$ it follows that $q_4 \in F_0$ and $q_1 \notin F_0$. Thus $F_0 \neq A^{(2)}$, and by Propositon 6 we have $F_0 \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and a smooth quotient. In both trivial cases the quotient is smooth.

7.2.8 $\Phi(A)$ of Type E_6, E_7, E_8, or G_2

In these cases F_0 is always trivial. By Corollary 7 and Corollary 8 the quotient is never smooth if $\Phi(A)$ is of type E_6 or E_7 and by Proposition 4 and Proposition 5 the quotient is smooth if $\Phi(A)$ is of type E_8 or G_2.

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