Introduction

In this introduction, we describe in detail the content of the second volume of the book, simultaneously highlighting its sources and the interconnections between various fragments of the book. Most of this volume is devoted to a direct exploration of the dynamics and geometry of elliptic functions. Indeed, all but one, the first, parts of the volume, i.e., Parts IV–VI, do this.

The first part of this volume, i.e., Part III, "Topological Dynamics of Meromorphic Functions," is devoted to the iteration of arbitrary meromorphic functions. Indeed, it provides a relatively short and condensed account of the topological dynamics of almost all meromorphic functions with an emphasis on Fatou domains, including a detailed account of Baker domains that are exclusive for transcendental functions and do not occur for rational functions. We actually do this for all meromorphic functions, occasionally restricting our attention to the class of transcendental meromorphic functions all of whose prepoles (that include poles) form an infinite set. Essentially, all results of this part are supplied with full proofs. In particular, we provide a complete proof of Fatou's classification of Fatou Periodic Components. We do a thorough analysis of the singular set of the inverse of a meromorphic function and all its iterates; in particular, we study at length asymptotic values and their relations to transcendental tracts. We analyze the structure of these components and the structure of their boundaries in greater detail. In particular, we provide a very detailed qualitative and quantitative description of the local behavior of locally and globally defined analytic functions around rationally indifferent periodic points and of the structure of corresponding Leau–Fatou flower petals, including the Fatou Flower Petal Theorem. Such an analysis will turn out to be an indispensable tool in the last three sections of Chapter 22 in Part VI, where we deal with the ergodic theory of parabolic elliptic functions. We also distinguish Speiser class $\mathcal{S}$ and Eremenko–Lyubich class $\mathcal{B}$ of meromorphic functions, which play a seminal role in the recent development

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of the theory of iteration of transcendental meromorphic functions, proving their fundamental properties, which include some structural theorems about their Fatou components such as no existence of Baker domains and wandering domains (the Sullivan Nonwandering Theorem) for class $\mathcal{S}$. The proof of the latter theorem, because of its length and high technicality, is however relegated to Appendix B.

To the best of our knowledge, there is no systematic book account of the topological dynamics of transcendental meromorphic functions. Some results, with and without proofs, can be found in [**BKL1**]–[**BKL4**] and in [**Ber1**]. As we have already said, essentially all results in Part III of our book are supplied with proofs.

In Part IV, we move on to elliptic functions and stay with them until the end of the book. The first chapter of this part, i.e., Chapter 16, which is interesting on its own, is devoted to presenting an account of the classical theory of elliptic functions. Almost no dynamics is involved here. We will actually not need this chapter anywhere else in the book except in Chapter 19, where we provide many examples of elliptic functions, including mainly but, we want to emphasize this, not only Weierstrass $\wp$ functions. Here, we primarily follow the classical books [**Du**] and [**JS**]. We would also like to draw the reader's attention to the books [**AE**] and [**La**].

Throughout the whole of Chapter 17, we deal with general nonconstant elliptic functions, i.e., we impose no constraints on a given nonconstant elliptic function. We first systematically deal with forward and, more importantly, backward images of open connected sets, especially those with connected components of the latter. We mean to consider such images under all iterates f^n, $n \geq 1$, of a given elliptic function f. We do a thorough analysis of the singular set of the inverse of a meromorphic function and all its iterates; in particular, we study at length asymptotic values and their relations to transcendental tracts. We also provide sufficient conditions for the restrictions of iterates f^n to such components to be proper or covering maps. Both of these methods, the latter allowing the use of the machinery of Section 8.6 from Volume I, are our primary tools to study the structure of connected component backward images of open connected sets. In particular, they prove the existence of holomorphic inverse branches if "there are no critical points." Holomorphic inverse branches will be one of the most common tools used throughout the rest of the book. We then apply these results to study images and backward images of connected components of the Fatou set.

Section 17.2 continues this theme, providing some structural theorems about Fatou and Julia sets of elliptic functions. Some of these are the immediate consequences of the results obtained in Part III, "Topological Dynamics of

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Meromorphic Functions," once we observed that each elliptic function belongs to Speiser class $\mathcal{S}$, while others are more technically complicated.

The rest of Chapter 17 is actually devoted to analyzing in greater detail the fractal properties of any nonconstant elliptic function. Following the paper [**KU3**], by associating with a given elliptic function an infinite alphabet conformal iterated function system, and heavily utilizing its θ number, we provide a strong, somewhat surprising, lower bound for the Hausdorff dimension of the Julia sets of all nonconstant elliptic functions. In particular, this estimate shows that the Hausdorff dimension of the Julia sets of any nonconstant elliptic function is strictly larger than 1. We also provide a simple closed formula for the Hausdorff dimension of the set of points escaping to infinity under iteration of an elliptic function. In the last section of this chapter, we prove that no conformal measure of an elliptic function charges the set of escaping points. However, the central focus of this chapter is Section 17.6, where we prove the existence of the Sullivan conformal measures with a minimal exponent for all elliptic functions and we characterize the value of this exponent in several dynamically significant ways. Section 17.6 depends on the preparatory work in Sections 17.4 and 17.5, which are also interesting on their own. It also heavily depends on Chapter 10 in the first volume.

In Part V, "Compactly Nonrecurrent Elliptic Functions: First Outlook," we define the class of nonrecurrent and, more notably, the class of compactly nonrecurrent elliptic functions. This is the class of elliptic functions that will be dealt with by us from the moment compactly nonrecurrent elliptic functions are defined until the end of the book. Its history goes back to the papers [**U3**], [**U4**], and [**KU4**]. One should also mention the paper [**CJY**]. Similarly to all the papers that our treatment of nonrecurrent elliptic functions is based on, the fact that this is possible at all is due to an appropriate version of the breakthrough Mañé's Theorem that was proven in [**M1**] in the context of rational functions. Without Mañé's Theorem, such treatment would not be possible. In our setting of elliptic functions, this is Theorem 18.1.6. The first section of Chapter 18 is entirely devoted to proving this theorem, its first most fundamental consequences, and some other results surrounding it. The next two sections of this chapter, also relying on Mañé's Theorem, provide us with further refined technical tools to study the structure of Julia sets and holomorphic inverse branches.

The last section of this chapter, i.e., Section 18.4, has a somewhat different character. It systematically defines and describes various subclasses of the, mainly compactly nonrecurrent, elliptic functions that we will be dealing with in Part VI of the book. Mostly, but not exclusively, these classes of elliptic functions are defined in terms of how strongly expanding these functions are. We would like to add that while in the theory of rational functions such classes pop up in a natural and fairly obvious way, e.g., metric and topological definitions of expanding rational functions describe the same class of functions, in the theory of iteration of transcendental meromorphic functions such a classification is by no means obvious as the topological and metric analogs of rational function concepts do not usually coincide and the definitions of expanding, hyperbolic, topologically hyperbolic, subhyperbolic, etc. functions vary from author to author. Our definitions seem to us to be quite natural and fit well with our purpose of the detailed investigation of the dynamical and geometric properties of the elliptic functions that they define. In this section, we also define the class of regular elliptic functions. The condition defining them is quite simple but, although very frequently holding, it does not look natural. It is, in fact, tailor-made for the proof of the possibly (in a sense) richest properties of the Sullivan conformal measures obtained in Section 20.3 to go through.

The purpose of Chapter 19 is to provide examples of elliptic functions with prescribed properties of the orbits of critical points (and values). We primarily focus on constructing examples of the various classes of compactly nonrecurrent elliptic functions discerned in Section 18.4. All these examples are either Weierstrass $\wp_\Lambda$ elliptic functions or their modifications. The dynamics of such functions depends heavily on the lattice Λ and varies drastically from Λ to Λ.

The first three sections of this chapter have a preparatory character and, respectively, describe the basic dynamical and geometric properties of all Weierstrass $\wp_\Lambda$ elliptic functions generated by square and triangular lattices Λ.

In Section 19.4, we provide simple constructions of many classes of elliptic functions discerned in Section 18.4. We essentially cover all of them. All these examples stem from Weierstrass $\wp$ functions.

We then, starting with Section 19.5, also provide some different, interesting on their own, and historically first examples of various kinds of Weierstrass $\wp$ elliptic functions and their modifications. These come from the series of papers [**HK1**], [**HK2**], [**HK3**], [**HKK**], and [**HL**] by Hawkins and her collaborators.

Part VI, "Compactly Nonrecurrent Elliptic Functions: Fractal Geometry, Stochastic Properties, and Rigidity," is entirely devoted to getting the dynamical, geometric/fractal, and stochastic properties of dynamical systems generated by compactly nonrecurrent elliptic functions, primarily subexpanding and parabolic ones.

In Chapter 20, we use the fruits of the existence of the Sullivan conformal measures with a minimal exponent proven in Section 17.6 and its dynamical characterizations obtained therein. Having compact nonrecurrence, we are able to prove in the first section of this chapter that this minimal exponent is equal

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to the Hausdorff dimension $\mathrm{HD}(J(f))$ of the Julia set $J(f)$, which we always denote by h. We also obtain in this section strong restrictions on the possible locations of atoms of such conformal measures.

Section 20.3, the last section in Chapter 20, is a culmination of our work on the Sullivan conformal measures for elliptic functions treated on their own. There, and from then onward, we assume that our compactly nonrecurrent elliptic function is regular, which is the concept introduced in Section 18.4. For this class of elliptic functions, we prove the uniqueness and atomlessness of h-conformal measures along with their first fundamental stochastic properties such as ergodicity and conservativity.

The results of Chapter 20 are not, however, the last word on the Sullivan conformal measures. Left alone, these measures would be a kind of curiosity that is perhaps only worthy of shrugging shoulders and raised eyebrows. Their true power, meaning, and importance come from their geometric characterizations, and, more accurately, from their usefulness – one could even say indispensability – for understanding geometric measures on Julia sets, i.e., their Hausdorff and packing h-dimensional measures, where, we recall, $h = \mathrm{HD}(J(f))$. This is fully achieved in Chapter 21 for compactly nonrecurrent regular elliptic functions. Having said this, Chapter 21 can be viewed from two perspectives. The first is that we provide therein a geometrical characterization of the h-conformal measure m_h, which, with the absence of parabolic points, turns out to be a normalized packing measure; the second is that we give a complete description of geometric, Hausdorff, and packing measures of the Julia sets $J(f)$. All of this is contained in Theorem 21.0.1, which gives a simple clear picture. Because of the fact that the Hausdorff dimension of the Julia set of an elliptic function is strictly larger than 1, this picture is even simpler than for nonrecurrent rational functions of [**U3**]; see also [**DU5**].

Throughout the whole of Chapter 22, $f\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is assumed to be a compactly nonrecurrent regular elliptic function. This chapter is, in a sense, the core of our book. Taking the fruits of what has been done in all previous chapters, we prove in Chapter 22 the existence and uniqueness, up to a multiplicative constant, of a σ-finite f-invariant measure μ_h equivalent to the h- conformal measure m_h. Furthermore, still heavily relying on what has been done in all previous chapters, particularly on conformal graph directed Markov systems, nice sets, first return map techniques, and Young towers, we provide here a systematic account of the ergodic and refined stochastic properties of the dynamical system (f, μ_h) generated by such subclasses of compactly nonrecurrent regular elliptic functions as normal subexanding elliptic functions of finite character and parabolic elliptic functions. By stochastic properties, we

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mean here the exponential decay of correlations, the Central Limit Theorem, the Law of the Iterated Logarithm for subexpanding functions, the Central Limit Theorem for those parabolic elliptic functions for which the invariant measure μ_h is finite (probabilistic after normalization), and an appropriate version of the Darling–Kac Theorem that establishes the strong convergence of weighted Birkhoff averages to Mittag–Leffler distributions for those parabolic elliptic functions for which the invariant measure μ_h is infinite.

In Chapter 23, the last actual chapter of the book, we deal with the problem of dynamical rigidity of compactly nonrecurrent regular elliptic functions. The issue at stake is whether a weak conjugacy such as a Lipschitz one on Julia sets can be promoted to a much better one such as an affine conjugacy on the whole complex plane $\mathbb{C}$. This topic has a long history and goes back at least to the seminal paper [**Su4**] by Sullivan, who treated, among others, the dynamical rigidity of conformal expanding repellers. Its systematical account can be found in [**PU2**]. A large variety, in many contexts, both smooth and conformal, of dynamical rigidity theorems have been proved. The literature abounds.

Our approach in this chapter stems from the original article by Sullivan [**Su4**]. It is also influenced by [**PU1**], where the case of tame rational functions has actually been done, and [**SU**], where the equivalence of statements (1) and (4) of Theorem 23.0.1 was established for all tame transcendental meromorphic functions. Being tame means that the closure of the postsingular set does not contain the whole Julia set; in particular, each nonrecurrent elliptic function is tame. We would, however, like to emphasize that, unlike [**SU**], we chose in our book the approach that does not make use of the dynamical rigidity results for conformal iterated function systems proven in [**MPU**].

In Appendix A, "A Quick Review of Some Selected Facts from Complex Analysis of a One Complex Variable," we collect for the convenience of the reader many basic and fundamental theorems of complex analysis. We provide no proofs, but we give detailed references (quite arbitrarily chosen) where the proofs can be found. We use these theorems throughout the book, frequently without directly referring to them. The content of Appendix B is clear from its title. It stems from the Sullivan breakthrough paper [**Su1**] and follows closely the proof presented in [**BKL4**].

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