Can. J. Math., Vol. XXVI, No. 4, 1974, pp. 893–907

CHROMATIC SUMS FOR ROOTED PLANAR TRIANGULATIONS, V: SPECIAL EQUATIONS

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Introduction. In **I** we obtained an equation, called the "chromatic equation", for the generating function $g(x, y, z, \lambda)$. In **II** and **III** we obtained special equations, valid in the cases $\lambda = \tau + 1$ and $\lambda = 3$ respectively, for the generating function $l(y, z, \lambda)$, defined as the coefficient of x^2 in $g(x, y, z, \lambda)$. The argument was independent of that in **I** and no attempt was made to derive the new formulae from the chromatic equation. In the present paper this omission is repaired. Moreover an equation for $l(y, z, \lambda)$ is derived from the chromatic equation for each of the values

$$2 + 2 \cos (2\pi/n)$$

of λ, n being any integer greater than 2. These values of λ are the "Beraha numbers" mentioned in **I**, Section 3, including $\tau + 1$ at $n = 5$ and 3 at $n = 6$.

The new equations are not solved. It is claimed however that their derivation establishes the Beraha numbers as being important in the theory of chromatic polynomials. Hitherto their significance, for $n > 7$, has been a matter of conjecture, supported by a rather small amount of numerical evidence.

1. Repeated differences. Let us say that a formal power series in z and a finite number of other independent variables is *z-restricted* if the coefficient of each power of z is a polynomial in the other variables. Then the sum or product of two z-restricted power series in the same variables is z-restricted. If the operator $\triangle$, defined in Section 2 of **I**, is applied to a z-restricted power series f involving y then another z-restricted power series $\triangle f$ is obtained. Moreover a given power of z can occur with non-zero coefficient in only a finite number of the z-restricted power series $\triangle^j f$, $(j = 0, 1, 2, 3, \ldots)$.

There can be only a finite number of combinatorially distinct rooted near-triangulations with a given number of faces. Hence the chromatic sums $g(x, y, z, \lambda)$, $q(x, z, \lambda)$ and $l(y, z, \lambda)$ are z-restricted.

The chromatic equation is given in **I**, (13) as

$$(1) \qquad xg = x^3y\lambda(\lambda - 1) + \lambda^{-1}yzgq + yz(g - x^2l) - x^2y^2z\triangle g.$$

Let us apply the operator $\triangle$ to this, using the rule $\triangle(yf) = f + \triangle f$. We

Received February 13, 1973.

obtain

$$(2)\qquad -(\lambda^{-1}zq + z)g + (x + x^2yz + x^2z - \lambda^{-1}zq - z)\triangle g + x^2z\triangle^2 g = x^3\lambda(\lambda - 1) - x^2z\triangle(yl).$$

A second application of $\triangle$ yields

$$(3)\qquad (x^2z - \lambda^{-1}zq - z)\triangle g + (2x^2z + x - \lambda^{-1}zq - z)\triangle^2 g + x^2z\triangle^3 g = -x^2z\triangle^2(yl).$$

By repeated application of the operator to (3) we obtain the general formula

$$(4)\qquad (x^2z - \lambda^{-1}zq - z)\triangle^n g + (2x^2z + x - \lambda^{-1}zq - z)\triangle^{n+1} g + x^2z\triangle^{n+2} g = -x^2z\triangle^{n+1}(yl),$$

valid for $n \geqq 1$.

To simplify the discussion of this recursion formula we write $x = zs$ and consider the generating functions g and q as power series in s, y, z and λ rather than x, y, z and λ. It is to be recalled that the terms of these series must all divide by λ, since the chromatic polynomials vanish at $\lambda = 0$. Hence $\lambda^{-1}q(zs, z, \lambda)$ is a well-defined z-restricted power series (without negative indices) in s, z and λ.

Let U denote the set of all formal power series in s, y, z and λ, and U_1 the set of all z-restricted ones. Let V denote the set of all such series in s, z and λ, and V_1 the set of all z-restricted ones. Then $V \subset U$ and $V_1 \subset U_1$. The functions g and l are members of U_1 and $\lambda^{-1}q$ belongs to V_1.

Dividing (4) by z we obtain

$$(5)\qquad (z^2s^2 - \lambda^{-1}q - 1)\triangle^n g + (2z^2s^2 + s - \lambda^{-1}q - 1)\triangle^{n+1} g + z^2s^2\triangle^{n+2} g = -z^2s^2\triangle^{n+1}(yl). \qquad (n \geqq 1).$$

Consider the quadratic equation

$$(6)\qquad (z^2s^2 - \lambda^{-1}q - 1)u^2 + (2z^2s^2 + s - \lambda^{-1}q - 1)u + z^2s^2 = 0.$$

Its discriminant D is given by

$$(7)\qquad D = (1 + \lambda^{-1}q - s - 2z^2s^2)^2 + 4z^2s^2(1 + \lambda^{-1}q - z^2s^2).$$

We observe that $\lambda^{-1}q - z^2s^2$ has constant term zero. Hence $1 + \lambda^{-1}q - z^2s^2$ has a reciprocal in V with constant term 1. Similarly we can use the binomial theorem to show that D has a square root in V with constant term 1. We conclude that the quadratic equation (6) has two roots θ and ϕ for u, each root being a member of V. We may suppose θ to have the constant term -1, and ϕ to have the constant term 0.

Let us multiply (5) by u^n and sum over n. We obtain

$$(8)\qquad u(z^2s^2 - \lambda^{-1}q - 1)\triangle g - z^2s^2\triangle^2 g = -z^2s^2 \sum_{n=1}^{\infty} u^n\triangle^{n+1}(yl).$$

The infinite summation is justified by the fact that a given power of z appears with non-zero coefficient in only a finite number of the differences $\triangle^j g$ and

https://doi.org/10.4153/CJM-1974-084-1 Published online by Cambridge University Press

$\Delta^k(yl)$. For the same reason the sum on the right of (8) is well-defined as a member of U.

We can substitute either θ or ϕ for u in (8). Subtracting the resulting equations we find

$$(9)\quad (\theta - \phi)(z^2s^2 - \lambda^{-1}q - 1)\Delta g = -z^2s^2 \sum_{n=1}^{\infty} (\theta^n - \phi^n)\Delta^{n+1}(yl).$$

By the definition of θ and ϕ as roots of (6) we have

$$(10)\quad (\theta + \phi)(z^2s^2 - \lambda^{-1}q - 1) = 1 + \lambda^{-1}q - s - 2z^2s^2,$$

$$(11)\quad \theta\phi(z^2s^2 - \lambda^{-1}q - 1) = z^2s^2.$$

By (9) and (11) we have

$$(12)\quad (\theta - \phi)\Delta g = -\theta\phi \sum_{n=1}^{\infty} (\theta^n - \phi^n)\Delta^{n+1}(yl).$$

2. Equations for $l(y, z, \lambda)$. We write W for the set of all formal power series in y, z and λ, and W_1 for the set of all z-restricted ones. Since $l(y, z, \lambda)$ is in W_1 we can write

$$(13)\quad yl(y, z, \lambda) = \sum_{r=0}^{\infty} f_r(y-1)^r,$$

where the f_r are power series in z and λ, and a given power of z occurs with non-zero coefficient in only a finite number of the f_r. Hence if we replace $y - 1$ on the right of (13) by an arbitrary member u of U the infinite sum will determine a definite member of U. We naturally denote this by $(1 + u)$ $l(1 + u, z, \lambda)$. We also deduce from (13) that

$$(14)\quad \Delta^n(yl) = \sum_{r=n}^{\infty} f_r(y-1)^{r-n} \quad (n = 0, 1, 2, \ldots).$$

Equation (12) can be rewritten as follows.

$$(15)\quad (\theta - \phi)(\Delta g + yl) = -\phi \sum_{n=0}^{\infty} \theta^n\Delta^n(yl) + \theta \sum_{n=0}^{\infty} \phi^n\Delta^n(yl).$$

But

$$\begin{aligned}\sum_{n=0}^{\infty} \theta^n\Delta^n(yl) &= \sum_{n=0}^{\infty} \theta^n \sum_{r=n}^{\infty} f_r(y-1)^{r-n}, \quad \text{by (14)},\\ &= \sum_{r\ 0}^{\infty} f_r \sum_{n=0}^{r} \theta^n(y-1)^{r-n}.\end{aligned}$$

Hence

$$\begin{aligned}(1 - y + \theta) \sum_{n=0}^{\infty} \theta^n\Delta^n(yl) &= \sum_{r=0}^{\infty} f_r\left\{\theta^{r+1} - (y-1)^{r+1}\right\}\\ &= \theta(1+\theta)l(1+\theta, z, \lambda) - (y-1)yl(y, z, \lambda),\end{aligned}$$

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by (13). Moreover this equation remains valid when θ is replaced by ϕ. Combining these results with (15) we have

$$(\theta - \phi)(1 - y + \theta)(1 - y + \phi)(\triangle g + yl(y, z, \lambda)) = -\phi\{(1 - y + \phi)\theta(1 + \theta)l(1 + \theta, z, \lambda) - (y - 1)yl(y, z, \lambda)\} + \theta\{(1 - y + \theta) \times \phi(1 + \phi)l(1 + \phi, z, \lambda) - (y - 1)yl(y, z, \lambda)\},$$

which is equivalent to

$$(16) \quad (\theta - \phi)(1 - y + \theta)(1 - y + \phi)\triangle g = -\theta\phi\{(1 - y + \phi)(1 + \theta)l(1 + \theta, z, \lambda) - (1 - y + \theta)(1 + \phi)l(1 + \phi, z, \lambda) + (\theta - \phi)yl(y, z, \lambda)\}$$

Another equation for $\triangle g$ can be obtained directly from (1) by using the identity

$$g = q + (y - 1)\triangle g.$$

Replacing x by zs we find that

$$(17) \quad \{s^2z^2y^2 + (y - 1)(s - \lambda^{-1}yq - y)\}\triangle g = s^3z^2y\lambda(\lambda - 1) - z^2s^2yl(y, z, \lambda) - (s - \lambda^{-1}yq - y)q.$$

From (10) and (11) we can derive equations giving s and q directly in terms of z, θ and ϕ. They are as follows.

$$(18) \quad z^2s(1 + \theta)(1 + \phi) = -\theta\phi,$$

$$(19) \quad z^2(\lambda^{-1}q + 1)(1 + \theta)^2(1 + \phi)^2 = (\theta\phi - 1)\theta\phi.$$

We deduce that

$$z^2(1 + \theta)^2(1 + \phi)^2\{s^2z^2y^2 + (y - 1)(s - \lambda^{-1}yq - y)\} = (1 + \theta)(1 + \phi)\theta\phi + y\{-(1 + \theta)(1 + \phi)\theta\phi + (\theta\phi - 1)\theta\phi\} + y^2\{\theta^2\phi^2 - (\theta\phi - 1)\theta\phi\} = \theta\phi(1 + \theta - y)(1 + \phi - y).$$

We can now rewrite (17) as

$$z^2\theta\phi(1 + \theta)^2(1 + \phi)^2(1 + \theta - y)(1 + \phi - y)\triangle g = z^4(1 + \theta)^4(1 + \phi)^4 \times \{\lambda(\lambda - 1)yz^2s^3 + \lambda y(\lambda^{-1}q + 1)^2 - \lambda y(\lambda^{-1}q + 1) - yz^2s^2l(y, z, \lambda) - \lambda s(\lambda^{-1}q + 1) + \lambda s\},$$

that is

$$z^2(1 + \theta)^2(1 + \phi)^2(1 + \theta - y)(1 + \phi - y)\triangle g = -z^2\theta\phi(1 + \theta)^2 \times (1 + \phi)^2yl(y, z, \lambda) + y\{-\theta^2\phi^2(1 + \theta)(1 + \phi)\lambda(\lambda - 1) + \lambda(\theta\phi - 1)^2\theta\phi - \lambda z^2(1 + \theta)^2(1 + \phi)^2(\theta\phi - 1)\} + \{\lambda\theta\phi(1 + \theta) \times (1 + \phi)(\theta\phi - 1) - \lambda z^2(1 + \theta)^3(1 + \phi)^3\}.$$

Let us multiply (16) by $z^2(1 + \theta)^2(1 + \phi)^2$ and the equation just obtained by $(\theta - \phi)$, and then let us subtract to eliminate $\triangle g$. We observe that this operation also eliminates $yl(y, z, \lambda)$. We are left with the following identity.

$$(20)\quad -\theta\phi z^2(1+\theta)^2(1+\phi)^2\{(1-y+\phi)(1+\theta)l(1+\theta, z, \lambda) - (1-y+\theta)(1+\phi)l(1+\phi, z, \lambda)\} = (\theta-\phi)\{y\{-\theta^2\phi^2(1+\theta) \times (1+\phi)\lambda(\lambda-1) + \lambda(\theta\phi-1)^2\theta\phi - \lambda z^2(1+\theta)^2(1+\phi)^2(\theta\phi-1)\} + \{\lambda\theta\phi(1+\theta)(1+\phi)(\theta\phi-1) - \lambda z^2(1+\theta)^3(1+\phi)^3\}\}.$$

Equating the terms independent of y and dividing by their common factor $(1+\theta)(1+\phi)$ we find that

$$(21)\quad z^2\theta\phi(1+\theta)^2(1+\phi)^2\{l(1+\theta, z, \lambda) - l(1+\phi, z, \lambda)\} = -\lambda(\theta-\phi)\{\theta\phi(\theta\phi-1) - z^2(1+\theta)^2(1+\phi)^2\}.$$

If instead we equate coefficients of y in (20) we get

$$(22)\quad z^2\theta\phi(1+\theta)^2(1+\phi)^2\{(1+\theta)l(1+\theta, z, \lambda) - (1+\phi)l(1+\phi, z, \lambda)\} = \lambda(\theta-\phi)\{-\theta^2\phi^2(1+\theta)(1+\phi)(\lambda-1) + (\theta\phi-1)^2\theta\phi - z^2(1+\theta)^2(1+\phi)^2(\theta\phi-1)\}.$$

We now multiply (21) by $(1+\phi)$ and subtract it from (22) .We find that

$$z^2\theta\phi(1+\theta)^2(1+\phi)^2(\theta-\phi)l(1+\theta, z, \lambda) = \lambda(\theta-\phi)\{-\theta^2\phi^2(1+\theta)(1+\phi)(\lambda-1) + \theta\phi(\theta\phi-1)(\theta\phi+\phi) - z^2(1+\theta)^2(1+\phi)^2(\theta\phi+\phi)\},$$

that is

$$(23)\quad z^2\theta(1+\theta)(1+\phi)^2l(1+\theta, z, \lambda) = \lambda\{-(\lambda-1)\theta^2\phi(1+\phi) + \theta\phi(\theta\phi-1) - z^2(1+\theta)^2(1+\phi)^2\}.$$

Had we eliminated $l(1+\theta, z, \lambda)$ instead of $l(1+\phi, z, \lambda)$ we would have obtained the equation derived from (23) by interchanging θ and ϕ.

It seems convenient to write $y_1 = 1+\theta$ and $y_2 = 1+\phi$. Then y_1 and y_2 are members of V, with constant terms 0 and 1 respectively.

We also write $\mu = \lambda - 2$. We may now put (23) and its analogue in the following form.

$$(24)\quad (1-y_1)y_1y_2^2z^2l(y_1, z, \lambda) = \lambda(1-y_1)(1-y_2)(y_1-\mu y_2+\mu y_1y_2) + \lambda y_1^2y_2^2z^2,$$

$$(25)\quad (1-y_2)y_1^2y_2z^2l(y_2, z, \lambda) = \lambda(1-y_1)(1-y_2)(y_2-\mu y_1+\mu y_1y_2) + \lambda y_1^2y_2^2z^2.$$

These equations can be used to compute $l(y, z, \lambda)$ and y_2 term by term as power series in z, the coefficients in y_2 being determined as rational functions of y_1 and λ. Knowing that the constant term in y_2 is 1 we can equate coefficients of z^2 in (25) and so determine the coefficient of z^2 in y_2 as

$$y_1^2/(1-y_1).$$

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Next we can equate coefficients of z^2 in (24) and so determine the constant term in $l(y, z, \lambda)$. Having this we determine the coefficient of z^4 in y_2 by equating coefficients of z^4 in (25), and so on in alternation between the two equations.

3. Invariants. Let F denote the field of quotients of members of U, and K the field of quotients of members of W_1. Let us also write W_0 for the set of all z-restricted power series in z and λ. Thus $W_0 \subset W_1$.

Consider an element

$$f(y, z, \lambda) = \frac{p_1(y, z, \lambda)}{p_2(y, z, \lambda)}$$

of K, where $p_1(y, z, \lambda)$ and $p_2(y, z, \lambda)$ are members of W_1. Substituting an arbitrary member u of U for y we replace $p_i(y, z, \lambda)$ by $p_i(u, z, \lambda)$, a member of U $(i = 1, 2)$. The quotient $f(y, z, \lambda)$ is thereby replaced by $f(u, z, \lambda)$, a member of F. In particular u may be y_1 or y_2.

We say that $f(y, z, \lambda)$ is an *invariant* of K if

$$f(y_1, z, \lambda) = f(y_2, z, \lambda).$$

Evidently the members of W_0 are invariants of K. We call them the *trivial* ones.

3.1. *The member*

$$J = \lambda^{-1}z^2 l(y, z, \lambda) - (1 - y)^{-1}yz^2 + y^{-2}(1 - y)$$

of K is an invariant.

Proof. Let J_1 and J_2, members of F, be obtained by substituting y_1 and y_2 respectively for y in J.

Let i and j be the integers 1 and 2 in some order. Then, by (24) and (25),

$$\begin{aligned} y_i^2y_j^2(1 - y_i)J_i &= y_i\,(1 - y_i)(1 - y_j)(y_i - \mu y_j + \mu y_iy_j) \\ &\qquad + y_i^3y_j^2z^2 - y_i^3y_j^2z^2 + (1 - y_i)^2y_j^2, \\ y_i^2y_j^2J_i &= y_i^2 - \mu y_iy_j + y_j^2 + (\mu - 1)(y_iy_j^2 + y_i^2y_j) - \mu y_i^2y_j^2. \end{aligned}$$

Since the expression on the right is symmetrical in y_i and y_j it follows that $J_1 = J_2$.

Let us define

$$\Lambda(v, w) = v^2 - \mu vw + w^2 + (\mu - 1)(v + w) - \mu \tag{26}$$

as a polynomial in two variables v and w, μ being assigned some constant value. By the proof of (3.1) we have

$$\Lambda(y_1^{-1}, y_2^{-1}) = J_1 = J_2. \tag{27}$$

The polynomial $\Lambda(v, w)$ has a further invariant property. Let us write

$$v_1 = y_1^{-1}, v_2 = y_2^{-1},$$

and let us define $v_3, v_4, v_5, \ldots$ recursively by

(28) $$v_{n+2} = \mu v_{n+1} - v_n + 1 - \mu \qquad (n \geqq 1).$$

3.2. $\Lambda(v_n, v_{n+1})$ *has the same value for all positive integers* n.

Proof. For any positive integer n we have, by (26),

$$\begin{aligned}\Lambda(v_{n+1}, v_{n+2}) &= (v_{n+1})^2 + v_{n+2}(v_{n+2} - \mu v_{n+1} + \mu - 1) \\ &\quad + (\mu - 1)v_{n+1} - \mu \\ &= (v_{n+1})^2 - v_n v_{n+2} + (\mu - 1)v_{n+1} - \mu \\ &= (v_{n+1})^2 - v_n(\mu v_{n+1} - v_n + 1 - \mu) + (\mu - 1)v_{n+1} - \mu \\ &= \Lambda(v_n, v_{n+1}).\end{aligned}$$

The theorem follows.

3.3. *Let* $p(y, z, \lambda)$ *be a member of* W_1, *and* n *a non-negative integer such that* $y^{-n}p(y, z, \lambda)$ *is an invariant of* K. *Then* $y^{-n}p(y, z, \lambda)$ *is in* W_0.

Proof. Let Z_m be the coefficient of z^m in $p(y, z, \lambda)$. We can write

(29) $$Z_m = \sum_{j=0}^{r} a_{mj} y^j$$

where the a_{mj} are polynomials in λ, and r is a positive integer depending on m.

We may suppose that there is a least value k of m such that Z_m is not of the form $a_{mn}y^n$. For otherwise $y^{-n}p(y, z, \lambda)$ would be in W_0 and the theorem would be satisfied. Let us form $p_1(y, z, \lambda)$ from $p(y, z, \lambda)$ by replacing the coefficient of each power of z lower than the kth by zero. It is evident that $y^{-n}p_1(y, z, \lambda)$ is also an invariant of K. Accordingly we have

(30) $$y_2^n p_1(y_1, z, \lambda) = y_1^n p_1(y_2, z, \lambda)$$

as an identity in U.

To find the coefficient of z^k in $p_1(y_i, z, \lambda)$, where i is 1 or 2, we must substitute for y in (29), with $m = k$, the value of y_i when z is set equal to zero. But for $z = 0$ Equation (6) reduces to

$$-u^2 + (s - 1)u = 0$$

and its two roots become $u = 0$ and $u = s - 1$. Thus y_1 and y_2 become s and 1, in that order since the constant terms in y_1 and y_2 are 0 and 1 respectively.

Equating coefficients of z^k in (30) we find that

$$\sum_{j=0}^{r} a_{kj} s^j = s^n \sum_{j=0}^{r} a_{kj}.$$

Hence $a_{kj} = 0$ unless $j = n$. But this is contrary to the definition of k.

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3.4. *If $p(y, z, \lambda)$ is in W_1 and if b and k are non-negative integers, then*

$$y^{-b}(1-y)^{-k}z^{2k}p(y, z, \lambda) = y^{-a}R(y, z, \lambda) + S,$$

where $a = \operatorname{Max}(b, 2k)$, $R(y, z, \lambda)$ is in W_1, and S is a polynomial in J of degree $\leqq k$, the coefficients in this polynomial being z-restricted power series in z and λ.

Proof. We may assume $k > 0$, for otherwise the theorem is trivial.

Let Z_m be the coefficient of z^m in $p(y, z, \lambda)$. It is a polynomial in y and λ. Using the identity $(1-y)^{-1} = y(1-y)^{-1} + 1$ we can write

$$y^{-b}(1-y)^{-k}Z_m = \sum_{j-1}^{k} b_{mj}y^j(1-y)^{-j} + y^{-b}c_m,$$

where b_{mj} is a polynomial in λ, and c_m is a polynomial in y and λ. Multiplying this equation by z^{m+2k} and summing over m we obtain

$$(31)\quad y^{-b}(1-y)^{-k}z^{2k}p(y, z, \lambda) = \sum_{j=1}^{k} \left\{B_j z^{2j}y^j(1-y)^{-j}\right\} + y^{-b}C.$$

Here B_j is a z-restricted power series in z and λ, and C is a z-restricted power series in y, z and λ. But by the definition of J we have

$$(32)\quad z^{2k}y^k(1-y)^{-k} = (-J)^k + \sum_{i=0}^{k-1} \binom{k}{i} P_i,$$

where

$$\begin{aligned} P_i &= (-J)^i\{\lambda^{-1}z^2l(y, z, \lambda) + y^{-2}(1-y)\}^{k-i} \\ &= \{(1-y)^{-1}yz^2 - \lambda^{-1}z^2l(y, z, \lambda) - y^{-2}(1-y)\}^i \\ &\qquad\qquad \times \{\lambda^{-1}z^2l(y, z, \lambda) + y^{-2}(1-y)\}^{k-i}. \end{aligned}$$

Applying the identity $(1-y)^{-1} = y(1-y)^{-1} + 1$ we deduce that

$$(33)\quad P_i = \sum_{j=1}^{k-1} \left\{E_j z^{2j}y^j(1-y)^{-j}\right\} + y^{-2k}E,$$

where E_j is a z-restricted power series, possibly zero, in z and λ, and E is a z-restricted power series in y, z and λ.

Substituting (33) in (32) and applying the result to (31) we obtain

$$(34)\quad y^{-b}(1-y)^{-k}z^{2k}p(y, z, \lambda) = (-1)^k B_k J^k + \sum_{j=1}^{k-1} \left\{N_j z^{2j}y^j(1-y)^{-j}\right\} + y^{-a}N.$$

Here N_j is a z-restricted power series in z and λ, and N is a z-restricted power series in y, z and λ.

This formula at once establishes the theorem in the case $k = 1$. But if the theorem is known to be true whenever k is less than some integer $q > 1$ we can apply it to the terms of the sum on the right of (34), with $k = q$, and so verify the theorem in the case $k = q$. The general theorem now follows by induction.

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3.5. *If $p(y, z, \lambda)$ is in W_1 and if b and k are non-negative integers such that*

$$Y = y^{-b}(1 - y)^{-k}z^{2k}p(y, z, \lambda)$$

is an invariant of K, then Y can be expressed as a polynomial in J of degree $\leqq k$, the coefficients in this polynomial being z-restricted power series in z and λ.

Proof.

$$Y = y^{-a}R(y, z, \lambda) + S,$$

where $R(y, z, \lambda)$, S and a are as in 3.4. Because S is a polynomial in J (of degree $\leqq k$) whose coefficients are z-restricted power series in z and λ it must be an invariant of K, by (3.1). Hence $Y - S$, that is $y^{-a}R(y, z, \lambda)$, is an invariant of K. It follows that $Y - S$ is in W_0, by (3.3). This result establishes the theorem.

4. Special invariants. We define some polynomials Q_n and A_n in μ. Here n can take any integral value, positive, negative or zero. The definitions are as follows.

(35) $\quad Q_0 = 0,\ Q_1 = 1,\ Q_n = \mu Q_{n-1} - Q_{n-2}.$

(36) $\quad A_0 = 1,\ A_{n+1} = A_n + (\mu - 1)Q_n.$

In particular we have $Q_2 = \mu$, $Q_3 = \mu^2 - 1$, $Q_4 = \mu^3 - 2\mu$, $A_1 = 1$, $A_2 = \mu$, $A_3 = \mu^2$ and $A_4 = \mu^3 - \mu + 1$.

It is clear from (35) and (36) that $A_n - \mu A_{n-1} + A_{n-2}$ is independent of n. We thus have a recursion formula,

(37) $\quad A_n = \mu A_{n-1} - A_{n-2} + 1.$

Let us apply these definitions to the functions v_j of (28).

4.1. *For $n \geqq 1$ we have*

$$v_n = Q_{n-1}v_2 - Q_{n-2}v_1 + 1 - A_{n-1}.$$

Proof. Write $Y_n = v_n - Q_{n-1}v_2 + Q_{n-2}v_1 - 1 + A_{n-1}$. It is easily verified that $Y_1 = Y_2 = 0$. But for $n > 2$ we have $Y_n - \mu Y_{n-1} + Y_{n-2} = 0$, by (35), (37) and (28). Hence $Y_n = 0$ for each n.

Let us write

$$\mu = 2 \cos \alpha.$$

Then it is easily verified, and well-known, that

(38) $\quad Q_n = \dfrac{\sin (n\alpha)}{\sin \alpha}.$

We can now use basic trigonometrical identities to verify that

(39) $\quad A_n = \dfrac{(\mu - 1)}{(\mu - 2)}\left\{\dfrac{\cos ((2n - 1)\alpha/2)}{\cos (\alpha/2)} - 1\right\} + 1.$

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4.2. *Let i and j be the integers 1 and 2 in some order. Then, for any positive integer m,*

$$(40)\quad Q_m^2\{\lambda^{-1}z^2l(y_i, z, \lambda) - (1 - y_i)^{-1}y_iz^2\} - \{Q_{m-1}v_i + A_m - Q_m\}\{-Q_{m+1}v_i + A_{m+1} + Q_m\} - \{Q_mv_j - A_m - Q_{m-1}v_i\}\{Q_mv_j + A_{m+1} - Q_{m+1}v_i\} = 0.$$

Proof. Formula (40) is equivalent to

$$(41)\quad Q_m^2y_i(1 - y_j)(y_i - \mu y_j + \mu y_iy_j) - y_j^2\{Q_{m-1} + (A_m - Q_m)y_i\}\{-Q_{m+1} + (A_{m+1} + Q_m)y_i\} - \{Q_my_i - A_my_iy_j - Q_{m-1}y_j\}\{Q_my_i + A_{m+1}y_iy_j - Q_{m+1}y_j\} = 0,$$

as we can verify by multiplying (40) by $\lambda y_i^2y_j^2(1 - y_i)$ and then applying (24) and (25).

The new formula can be verified by showing that the coefficients of y_i^2, y_j^2, y_iy_j, $y_iy_j^2$, $y_i^2y_j$ and $y_i^2y_j^2$ in the expression on the left are all zero. For y_i^2 and y_j^2 this is trivial.

For y_iy_j the coefficient is

$$-\mu Q_m^2 + Q_{m+1}Q_m + Q_{m-1}Q_m = 0 \qquad \text{by (35).}$$

For $y_iy_j^2$ the coefficient is

$$\mu Q_m^2 - Q_{m-1}(A_{m+1} + Q_m) + Q_{m+1}(A_m - Q_m) - Q_mA_{m+1} - Q_{m+1}A_m = \mu Q_m^2 - Q_{m-1}Q_m - Q_{m+1}Q_m = 0 \qquad \text{by (35).}$$

For $y_i^2y_j$ the coefficient is

$$(\mu - 1)Q_m^2 - Q_mA_{m+1} + Q_mA_m = 0 \qquad \text{by (36).}$$

For $y_i^2y_j^2$ the coefficient is

$$-\mu Q_m^2 - (A_m - Q_m)(A_{m+1} + Q_m) + A_mA_{m+1} = Q_m(-\mu Q_m + A_{m+1} - A_m + Q_m) = 0 \qquad \text{by (36).}$$

This completes the proof of the theorem.

Let us write

$$(42)\quad H(m) = Q_m^2\{\lambda^{-1}z^2l(y, z, \lambda) - (1 - y)^{-1}yz^2\} - \{Q_{m-1}v + A_m - Q_m\}\{-Q_{m+1}v + A_{m+1} + Q_m\},$$

where $v = y^{-1}$ and m can be any positive integer. It is a member of K. Substituting y_1 and y_2 for y in it we obtain $H_1(m)$ and $H_2(m)$ respectively, elements of F. By (40),

$$(43)\quad H_i(m) = -(Q_mv_j + A_{m+1} - Q_{m+1}v_i)(Q_{m-1}v_i + A_m - Q_mv_j).$$

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Hence, for any positive integer r,

$$(44)\quad (-1)^r \prod_{m=1}^{r} H_i(m) = (1 - v_j)(Q_r v_j + A_{r+1} - Q_{r+1} v_i) \times \prod_{m=1}^{r-1} \{(Q_m v_j + A_{m+1} - Q_{m+1} v_i)(Q_m v_i + A_{m+1} - Q_{m+1} v_j)\},$$

where an empty product is to be interpreted as unity.

We now propose to assign a constant value to μ. We put

$$\alpha = 2\pi/n, \qquad \mu = 2\cos(2\pi/n),$$

where the integer n is at least 3. Referring to Equations (6) and (7) we see that θ and ϕ, and therefore y_1 and y_2, remain well-defined as power series in s, y and z when a constant value is assigned to λ. They are still to be counted as members of U, even though no term with a non-zero coefficient now involves the variable λ. It is likewise clear that the substitution transforms members of W_1, W_0 and K into members of W_1, W_0 and K respectively.

Suppose first that n is an even number $2M$, $(M \geqq 2)$. Then, by (38),

$$Q_{M+m} = -Q_m$$

for each integer m. Thus $Q_M = -Q_0 = 0$ and $Q_{M-1} = -Q_{-1} = 1$. We note that if $r = M - 1$ then

$$(45)\quad Q_r v_j + A_{r+1} - Q_{r+1} v_i = A_M + v_j.$$

We write

$$(46)\quad I(2M) = (1 - v)(A_M + v) \prod_{m=1}^{M-1} H(m).$$

Then whether we substitute y_1 or y_2 for y in $I(2M)$ the result is, by (44) with $r = M - 1$,

$$(-1)^{M-1}(1 - v_1)(1 - v_2)(A_M + v_1)(A_M + v_2) \times \prod_{m=1}^{M-2} (Q_m v_1 + A_{m+1} - Q_{m+1} v_2)(Q_m v_2 + A_{m+1} - Q_{m-1} v_1).$$

Thus $I(2M)$ is an invariant of K.

Suppose next that n is an odd number $2M - 1$, $(M \geqq 2)$. Then, by (38), $Q_{M-1} = -Q_M$. Write

$$(47)\quad I(2M - 1) = (1 - v) \prod_{m=1}^{M-1} H(m).$$

Whether we substitute y_1 or y_2 for y in $I(2M - 1)$ the result is, by (44) with $r = M - 1$,

$$(-1)^{M-1}(1 - v_1)(1 - v_2)(Q_{M-1} v_1 + A_M + Q_{M-1} v_2) \times \prod_{m=1}^{M-2} (Q_m v_1 + A_{m+1} - Q_{m+1} v_2)(Q_m v_2 + A_{m+1} - Q_{m+1} v_1).$$

Thus $I(2M - 1)$ is an invariant of K.

By (42), (46) and (47) the invariants $I(2M)$ and $I(2M-1)$ can each be expressed in the form

$$y^{-2M}(1-y)^{-M+2}z^{2M-4}p(y, z),$$

where $p(y, z)$ is a member of W_1 not involving the variable λ. Applying 3.5 we obtain the following theorem.

4.3. *If $M \geqq 2$ then each of the invariants $I(2M-1)$ and $I(2M)$ can be expressed as a polynomial in J of degree $\leqq M-2$, the coefficients in this polynomial being power series in z alone.*

The symmetry of the invariants is perhaps best shown by expressing them in terms of the functions v_j. We note that when $\mu = 2\cos(2\pi/n)$ the sequence $(v_1, v_2, v_3, \ldots)$ is periodic with period n, by (38) and (39). Moreover we can deduce from these formulae that $Q_{n-m} = -Q_m$ and $A_{n-m} = A_{m+1}$ for any integer m. We have

$$Q_m v_1 + A_{m+1} - Q_{m+1}v_2 = 1 - v_{m+2}, \qquad (m \geqq 0),$$

by (4.1). Hence

$$Q_m v_2 + A_{m+1} - Q_{m+1}v_1 = Q_{n-m-1}v_1 + A_{n-m} - Q_{n-m}v_2 = 1 - v_{n-m+1},$$

by (4.1). Here $0 \leqq m \leqq n$. We apply these results to the formulae giving $I(2M-1)$ and $I(2M)$ symmetrically in terms of y_1 and y_2, y being set equal to y_1 or y_2. Using (45) we find that

$$(48) \quad I(n) = (-1)^{[(n-1)/2]} \prod_{j=1}^{n} (1 - v_j), \quad \text{if } y = y_1 \text{ or } y_2.$$

Here $[(n-1/2)]$ denotes the integral part of $(n-1)/2$.

5. Special equations. Theorem 4.3 gives us our special equations for the cases $\mu = 2\cos(2\pi/n)$, $n = 2M$ or $2M-1$, $M \geqq 2$. We can write the typical one as

$$(49) \quad I(n) = \sum_{i=0}^{M-2} f_i J^i,$$

where each f_i is a power series in z. Or we can write it at greater length as follows.

$$\begin{aligned}(50) \quad (1 - y^{-1})B \prod_{m=1}^{M-1} [Q_m^{\,2}\{\lambda^{-1}z^2 l(y, z) - (1-y)^{-1}yz^2\} \\ - \{Q_{m-1}y^{-1} + A_m - Q_m\}\{-Q_{m+1}y^{-1} + A_{m+1} + Q_m\}] \\ = \sum_{i=0}^{M-2} f_i\{\lambda^{-1}z^2 l(y, z) - (1-y)^{-1}yz^2 + y^{-2} - y^{-1}\}^i.\end{aligned}$$

Here $B = 1$ if $n = 2M-1$ and $B = A_M + y^{-1}$ if $n = 2M$. We write $l(y, z)$ for the function $l(y, z, \lambda)$ with λ given its constant value.

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We observe that if $m = 1$ the expression $Q_{m-1}y^{-1} + A_m - Q_m$ vanishes. Hence the factor for $m = 1$ under the product sign on the left of (50) always reduces to

$$\lambda^{-1}z^2 l(y, z) - (1 - y)^{-1}yz^2.$$

Let us call this the *leading factor* of the equation. We can combine it with the factor $(1 - y^{-1})$ and so reduce the number of occurrences of $(1 - y)^{-1}$.

In the cases $n = 3$ and $n = 4$ Equation (50) reduces to

$$z^2(1 - y^{-1})l(y, z) + z^2 = f_0$$

and

$$2z^2y^{-1}(1 - y^{-1})l(y, z) + y^{-1}z^2 = f_0$$

respectively. Putting $y = 1$ we find that $f_0 = z^2$ in each case. We then determine $l(y, z)$ as 0 and $2y$ respectively, as was to be expected.

In the cases $n = 5$ and $n = 6$ Equation (50) reduces to the equations for $l(y, z)$ discussed in Papers **II** and **III**. The author has made a direct verification of (50) in the case $n = 7$. A special identity between chromials of triangulations, valid for $\lambda = 2 + 2 \cos (2\pi/7)$, was obtained by substituting this simplifying value of λ in the equations for the six-ring given by D. W. Hall and D. C. Lewis [**1**]. This identity was converted into an equation for $l(y, z)$ by an appropriate process of summation.

As stated in the Introduction we do not attempt the general solution of (50) in this paper, being content to exhibit the equation as evidence for the chromatic significance of the "Beraha numbers". We ought however to satisfy ourselves that Equation (50) determines $l(y, z)$ uniquely.

Putting $z = 0$ in (50) and leaving y unrestricted we find that each of the power series f_i divides by z^2.

We can multiply out both sides of (50) and then equate the numerators of the terms in $(1 - y)^{M-2}$, after putting $y = 1$ in them. We thus obtain

$$(51) \quad f_{M-2} = z^2 \prod_{m=1}^{M-1} Q_m^2 \quad \textit{or} \quad (1 + A_m)z^2 \prod_{m=1}^{M-1} Q_m^2$$

according as n is odd or even.

Let l_j denote the coefficient of z^{2j} in $l(y, z)$, and let f_{ij} denote the coefficient of z^{2j} in f_i.

Let P_r, where r is any non-negative integer, denote the following proposition. *The coefficient l_j is uniquely determined by* (50) *if $j < r$, and f_{ij} is uniquely determined by* (50) if $j \leqq r$.

Since each f_i divides by z^2, as a consequence of (50), the proposition P_0 is true. Let us assume that P_r is true whenever r is less than some positive integer b, and let us discuss P_b. We may assume $n \geqq 5$.

First we rewrite (50) as follows, multiplying by

$$y^{2M-3}(1 - y)^{M-2}$$

and isolating the leading factor.

$$(52)\quad By\cdot[\lambda^{-1}z^2(y-1)y^{-1}l(y,z)+z^2]$$
$$\times\prod_{m=2}^{M-1}[Q_m{}^2\{\lambda^{-1}z^2y^2(1-y)l(y,z)-y^3z^2\}$$
$$+(y-1)\{Q_{m-1}+y(A_m-Q_m)\}\{-Q_{m+1}+y(A_{m+1}+Q_m)\}]$$
$$=\sum_{i=0}^{M-2}[f_i\{\lambda^{-1}z^2y^2(1-y)l(y,z)-y^3z^2+(1-y)^2\}^i$$
$$\times y^{2M-3-2i}(1-y)^{M-2-i}].$$

We note that the expression on the right divides by y. Equating coefficients of y^0 we find that each l_j divides by y. This is of course evident from the definition of $l(y,z)$ as a chromatic sum, but we wish to show that all necessary information about $l(y,z)$, granted only that it is a z-restricted power series in y and z, is contained in (50).

We now equate coefficients of z^{2b} in (52), finding that

$$(53)\quad \{\lambda^{-1}(y^{-1}l_{b-1})(y-1)+\delta_{1,b}\}(y-1)^{M-2}Y=(1-y)^{M-2}Z+A,$$

where

$$(54)\quad Y=By\cdot\prod_{m=2}^{M-1}[\{Q_{m-1}+y(A_m-Q_m)\}\{-Q_{m+1}+y(A_{m+1}+Q_m)\}],$$

$$(55)\quad Z=\sum_{i=0}^{M-2}f_{ib}(1-y)^iy^{2M-3-2i}$$

and A is a polynomial in y that can be calculated from those coefficients l_j and f_{ij} that are supposed known.

Since $(y^{-1}l_{b-1})$ is a polynomial in y we deduce from (53) that $(1-y)^{M-2}$ divides A. We can therefore determine a polynomial X_0 in y such that

$$(56)\quad A-\delta_{1,b}(y-1)^{M-2}Y=(1-y)^{M-2}X_0.$$

We now observe that $(y-1)Y$ divides $Z+X_0$. Before we can use this result we must make a more detailed study of the polynomial Y.

We note first that yB is y or $1+A_My$ according as n is odd or even. In the latter case

$$A_M=-\mu/(\mu-2)\neq 0,$$

by (39).

We note next that $A_2-Q_2=\mu-\mu=0$. Suppose on the other hand that $A_m-Q_m=0$ for some integer m satisfying $0\leqq m\leqq n$. Then $A_{m+1}=\mu Q_m=\mu A_m$, by (36), and $A_{m-1}=1$, by (37). It follows from (39) that

$$\cos((2m-3)\alpha/2)=\cos(\alpha/2),$$

i.e., $m=1$ or 2.

We have $A_{m+1} + Q_m = A_{n-m} - Q_{n-m}$. Hence if $A_{m+1} + Q_m = 0$ for $0 \leqq m \leqq n$ we must have $m = n - 1$ or $n - 2$, by the preceding result.

We deduce that $A_m - Q_m$ is non-zero for $3 \leqq m \leqq M - 1$, and that $A_{m+1} + Q_m$ is non-zero for $2 \leqq m \leqq M - 1$. It follows that the degree of Y is $2M - 4$.

We note finally that if $2 \leqq m \leqq M - 1$ then Q_{m-1} and Q_{m+1} are non-zero, by (38), except that if n is even and $m = M - 1$ then $Q_{m+1} = Q_M = 0$. We deduce that Y always divides by y, and that the coefficient of y^1 in Y is always non-zero.

The degree of Z is at most $2M - 3$, by (55), and that of $(y - 1)Y$ is exactly $2M - 3$. The remainder X_1 after division of X_0 by $(y - 1)Y$ has degree at most $2M - 4$. Since $(y - 1)Y$ divides $X_0 + Z$ there must be a number η such that

$$(57) \quad X_1 + Z = \eta(y - 1)Y.$$

But the coefficient of y^1 in $(y - 1)Y$ is non-zero, and the coefficients of y^1 in X_1 and Z are known. (See (51).) Hence η is uniquely determined, and the coefficients f_{ib} can be found from (57). The coefficient l_{b-1} is now determined by (53).

We have deduced from our assumptions that P_b is true. It follows by induction that Equation (50) completely determines the z-restricted power series $l(y, z)$.

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