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GROWTH OF HYPERCYCLIC FUNCTIONS: A CONTINUOUS PATH BETWEEN $\mathcal{U}$-FREQUENT HYPERCYCLICITY AND HYPERCYCLICITY

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Abstract We are interested in the optimal growth in terms of L^p-averages of hypercyclic and $\mathcal{U}$-frequently hypercyclic functions for some weighted Taylor shift operators acting on the space of analytic functions on the unit disc. We unify the results obtained by considering intermediate notions of upper frequent hypercyclicity between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity.

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1. Introduction

A linear operator on a Fréchet space X is said to be *hypercyclic* if there is a vector $x \in X$ such that for every non-empty open set $U \subset X$ the set $N(x,U) := \{n \in \mathbb{N} : T^n x \in U\}$ is infinite, where (T^n) is the sequence of iterates of T. In this situation, x is called a *hypercyclic* vector. Further there are more precise and stringent notions that allow to quantify how often a hypercyclic vector visits a non-empty open set. A linear operator on a Fréchet space X is said to be *frequently hypercyclic* (resp. *$\mathcal{U}$-frequently hypercyclic*) if there is a vector $x \in X$ such that for every non-empty open set $U \subset X$ the set $N(x,U)$ has positive lower (resp. upper) density, where the lower and upper densities of a subset $A \subset \mathbb{N}$ are defined respectively as follows:

$$\underline{d}(A) = \liminf_{n\to+\infty} \frac{\#A \cap \{1,\dots,n\}}{n} \quad \text{and} \quad \overline{d}(A) = \limsup_{n\to+\infty} \frac{\#A \cap \{1,\dots,n\}}{n}.$$

These notions were introduced by Bayart and Grivaux [1] and Shkarin [26]. The dynamics of linear operators is a very active branch of research: we refer the reader to [2, 21] and the references therein for background in linear dynamics. Clearly a frequently hypercyclic vector is $\mathcal{U}$-frequently hypercyclic and a $\mathcal{U}$-frequently hypercyclic vector is hypercyclic. Classical examples of frequently or $\mathcal{U}$-frequently hypercyclic operators are given by suitable weighted shifts. As usual we denote by $\mathbb{D}$ the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane and by $H(\mathbb{D})$ the set of analytic functions in $\mathbb{D}$. It is well known that $H(\mathbb{D})$ endowed with the topology of uniform convergence on compact subsets is a Fréchet space. For $\alpha \in \mathbb{R}$, let $w(\alpha) = (w_n(\alpha))$ be the weighted sequence of non-zero complex numbers given by, for all $n \geq 1$,

$$w_n(\alpha) = \left(1 + \frac{1}{n}\right)^{\alpha}.$$

In the present paper, we consider the associated weighted Taylor shift:

$$T_\alpha : H(\mathbb{D}) \to H(\mathbb{D}) \text{ given by } T_\alpha\Big(\sum_{k\geq 0} a_k z^k\Big) = \sum_{k\geq 0} a_{k+1} w_{k+1}(\alpha) z^k.$$

For $\alpha = 0$, T_0 is the classical Taylor shift operator. It is easy to check that for every real number α, T_α is a frequently hypercyclic operator. For instance, we refer the reader to [4, 20, 25]. The problem of determining possible rates of growth of frequently hypercyclic functions for T_α in terms of L^p averages was studied in [25] (see [24] for the case $\alpha = 0$ too). For $0 < r < 1$ and $f \in H(\mathbb{D})$, we consider the classical integral means:

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\right)^{1/p} (1 \leq p < \infty) \text{ and } M_\infty(f,r) = \sup_{0\leq t\leq 2\pi} |f(r\mathrm{e}^{\mathrm{i}t})|.$$

In the same way, for any holomorphic polynomial P let us define, for $p \geq 1$,

$$\|P\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |P(\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\right)^{1/p} \text{ and } \|P\|_\infty = \sup_{0\leq t\leq 2\pi} |P(\mathrm{e}^{\mathrm{i}t})|.$$

In the following, for all $p > 1$, q will stand for the exponent conjugate to p, i.e. $\frac{1}{p} + \frac{1}{q} = 1$ and we will adopt the convention $q = \infty$ if $p = 1$. For $1 \leq p \leq \infty$, the authors recently highlighted a *critical exponent*, i.e. a value of the parameter α from which the L^p-growth of a frequently hypercyclic function for T_α no longer has the same behaviour. In the case of frequent hypercyclicity for T_α, the critical exponent is equal to $\alpha = \frac{1}{\max(2,q)}$. Indeed the authors obtained the following statements. First for $p > 1$ they proved the following result.

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Theorem 1.1. **([25, Theorem 1.2])** *Let $\alpha \in \mathbb{R}$. The following assertions hold*

(1) For any $1 < p < +\infty$ there is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C>0$ such that for every $0 < r < 1$

$$M_p(f,r) \leq \begin{cases} C(1-r)^{\alpha - \frac{1}{\max(2,q)}} & \text{if } \alpha < \frac{1}{\max(2,q)}, \\ C|\log(1-r)|^{\frac{1}{p}} & \text{if } \alpha = \frac{1}{\max(2,q)}, \\ C & \text{if } \alpha > \frac{1}{\max(2,q)}. \end{cases}$$

These estimates are optimal: every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α is bounded from below by the corresponding previous estimate depending on α.

(2) There is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C>0$ such that for every $0 < r < 1$

$$M_\infty(f,r) \leq \begin{cases} C(1-r)^{\alpha - \frac{1}{2}} & \text{if } \alpha < 1/2, \\ C|\log(1-r)| & \text{if } \alpha = 1/2, \\ C & \text{if } \alpha > 1/2. \end{cases}$$

For $\alpha \neq 1/2$, these estimates are optimal: every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α is bounded from below by the corresponding previous estimate depending on α.

For $p=1$, the following result holds. For any positive integer $\ell \geq 1$, $\log_\ell$ stands for $\log \circ \cdots \circ \log$ where log appears ℓ times.

Theorem 1.2. **([25, Proposition 4.1 and Theorem 4.4])**
For any $\ell \geq 1$, there is a frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfying the following estimates: there exists $C>0$ such that for every $0 < r < 1$ sufficiently large

$$M_1(f,r) \leq \begin{cases} C(1-r)^{\alpha}\log_\ell(-\log(1-r)) & \text{if } \alpha < 0, \\ C|\log(1-r)|\log_\ell(-\log(1-r)) & \text{if } \alpha = 0, \\ C & \text{if } \alpha > 0. \end{cases}$$

Moreover every frequently hypercyclic function f in $H(\mathbb{D})$ for T_α satisfies the following estimates:

$$\liminf_{r\to 1^-}\left[M_1(f,r)(1-r)^{-\alpha}\right] > 0 \text{ if } \alpha < 0, \quad \liminf_{r\to 1^-}\left[\frac{M_1(f,r)}{-\log(1-r)}\right] > 0 \text{ if } \alpha = 0,$$

$$\liminf_{r\to 1^-}\left[M_1(f,r)\right] > 0 \text{ if } \alpha > 0.$$

It should be noted that the study of the growth of hypercyclic or frequently hypercyclic functions started with those related to the differentiation operator on $H(\mathbb{C})$

(see for instance [6, 8, 18, 19]) but was also recently extended to the partial differentiation operator [16] or the Dunkl operator [3]. Here, as a first step, we obtain sharp results on the permissible rates of L^p-growth of hypercyclic and $\mathcal{U}$-frequently hypercyclic functions for T_α. On one hand, for hypercyclicity, for any $1 \leq p \leq \infty$ we find that the rate of growth $(1-r)^{\min(\alpha,0)}$ turns out to be critical and hence, for any $1 \leq p \leq \infty$, $\alpha = 0$ is the critical exponent. Observe that in this case the critical exponent does not depend on p. We refer to Theorem 2.2. In particular this result states that for $1 \leq p \leq \infty$ there is no hypercyclic function f for T_α satisfying $\limsup_{r\to 1^-}((1-r)^{-\alpha} M_p(f,r)) < +\infty$ if $\alpha \leq 0$ while if $\alpha > 0$ there exist hypercyclic functions f for T_α such that the average $M_p(f,r)$ is bounded. In passing Theorem 2.2 gives a negative answer to a question of [25] which asked if for $\alpha < 0$ there is a frequently hypercyclic function g_α for T_α such that $\limsup_{r\to 1^-}((1-r)^{-\alpha} M_1(g_\alpha, r)) < +\infty$. On the other hand, for $\mathcal{U}$-frequent hypercyclicity, we find the same critical exponent $\alpha = \frac{1}{\max(2,q)}$ as for frequent hypercyclicity. Therefore contrary to the previous case this exponent depends on p. Moreover we show that the $\mathcal{U}$-frequently hypercyclic functions and the frequently hypercyclic functions for T_α share the same admissible (and optimal) L^p-growth when α is different from the critical exponent, i.e. $\alpha \neq \frac{1}{\max(2,q)}$. Concerning the case $\alpha = \frac{1}{\max(2,q)}$ with $1 \leq p \leq \infty$, we prove that every $\mathcal{U}$-frequently hypercyclic vector f for T_α satisfies $\limsup_{r\to 1^-} M_p(f,r) = +\infty$, without a priori additional information on the growth of the function. We refer to Theorems 3.3 and 3.5. Nevertheless several questions remain and need to be addressed. The first question that comes to mind is the following: what is the optimal boundary growth of $\mathcal{U}$-frequently hypercyclic functions for T_α when α is the critical exponent? Further if we go back to what was just said, we see that for $p = 1$ the critical exponent is always equal to 0 for the hypercyclic case, the $\mathcal{U}$-frequently case and the frequently hypercyclic case. But surprisingly for $p > 1$ this critical exponent is equal to $\frac{1}{\max(2,q)}$ for the $\mathcal{U}$-frequently or frequently hypercyclic cases and is equal to zero for the hypercyclic case. Thus, as a second question, we can wonder about what happens between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity. Why does the critical exponent go from $\frac{1}{\max(2,q)}$ to zero? In order to understand this phenomenon, we introduce intermediate notions of linear dynamics between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity: $\mathcal{U}_{\beta\gamma}$-frequent hypercyclicity related to notions of upper weighted densities $\overline{d}_{\beta\gamma}$, with $0 \leq \gamma \leq 1$ a continuous parameter, where we replace in the definition of $\mathcal{U}$-frequent hypercyclicity the natural upper density $\overline{d}$ by $\overline{d}_{\beta\gamma}$. Moreover for $\gamma = 0$ $\mathcal{U}_{\beta 0}$-frequent hypercyclicity will coincide with frequent hypercyclicity and for $\gamma = 1$ $\mathcal{U}_{\beta 1}$-frequent hypercyclicity will coincide with hypercyclicity. Further for any $0 \leq \gamma \leq \gamma' \leq 1$ and for any subset $E \subset \mathbb{N}$, the following chain of inequalities $\overline{d}(E) \leq \overline{d}_{\beta\gamma}(E) \leq \overline{d}_{\beta\gamma'}(E) \leq \overline{d}_{\beta 1}(E)$ will show that the notions of $\mathcal{U}_{\beta\gamma}$-frequent hypercyclicity for $\gamma \in (0,1)$ furnish refined notions of linear dynamics between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity. We refer the reader to the beginning of § 4 for the main definitions and properties. Similar notions of weaker densities have been recently studied in the context of linear dynamics (see for instance [5, 7, 11–13, 23] and the references therein). In the present paper, we investigate the growth in terms of L^p-averages of $\mathcal{U}_{\beta\gamma}$-frequently hypercyclic functions for T_α. In particular, for $0 < \gamma < 1$, and for $p > 1$ we find that the critical exponent is given by $\alpha = \frac{1-\gamma}{\max(2,q)}$. Hence let us observe that this critical exponent:

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- tends to $\frac{1}{\max(2,q)}$ as γ tends to zero, i.e. tends to the critical exponent for $\mathcal{U}$-frequent hypercyclicity;
- tends to 0 as γ tends to 1, i.e. tends to the critical exponent for hypercyclicity.

These estimates thus allow to highlight *a continuous path* between the rate of growth of hypercyclic and $\mathcal{U}$-frequently hypercyclic functions: the growth (in terms of L^p-averages) of a hypercyclic function for T_α continuously depends on the frequency of visits (measured by the densities $\overline{d}_{\beta\gamma}$, $0 \le \gamma \le 1$) of non-empty open subsets by its orbit under the action of T_α. We also show that the estimates on the growth of $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions that we obtained are optimal. To do this, we apply a method based on the use of Rudin–Shapiro polynomials and inspired by a construction of frequently hypercyclic functions with optimal growth for differentiation operator on $H(\mathbb{C})$ due to Drasin and Saksman [8] and that has also been adapted for the proofs of Theorems 1.1 and 1.2 in [24, 25]. For all these results, we refer the reader to Theorems 4.5, 4.10 and 4.12. Finally let us return to the first question mentioned above. In the last section, we answer it by showing that the optimal growth of $\mathcal{U}$-frequently and $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions for T_α coincides whenever α is the critical exponent: actually the L^p-growth can be arbitrarily slow as in the hypercyclic case. We refer to Theorem 5.12.

The paper is organized as follows. In § 2 and 3, we establish the boundary behaviour of hypercyclic functions and $\mathcal{U}$-frequently hypercyclic functions for T_α respectively. In § 4, we deal with the $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions for T_α. In § 5, we turn our attention to the specific case of critical exponent.

Throughout the paper, whenever A and B depend on some parameters, we will use the notation $A \lesssim B$ (resp. $A \gtrsim B$) to mean $A \le CB$ (resp. $A \ge CB$) for some constant $C > 0$ that does not depend on the involved parameters.

2. Growth of hypercyclic functions

In this section, we are going to establish the rate of growth of hypercyclic functions with respect to the weighted Taylor shift operator T_α. To do this, inspired by the proofs of [18, Theorem (A)] and [3, Theorem 3], where the authors are interested in the rate of growth of hypercyclic functions with respect to the Mac-Lane operator or the Dunkl operator respectively, we need an important tool in linear dynamics: the Universality Criterion. Indeed a natural extension of the notion of hypercyclicity is the concept of universality. A sequence of continuous linear mappings $L_n : X \to Y$ between topological vector spaces X, Y is said to be *universal* whenever there exists a vector $x \in X$ such that the set $\{L_n x\ ;\ n \in \mathbb{N}\}$ is dense in Y. Such a vector x is called a *universal* vector for (L_n). Observe that an operator $T : X \to X$ is hypercyclic if and only if the sequence (T^n) is universal. The following result which is known as the Universality Criterion furnishes a sufficient condition for universality [17]. It is a refined version of the hypercyclicity criterion [15, 22].

Theorem 2.1. (Universality Criterion) *Assume that X and Y are topological vector spaces, such that X is a Baire space and Y is separable and metrizable. Let $L_j : X \to Y$ be a sequence of continuous linear mappings. Suppose that there are dense subsets X_0 of X and Y_0 of Y and mappings $S_j : Y_0 \to X$ such that*

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(i) for every $x \in X_0$, $L_j x \to 0$,
(ii) for every $y \in Y_0$, $S_j y \to 0$,
(iii) for every $y \in Y_0$, $(L_j S_j)y \to y$.

Then (L_j) *is universal and the set of universal vectors for* (L_j) *is residual in* X.

Now we are ready to obtain the critical rate of growth for hypercyclic functions with respect to the weighted Taylor shift operator T_α. The following statement holds.

Theorem 2.2. *Let* $1 \le p \le \infty$.

(1) *Let* $\alpha \le 0$.
   (a) *For any function* $\varphi : [0,1) \to \mathbb{R}_+$ *with* $\varphi(r) \to \infty$ *as* $r \to 1^-$ *there is a hypercyclic function* f *for* T_α *with*

$$M_p(f,r) \lesssim \varphi(r)(1-r)^\alpha \quad \text{for } 0<r<1 \text{ sufficiently close to } 1.$$

   (b) *There is no hypercyclic function* f *for* T_α *that satisfies, for* $0<r<1$

$$M_p(f,r) \le C(1-r)^\alpha,$$

   where $C>0$.
(2) *Let* $\alpha>0$.
   (a) *There is a hypercyclic function* f *for* T_α *with*

$$M_p(f,r) \le C$$

   for some $C>0$.
   (b) *For any function* $\varphi : [0,1) \to \mathbb{R}_+$ *with* $\varphi(r) \to 0$ *as* $r \to 1^-$, *there is no hypercyclic function* f *for* T_α *that satisfies, for* $0<r<1$

$$M_p(f,r) \le \varphi(r).$$

Proof. Since we have for $1 \le p \le l$

$$M_p(f,r) \le M_l(f,r), \qquad \text{for } 0<r<1,$$

it suffices to prove assertions (1a) and (2a) for $M_\infty(f,r)$ and assertions (1b) and (2b) for $M_1(f,r)$.

(1) We begin by the case $\alpha \le 0$.
First we can assume without loss of generality that the function φ is increasing and continuous with $\varphi(0)>0$. Let us consider the space X of all functions f in $H(\mathbb{D})$

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with $f(z) = \sum_{k\geq 0} a_k z^k$ satisfying for any $n \geq 0$, $\rho_n(f) < +\infty$ and $\rho_n(f) \to 0$ as $n \to +\infty$, where

$$\rho_n(f) = \sup_{|z|<1} \left\{ \left| \sum_{k=n}^{+\infty} a_k z^k \right| (1-|z|)^{-\alpha} [\varphi(|z|)]^{-1} \right\}.$$

It is easy to check that X endowed with the norm $\|.\| = \sup_n \rho_n(.)$ is a Banach space. Therefore $(X, \|.\|)$ is a Baire space. For all integer j, let $L_j : X \to H(\mathbb{D})$ be the operator given by $L_j f = T^j f$. Clearly (L_j) is a sequence of continuous linear operators. We choose $X_0 = Y_0 = \mathcal{P}$ the set of polynomials. The set $\mathcal{P}$ is dense in $H(\mathbb{D})$. Moreover, setting the polynomial $s_N(f) = \sum_{k=0}^N a_k z^k$ we get

$$\rho_n(f - s_N(f)) = \begin{cases} \rho_n(f) & \text{for } n \geq N+1 \\ \rho_{N+1}(f) & \text{for } n \leq N \end{cases}$$

which implies $\|f - s_N(f)\| = \sup_{n\geq N+1} \rho_n(f) \to 0$ as N tends to infinity. Hence $\mathcal{P}$ is dense in X. Then we define the operators S_j as follows

$$S_j : \mathcal{P} \to X, \quad S_j\left(\sum_{k=0}^n a_k z^k\right) = \sum_{k=0}^n a_k \frac{(k+1)^\alpha}{(k+j+1)^\alpha} z^{k+j}.$$

Clearly we have, for all $P \in \mathcal{P}$,

$$L_j(P) \to 0, \text{ as } j \to +\infty, \quad \text{and } L_j S_j(P) = P.$$

Now we prove that, for all $P \in \mathcal{P}$, $S_j(P) \to 0$, as $j \to +\infty$. Since $S_j(z^k) = (k+1)^\alpha S_{j+k}(1)$, it suffices to show that $S_j(1) \to 0$, as $j \to +\infty$. To do this, observe that

$$\|S_j(1)\| = \sup_{0<r<1} \frac{r^j (1-r)^{-\alpha}}{(j+1)^\alpha \varphi(r)}.$$

Let us define $h_j : [0,1) \to \mathbb{R}_+$ given by $h_j(r) = \frac{r^j(1-r)^{-\alpha}}{\varphi(r)}$. We have $h_j(0) = 0 = \lim_{r\to 1^-} h_j(r)$. Let $0 < r_j < 1$ with $h_j(r_j) = \sup_{0<r<1} \frac{r^j(1-r)^{-\alpha}}{\varphi(r)}$. If $r_{j+1} < r_j$, we get

$$h_{j+1}(r_{j+1}) = r_{j+1} h_j(r_{j+1}) < r_j h_j(r_{j+1}) \leq r_j h_j(r_j) = h_{j+1}(r_j)$$

which gives a contradiction. Hence the sequence (r_j) is increasing. If $r_j \to \gamma$ with $\gamma < 1$, then

$$\|S_j(1)\| = (j+1)^{-\alpha} h_j(r_j) \leq (j+1)^{-\alpha} \frac{\gamma^j}{\varphi(0)} \to 0, \text{ as } j \to +\infty.$$

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Otherwise $r_j \to 1$ and

$$\|S_j(1)\| \le \frac{(j+1)^{-\alpha}}{\varphi(r_j)} \left(\frac{j}{j-\alpha}\right)^j \left(1 - \frac{j}{j-\alpha}\right)^{-\alpha} \to 0, \text{ as } j \to +\infty.$$

Thus we have $\|S_j(1)\| \to 0$ as j tends to infinity. We apply the universality criterion to obtain universal elements for the sequence (L_j) that are hypercyclic functions for T_α satisfying the growth condition required.

For assertion (1b), assume that $f = \sum_{k\ge 0} a_k z^k$ is a function in $H(\mathbb{D})$ with, for all $0 < r < 1$, $M_1(f,r) \le C(1-r)^\alpha$, for some $C > 0$. By Cauchy estimates we get

$$|a_n| \le \frac{M_1(f,r)}{r^n}.$$

We obtain, for all $n \ge 0$ and all $0 < r < 1$,

$$|a_n w_1(\alpha)\dots w_n(\alpha)| \le C \frac{|w_1(\alpha)\dots w_n(\alpha)|}{r^n}(1-r)^\alpha.$$

Hence we get for all $n \ge 0$,

$$|a_n(n+1)^\alpha| \le C \frac{(n+1)^\alpha}{e^{-n/(n+1)}}(1 - e^{-1/(n+1)})^\alpha$$

which is bounded. Hence f cannot be hypercyclic for T_α.

(2) Now let us consider the case $\alpha > 0$.
First we can assume without loss of generality that the function φ is increasing and continuous with $\varphi(0) > 0$. Let us consider the Banach space $(H^\infty(\mathbb{D}), \|.\|)$

$$H^\infty(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) \; ; \; \|f\| := \sup_{0<r<1} M_\infty(f,r) < \infty \right\}$$

which is continuously embedded in $H(\mathbb{D})$. We set X the closure of the polynomials in $H^\infty(\mathbb{D})$. Let us define the sequences (L_j) and (S_j) of linear operators as in the previous case. Clearly we have, for all $P \in \mathcal{P}$, where $\mathcal{P}$ is the set of polynomials,

$$L_j(P) \to 0, \text{ as } j \to +\infty, \quad \text{ and } L_j S_j(P) = P.$$

Now we prove that, for all $P \in \mathcal{P}$, $S_j(P) \to 0$, as $j \to +\infty$. To do this, it suffices to show that $S_j(1) \to 0$, as $j \to +\infty$. Since

$$\|S_j(1)\| \le \frac{1}{(j+1)^\alpha}$$

and $\alpha > 0$ we have $\|S_j(1)\| \to 0$ as j tends to infinity. We apply the universality criterion to obtain universal elements for the sequence (L_j). These universal vectors are clearly hypercyclic functions for T_α satisfying the growth condition required.

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For assertion (2b), assume that $f = \sum_{k\geq 0} a_k z^k$ is a function in $H(\mathbb{D})$ with, for all $0 < r < 1$, $M_1(f, r) \leq \varphi(r)$, where $\varphi : [0, 1) \to \mathbb{R}_+$ is a function such that $\varphi(r) \to 0$ as $r \to 1^-$. We obviously may assume that φ is continuous and decreasing. By Cauchy estimates, we get

$$|a_n| \leq \frac{\varphi(r)}{r^n}.$$

We obtain, for all $n \geq 0$ and all $0 < r < 1$,

$$|a_n(n+1)^\alpha| \leq \frac{(n+1)^\alpha \varphi(r)}{r^n}.$$

Let us choose a sequence (r_n) such that $r_n \geq \max(1 - 1/n, \varphi^{-1}((n+1)^{-\alpha}))$. Hence we get, for all $n \in \mathbb{N}$,

$$|a_n(n+1)^\alpha| \leq \frac{(n+1)^\alpha \varphi(r_n)}{r_n^n} \leq \mathrm{e}^{-n\log(1-1/n)}$$

which is bounded. Hence f cannot be hypercyclic for T_α.

□

Remark 2.3. For $\alpha < 0$, Theorem 2.2 ensures that every hypercyclic function f for T_α satisfies

$$\limsup_{r\to 1^-} \left[(1-r)^{-\alpha} M_1(f, r)\right] = +\infty.$$

Since a frequently hypercyclic function is necessarily hypercyclic, this observation gives a negative answer to the first part of the question from Remark 4.5 of [25] which asked if for $\alpha < 0$ there is a frequently hypercyclic function g_α for T_α such that $\limsup_{r\to 1^-}((1-r)^{-\alpha} M_1(g_\alpha, r)) < +\infty$.

3. Growth of $\mathcal{U}$-frequently hypercyclic functions

In this section, we are interested in the growth of $\mathcal{U}$-frequently hypercyclic functions for T_α. First of all, in the sequel, we will need the following easy lemmas.

Lemma 3.1. *Let $N \in \mathbb{N}$. Let A_N be a subset of $\{1, \ldots, N\}$. For all $\gamma \in \mathbb{R} \setminus \{-1\}$ the following estimate holds*

$$\sum_{k\in A_N} (k+1)^\gamma \geq \begin{cases} \dfrac{(\# A_N + 1)^{\gamma+1} - 1}{\gamma + 1} & \text{if } \gamma \geq 0, \\ \dfrac{(N+2)^{\gamma+1}}{\gamma+1}\left(1 - \left(1 - \dfrac{\# A_N}{N+2}\right)^{\gamma+1}\right) & \text{if } \gamma < 0,\ \gamma \neq -1. \end{cases}$$

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Proof. We begin by the case $\gamma \geq 0$. We write

$$\sum_{k\in A_N}(k+1)^\gamma \geq \sum_{k=1}^{\#A_N}(k+1)^\gamma \geq \int_0^{\#A_N}(t+1)^\gamma\,\mathrm{d}t,$$

which gives the announced result.

For $\gamma < 0$ with $\gamma \neq -1$, we obtain in an analogue way

$$\sum_{k\in A_N}(k+1)^\gamma \geq \sum_{k=N-\#A_N+1}^{N}(k+1)^\gamma \geq \int_{N-\#A_N+1}^{N+1}(t+1)^\gamma\,\mathrm{d}t,$$

which allows to finish the proof. □

Lemma 3.2. *Let (u_k) and (v_k) be two sequences of non-negative real numbers. Assume that (v_k) is decreasing. For any increasing sub-sequence $(N_j)\subset\mathbb{N}$, the following inequality holds, for all $l \geq 1$:*

$$\sum_{k=1+N_0}^{N_l}u_kv_k \geq S_{N_l}v_{N_l} - S_{N_0}v_{N_0} + \sum_{j=1}^{l}S_{N_{j-1}}(v_{N_{j-1}} - v_{N_j}),$$

with $S_N = \sum\limits_{k=1}^{N}u_k$.

Proof. Observe that (S_N) is increasing and (v_k) is decreasing. Thus by using a summation by parts we derive

$$\begin{aligned}\sum_{k=1+N_0}^{N_l}u_kv_k &= \sum_{j=1}^{l}\sum_{k=1+N_{j-1}}^{N_j}u_kv_k\\ &= \sum_{j=1}^{l}\left(S_{N_j}v_{N_j} - S_{N_{j-1}}v_{N_{j-1}} + \sum_{k=N_{j-1}}^{N_j-1}S_k(v_k - v_{k+1})\right)\\ &= S_{N_l}v_{N_l} - S_{N_0}v_{N_0} + \sum_{j=1}^{l}\sum_{k=N_{j-1}}^{N_j-1}S_k(v_k - v_{k+1})\\ &\geq S_{N_l}v_{N_l} - S_{N_0}v_{N_0} + \sum_{j=1}^{l}S_{N_{j-1}}(v_{N_{j-1}} - v_{N_j}).\end{aligned}$$

□

We are ready to establish the boundary behaviour of $\mathcal{U}$-frequently hypercyclic functions for T_α. Actually we are going to prove that these functions share the same optimal growth

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as frequently hypercyclic functions except in the case where α is the critical exponent, for which we will show in § 5 that the growth can be arbitrarily slow. We begin by the case $p>1$.

Theorem 3.3. *Let f be a $\mathcal{U}$-frequently hypercyclic function for the operator T_α and $1<p\leq\infty$. Then the following estimates hold*

$$\limsup_{r\to 1^-}\left((1-r)^{\frac{1}{\max(2,q)}-\alpha}M_p(f,r)\right)>0,\qquad \text{if } \alpha<\frac{1}{\max(2,q)},$$
$$\limsup_{r\to 1^-}(M_p(f,r))=+\infty,\qquad \text{if } \alpha=\frac{1}{\max(2,q)},$$
$$\limsup_{r\to 1^-}M_p(f,r)>0,\qquad \text{if } \alpha>\frac{1}{\max(2,q)}.$$

For $\alpha\neq\frac{1}{\max(2,q)}$, these results are optimal in the following sense: for all $p>1$, there exists a $\mathcal{U}$-frequently hypercyclic function for T_α such that, for every $0<r<1$,

$$M_p(f,r)\lesssim\begin{cases}(1-r)^{\alpha-\frac{1}{\max(2,q)}} & \text{if } \alpha<\frac{1}{\max(2,q)}\\ 1 & \text{if } \alpha>\frac{1}{\max(2,q)}.\end{cases}$$

Remark 3.4. For the critical value $\alpha=\frac{1}{\max(2,q)}$, the optimality of the rate of growth in this theorem will be obtained in Theorem 5.7 further in the paper.

Proof. We write $f=\sum_{k\geq 0}\frac{a_k}{(k+1)^\alpha}z^k$. Since f is $\mathcal{U}$-frequently hypercyclic there exists an increasing sub-sequence $(n_k)\subset\mathbb{N}$ with positive upper density such that for all $k\geq 1$

$$|T_\alpha^{n_k}f(0)-3/2|=|a_{n_k}-3/2|<1/2.$$

We get, for all $k\geq 1$, $|a_{n_k}|\geq 1$. Set $I=\{n_k\ :\ k\geq 1\}$ and for all $N\geq 1$, $I_N=I\cap\{1,\dots,N\}$. The hypothesis $\overline{d}(I)>0$ ensures that there exist $0<C<1$ and an increasing sequence (N_l) of positive integers such that

$$\#I_{N_l}\geq CN_l. \tag{1}$$

Up to take a sub-sequence, we can also assume that

$$C(N_{l+1}+1)\geq N_l+1. \tag{2}$$

Let us consider, for all $l\geq 1$, $1-\frac{1}{N_l-1}\leq r_l<1-\frac{1}{N_l}$. Thus, we derive

$$N_l-1\leq\frac{1}{1-r_l}<N_l. \tag{3}$$

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(1) **Case** $2 \leq p \leq \infty$:
Jensen's inequality and Parseval's Theorem give

$$[M_p(f, r_l)]^2 \geq [M_2(f, r_l)]^2 = \sum_{k \geq 0} \frac{|a_k|^2}{(k+1)^{2\alpha}} r_l^{2k} \geq \sum_{k=1}^{N_l} \frac{|a_k|^2}{(k+1)^{2\alpha}} r_l^{2k}.$$

Thus, we deduce

$$[M_p(f, r_l)]^2 \geq \left(1 - \frac{1}{N_l - 1}\right)^{2N_l} \sum_{k=1}^{N_l} \frac{|a_k|^2}{(k+1)^{2\alpha}} \gtrsim \sum_{k=1}^{N_l} \frac{|a_k|^2}{(k+1)^{2\alpha}} \tag{4}$$

and using the inequality $|a_k| \geq 1$ for $k \in I_{N_l}$

$$[M_p(f, r_l)]^2 \gtrsim \sum_{k \in I_{N_l}} \frac{1}{(k+1)^{2\alpha}}. \tag{5}$$

(a) *Case* $\alpha \leq 0$: Combining Lemma 3.1 with (1), (3) and (5) we get

$$[M_p(f, r_l)]^2 \gtrsim N_l^{-2\alpha+1} \geq (1 - r_l)^{2\alpha-1}.$$

(b) *Case* $0 < \alpha < 1/2$: using (5) and Lemma 3.1 again, we get

$$[M_p(f, r_l)]^2 \geq \frac{(2 + N_l)^{-2\alpha+1}}{-2\alpha+1} \left(1 - \left(1 - \frac{\# I_{N_l}}{2 + N_l}\right)^{-2\alpha+1}\right).$$

The inequality (1) ensures $\left(1 - \frac{\# I_{N_l}}{2+N_l}\right)^{-2\alpha+1} \leq \left(1 - \frac{CN_l}{2+N_l}\right)^{-2\alpha+1}$. We deduce by using (1) and (3) again

$$[M_p(f, r_l)]^2 \gtrsim N_l^{-2\alpha+1} \geq (1 - r_l)^{2\alpha-1}.$$

Hence we conclude

$$\limsup_{r \to 1^-} \left((1 - r)^{\frac{1}{2} - \alpha} M_p(f, r)\right) > 0.$$

(c) *Case* $\alpha = \frac{1}{2}$: using (4), we know

$$[M_p(f, r_l)]^2 \gtrsim \sum_{j=1}^{l} \sum_{k=1+N_{j-1}}^{N_j} \frac{|a_k|^2}{k+1}.$$

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Hence applying Lemma 3.2 with $u_k = |a_k|^2$ and $v_k = 1/(k+1)$, we get, for all $l \geq 1$,

$$[M_p(f, r_l)]^2 \gtrsim \frac{S_{N_l}}{N_l + 1} - \frac{S_{N_0}}{N_0 + 1} + \sum_{j=1}^{l} S_{N_{j-1}} \left(\frac{1}{N_{j-1} + 1} - \frac{1}{N_j + 1} \right).$$

By construction $S_{N_{k-1}} \gtrsim N_{k-1} + 1$. Thus taking into account (2) we derive, for all $l \geq 1$, the inequality

$$[M_p(f, r_l)]^2 \gtrsim \sum_{j=1}^{l} \left(1 - \frac{N_{j-1} + 1}{N_j + 1}\right) \gtrsim l,$$

that allows to obtain $\limsup_{r \to 1^-} M_p(f, r) = +\infty$.

(2) **Case** $1 < p < 2$:
From Hausdorff–Young inequality (see [9]) we get

$$[M_p(f, r_l)]^q \geq \sum_{k \geq 0} \frac{|a_k|^q}{(k+1)^{q\alpha}} r_l^{qk} \geq \sum_{k=1}^{N_l} \frac{|a_k|^q}{(k+1)^{q\alpha}} r_l^{qk}.$$

Thus, we deduce

$$[M_p(f, r_l)]^q \geq \left(1 - \frac{1}{N_l - 1}\right)^{qN_l} \sum_{k=1}^{N_l} \frac{|a_k|^q}{(k+1)^{q\alpha}} \gtrsim \sum_{k=1}^{N_l} \frac{|a_k|^q}{(k+1)^{q\alpha}}.$$

Using the same strategy as in the case $2 \leq p \leq \infty$, we obtain,

$$\limsup_{r \to 1^-} \left((1-r)^{\frac{1}{q} - \alpha} M_p(f, r)\right) > 0, \qquad \text{for } \alpha < 1/q,$$
$$\limsup_{r \to 1^-} (M_p(f, r)) = +\infty, \qquad \text{for } \alpha = 1/q.$$

Moreover, since a $\mathcal{U}$-frequently hypercyclic function is necessarily hypercyclic, the assertion for the case $\alpha > \frac{1}{\max(2,q)}$ of the statement is given by the assertion (2b) of Theorem 2.2.

Finally, since a frequently hypercyclic function is necessarily $\mathcal{U}$-frequently hypercyclic, Theorem 1.1 ensures that the estimates we have proved are optimal when $\alpha \neq \frac{1}{\max(2,q)}$. □

Now we deal with the case $p = 1$.

Theorem 3.5. *Let f be a $\mathcal{U}$-frequently hypercyclic function for the operator T_α. Then, the following estimates hold*

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$$\limsup_{r\to 1^-}\big((1-r)^{-\alpha}M_1(f,r)\big)=+\infty, \qquad \text{if } \alpha\le 0,$$
$$\limsup_{r\to 1^-} M_1(f,r)>0, \qquad \text{if } \alpha>0.$$

For $\alpha\neq 0$, these results are optimal in the following sense: for any positive integer $l\geq 1$, there exists a $\mathcal{U}$-frequently hypercyclic function for the operator T_α such that for every $0<r<1$ sufficiently large

$$M_1(f,r)\lesssim\begin{cases}(1-r)^{\alpha}\log_l(-\log(1-r)) & \text{if } \alpha<0\\ 1 & \text{if } \alpha>0.\end{cases}$$

Proof. First since a $\mathcal{U}$-frequently hypercyclic function is necessarily hypercyclic, the assertions (1b) and (2b) of Theorem 2.2 ensure that,

$$\limsup_{r\to 1^-}\big((1-r)^{-\alpha}M_1(f,r)\big)=+\infty,\ \text{if } \alpha\le 0, \qquad \text{and} \qquad \limsup_{r\to 1^-} M_1(f,r)>0,\ \text{if } \alpha>0.$$

Moreover, since a frequently hypercyclic function is necessarily $\mathcal{U}$-frequently hypercyclic, Theorem 1.2 shows that the previous estimates are optimal when $\alpha\neq 0$. □

Remark 3.6. For the critical value $\alpha=0$, the optimality of the rate of growth in Theorem 3.5 will be obtained in Theorem 5.7 again.

4. Between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity

Let $1\le p\le\infty$. In view of Theorems 1.1, 1.2, 3.3 and 3.5, the critical exponent related to the L^p growth of frequently hypercyclic functions for T_α is the same as that related to the L^p growth of $\mathcal{U}$-frequently hypercyclic functions. It is equal to $\frac{1}{\max(2,q)}$. Nevertheless, this critical exponent is always equal to 0 in the case of hypercyclic functions, and hence it does not depend on p. In this section, we are interested in what happens between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity. In particular, when $p>1$, we will try to understand why and how the critical exponent goes from $\frac{1}{\max(2,q)}$ in the case of L^p-norm of $\mathcal{U}$-frequently hypercyclic functions to 0 for the L^p-norm of hypercyclic functions. To do this, we introduce intermediate notions of linear dynamics that link $\mathcal{U}$-frequent hypercyclicity and hypercyclicity. First of all, we need some definitions and results.

4.1. Some weighted densities

First, we introduce a refined notion of upper densities.

Definition 4.1. *Let $\beta=(\beta_n)$ be a non-decreasing sequence of positive real numbers tending to infinity. For a subset $E\subset\mathbb{N}$, its upper β-density is given by*

$$\overline{d}_\beta(E)=\limsup_{n\to+\infty}\frac{\sum_{k=1;k\in E}^{n}\beta_k}{\sum_{k=1}^{n}\beta_k}.$$

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These quantities enjoy all the classical properties of densities (see [12, 14]) and allow to define dynamical notions of the same nature as hypercyclicity or $\mathcal{U}$-frequent hypercyclicity.

Definition 4.2. *Let $\beta = (\beta_n)$ be a non-decreasing sequence of positive real numbers tending to infinity and let E be a subset of $\mathbb{N}$. An operator $T : X \to X$, where X is a Fréchet space, is said to be $\mathcal{U}_\beta$-frequently hypercyclic if there exists $x \in X$ such that for every non-empty open subset $U \subset X$,*

$$\overline{d}_\beta(\{n \in \mathbb{N} \ : \ T^n x \in U\}) > 0.$$

In the sequel, we are interested in densities given by the weighted sequence denoted by β^γ and defined by $\beta^\gamma = (e^{n^\gamma})$, where γ is a parameter with $0 \le \gamma \le 1$. First of all, let us notice that:

(i) the density $\overline{d}_{\beta^0}$ coincides with the upper natural density $\overline{d}$,

(ii) for any subset $E \subset \mathbb{N}$, $\overline{d}_{\beta^1}(E) > 0$ if and only if E is infinite.

Moreover for $0 < \gamma < 1$ an integral comparison test leads to the estimate

$$\sum_{k=1}^{n} e^{k^\gamma} \sim \frac{n^{1-\gamma}}{\gamma} e^{n^\gamma}, \qquad \text{as } n \text{ tends to infinity},$$

that we will use regularly in the rest of the paper. In addition, according to Lemma 2.8 of [12], the following inequalities hold. For the sake of clarity, let us mention that the density $\overline{d}_{\beta^\gamma}$ is denoted by $\overline{d}_{A_\gamma}$ in [12].

Lemma 4.3. *For any $0 \le \gamma_1 \le \gamma_2 \le 1$ and for any subset E of $\mathbb{N}$, we have*

$$\overline{d}(E) \le \overline{d}_{\beta^{\gamma_1}}(E) \le \overline{d}_{\beta^{\gamma_2}}(E) \le \overline{d}_{\beta^1}(E).$$

Therefore, the densities $\overline{d}_{\beta^\gamma}$ can give very different notions of dynamics that are intermediate between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity. In particular, the following lemma holds.

Lemma 4.4. *Let $0 < \gamma \le 1$. There exists a subset $E_\gamma \subset \mathbb{N}$ such that, for any $0 \le \gamma' < \gamma \le 1$, $\overline{d}_{\beta^\gamma}(E) > 0$ and $\overline{d}_{\beta^{\gamma'}}(E) = 0$.*

Proof. First observe that, for all $0 < t < 1$,

$$\frac{\sum_{k=1}^{2^{n-1}} \mathrm{e}^{k^t}}{\sum_{k=1}^{2^n} \mathrm{e}^{k^t}} \sim 2^{-(1-t)} \mathrm{e}^{-2^{nt}(1-2^{-t})} \to 0 \text{ as } n \to +\infty. \tag{6}$$

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Let $\gamma \neq 1$. Set $E_\gamma = \mathbb{N} \bigcap \left(\bigcup_{n \geq \lfloor \frac{1}{\gamma} \rfloor + 1} \left[2^n - \lfloor 2^{n(1-\gamma)} \rfloor ; 2^n \right] \right)$. Clearly, for all n large enough, we have

$$\sum_{k=1; k \in E_\gamma}^{2^n} \mathrm{e}^{k^\gamma} \geq \sum_{k=2^n - \lfloor 2^{n(1-\gamma)} \rfloor + 1}^{2^n} \mathrm{e}^{k^\gamma}. \tag{7}$$

Moreover, we get

$$\frac{\sum_{k=2^n - \lfloor 2^{n(1-\gamma)} \rfloor + 1}^{2^n} \mathrm{e}^{k^\gamma}}{\sum_{k=1}^{2^n} \mathrm{e}^{k^\gamma}} = 1 - \frac{\sum_{k=1}^{2^n - \lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{k^\gamma}}{\sum_{k=1}^{2^n} \mathrm{e}^{k^\gamma}}.$$

But we compute

$$\frac{\sum_{k=1}^{2^n - \lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{k^\gamma}}{\sum_{k=1}^{2^n} \mathrm{e}^{k^\gamma}} \sim \left(1 - \frac{\lfloor 2^{n(1-\gamma)} \rfloor}{2^n} \right)^{1-\gamma} \mathrm{e}^{2^{n\gamma}((1-2^{-n}\lfloor 2^{n(1-\gamma)} \rfloor)^\gamma - 1)} \to \mathrm{e}^{-\gamma} \text{ as } n \to +\infty.$$

Taking into account (6) and (7), we deduce

$$\overline{d}_{\beta\gamma}(E_\gamma) > 0.$$

Now let $0 \leq \gamma' < \gamma \leq 1$. Clearly keeping in mind that

$$\sum_{k=1; k \in E_\gamma}^{2^n} \mathrm{e}^{k^{\gamma'}} \leq \sum_{k=1}^{2^{n-1}} \mathrm{e}^{k^{\gamma'}} + \sum_{k=2^n - \lfloor 2^{n(1-\gamma)} \rfloor}^{2^n} \mathrm{e}^{k^{\gamma'}}$$

and by using both $\gamma' - \gamma < 0$, the estimate

$$\frac{\sum_{k=1}^{2^n - \lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{k^{\gamma'}}}{\sum_{k=1}^{2^n} \mathrm{e}^{k^{\gamma'}}} \sim \left(1 - \frac{\lfloor 2^{n(1-\gamma)} \rfloor}{2^n} \right)^{1-\gamma'} \mathrm{e}^{2^{n\gamma'}((1-2^{-n}\lfloor 2^{n(1-\gamma)} \rfloor)^{\gamma'} - 1)} \to 1 \text{ as } n \to +\infty,$$

and (6) we derive

$$\overline{d}_{\beta\gamma'}(E_\gamma) = 0.$$

Finally, for $\gamma = 1$, the lemma is easy to establish since a subset $E \subset \mathbb{N}$ satisfies $d_{\beta 1}(E) > 0$ if and only if E is infinite. This finishes the proof. □

In some sense, the densities $\underline{d}_{\beta\gamma}$, $0 \leq \gamma \leq 1$, will us allow to interpolate the behaviour of hypercyclic vectors between $\mathcal{U}$-frequent hypercyclicity and hypercyclicity.

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4.2. Rate of growth of $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions

First, we deal with the case $p>1$. We will discuss the case $p=1$ at the end of the section. We are ready to state the result that highlights both the continuous variation of the critical exponent and that of the growth (in term of L^p averages) of a hypercyclic function for T_α according to the frequency of visits of non-empty open subsets by its orbit under the action of T_α.

Theorem 4.5. *Let $0<\gamma<1$ and $1<p\leq\infty$. Let f be a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for the operator T_α. Then, the following hold*

$$\begin{aligned}
&\textit{if } \alpha<\frac{1-\gamma}{\max(2,q)}, && \limsup_{r\to 1^-}\left([1-r]^{\frac{1-\gamma}{\max(2,q)}-\alpha}M_p(f,r)\right)>0,\\
&\textit{if } \alpha=\frac{1-\gamma}{\max(2,q)}, && \limsup_{r\to 1^-}(M_p(f,r))=+\infty,\\
&\textit{if } \alpha>\frac{1-\gamma}{\max(2,q)}, && \limsup_{r\to 1^-}(M_p(f,r))>0.
\end{aligned}$$

Proof. Let f be a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for T_α. We write $f=\sum_{k\geq 0}\frac{a_k}{(k+1)^\alpha}z^k$. Since f is $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic, there exists an increasing subsequence $(n_k)\subset\mathbb{N}$ with positive upper β^γ-density such that, for all $k\geq 1$,

$$|T_\alpha^{n_k}f(0)-3/2|=|a_{n_k}-3/2|<1/2.$$

We get, for all $k\geq 1$, $|a_{n_k}|\geq 1$. Set $I=\{n_k\ :\ k\geq 1\}$ and for all $N\geq 1$, $I_N=I\cap\{1,\dots,N\}$. The hypothesis $\overline{d}_{\beta^\gamma}(I)>0$ ensures that there exist $0<C<1$ and an increasing sequence (N_l) of positive integers such that

$$\sum_{k\in I_{N_l}}\mathrm{e}^{k^\gamma}\geq C\frac{N_l^{1-\gamma}}{\gamma}\mathrm{e}^{N_l^\gamma}. \tag{8}$$

Up to take a sub-sequence, we can suppose that

$$C(N_{k+1}+1)\geq N_k+1. \tag{9}$$

Let us consider, for all $l\geq 1$, a sequence (r_l) with $1-\frac{1}{N_l-1}\leq r_l<1-\frac{1}{N_l}$. Observe that

$$N_l-1\leq\frac{1}{1-r_l}<N_l. \tag{10}$$

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(1) **Case** $2 \leq p \leq \infty$:
Jensen's inequality and Parseval's Theorem give

$$[M_p(f,r_l)]^2 \geq [M_2(f,r_l)]^2 = \sum_{k\geq 0} \frac{|a_k|^2}{(k+1)^{2\alpha}} r_l^{2k} \geq \sum_{k=1}^{N_l} \frac{|a_k|^2}{(k+1)^{2\alpha}} r_l^{2k} \gtrsim \sum_{k=1}^{N_l} \frac{|a_k|^2}{(k+1)^{2\alpha}}. \tag{11}$$

Let us choose $j_0 \in \mathbb{N}$ such that the function $t \mapsto (t+1)^{-2\alpha} e^{-t^\gamma}$ is decreasing for $t \geq N_{j_0}$. Thus, we can write, for all $l \geq j_0 + 1$,

$$[M_p(f,r_l)]^2 \gtrsim \sum_{j=1+j_0}^{l} \sum_{k=1+N_{j-1}}^{N_j} \frac{|a_k|^2}{(k+1)^{2\alpha}} \mathrm{e}^{k^\gamma} \mathrm{e}^{-k^\gamma}.$$

Then applying Lemma 3.2 with $u_k = |a_k|^2 \mathrm{e}^{k^\gamma}$, we get

$$\begin{aligned}[M_p(f,r_l)]^2 \gtrsim \quad & S_{N_l}(N_l+1)^{-2\alpha}\mathrm{e}^{-N_l^\gamma} - S_{N_{j_0}}(N_{j_0}+1)^{-2\alpha}\mathrm{e}^{-N_{j_0}^\gamma} \\ & + \sum_{j=1+j_0}^{l} S_{N_{j-1}} \left((N_{j-1}+1)^{-2\alpha}\mathrm{e}^{-N_{j-1}^\gamma} - (N_j+1)^{-2\alpha}\mathrm{e}^{-N_j^\gamma} \right). \end{aligned} \tag{12}$$

Since $S_{N_i} = \sum_{k \leq N_i} |a_k|^2 \mathrm{e}^{k^\gamma}$, by construction and by (8), we get, for all $i \geq 1$,

$$S_{N_i} \geq \sum_{k \in I_{N_i}} \mathrm{e}^{k^\gamma} \gtrsim N_i^{1-\gamma} \mathrm{e}^{N_i^\gamma} \gtrsim (N_i^{1-\gamma}+1)\mathrm{e}^{N_i^\gamma}. \tag{13}$$

From (12) and (13), we deduce

$$\begin{aligned}[M_p(f,r_l)]^2 \gtrsim & (N_l+1)^{(1-\gamma)-2\alpha} \\ & + \sum_{j=1+j_0}^{l} (N_{j-1}+1)^{1-\gamma}\mathrm{e}^{N_{j-1}^\gamma} \Big((N_{j-1}+1)^{-2\alpha}\mathrm{e}^{-N_{j-1}^\gamma} \\ & \qquad - (N_j+1)^{-2\alpha}\mathrm{e}^{-N_j^\gamma} \Big). \end{aligned} \tag{14}$$

(a) *Case* $\alpha < \frac{1-\gamma}{2}$: From (14), we get, for l large enough,

$$[M_p(f,r_l)]^2 \gtrsim (N_l+1)^{(1-\gamma)-2\alpha}. \tag{15}$$

Thanks to (10) and (15), we deduce

$$[M_p(f,r_l)]^2 \gtrsim (1-r_l)^{2\alpha-(1-\gamma)}.$$

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Hence we conclude

$$\limsup_{r\to 1^-}\left[(1-r)^{-\alpha+\frac{1-\gamma}{2}}M_p(f,r)\right]>0.$$

(b) *Case* $\alpha=\frac{1-\gamma}{2}$: taking into consideration (14), we can write, for all $l\geq 1+j_0$,

$$[M_p(f,r_l)]^2\gtrsim\sum_{j=1}^{l}\left(1-\left(\frac{N_{j-1}+1}{N_j+1}\right)^{1-\gamma}\mathrm{e}^{N_{j-1}^{\gamma}-N_j^{\gamma}}\right).$$

Thus taking into account (9), we derive, for all $l\geq 1+j_0$, $[M_p(f,r_l)]^2\gtrsim l$, which allows to obtain

$$\limsup_{r\to 1^-}M_p(f,r)=+\infty.$$

(c) *Case* $\alpha>\frac{1-\gamma}{2}$: since f is hypercyclic, the conclusion is given by Theorem 2.2.

(2) **Case** $1<p<2$:
It suffices to combine the arguments of the proof of the preceding case with those of the proof of (2) of Theorem 3.3 to obtain the desired conclusions.

□

4.3. Optimal growth of $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions: a constructive proof

In this subsection, we intend to prove that the estimates given by Theorem 4.5 whenever α is different from the critical exponent, i.e. $\alpha\neq\frac{1-\gamma}{\max(2,q)}$, are optimal. The case $\alpha=\frac{1-\gamma}{\max(2,q)}$ will be treated separately in § 5. Thus for all $0<\gamma<1$ and for $\alpha\neq\frac{1-\gamma}{\max(2,q)}$, we propose to build $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions for T_α that have the required L^p growth and no more. To do this, we follow the construction of frequently hypercyclic functions for T_α given in [25] which itself was partly inspired by [8]. In particular, we will need the so-called Rudin–Shapiro polynomials (combined with the de la Vallée–Poussin polynomials), which have coefficients ±1 (or bounded by 1) and an optimal growth of L^p-norm. Let us recall the associated result in the form of Lemma 2.1 of [8] that summarized the result of Rudin–Shapiro [27].

Lemma 4.6.

(1) *For each $N\geq 1$, there is a trigonometric polynomial $p_N=\sum_{k=0}^{N-1}\varepsilon_{N,k}\mathrm{e}^{\mathrm{i}k\theta}$ where $\varepsilon_{N,k}=\pm1$ for all $0\leq k\leq N-1$ with at least half of the coefficients being $+1$ and with*

$$\|p_N\|_p\leq 5\sqrt{N}\ \textit{for}\ p\in[2,+\infty].$$

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(2) For each $N \geq 1$*, there is a trigonometric polynomial* $p_N^* = \sum_{k=0}^{N-1} a_{N,k} e^{ik\theta}$ *where* $|a_{N,k}| \leq 1$ *for all* $0 \leq k \leq N-1$ *with at least* $\lfloor \frac{N}{4} \rfloor$ *coefficients being* $+1$ *and with*

$$\|p_N^*\|_p \leq 3N^{1/q} \textit{ for } p \in [1,2].$$

For any given polynomial q with $q(z) = \sum_{j=0}^{d} b_j z^j$ with $b_d \neq 0$, we denote $d = \deg(q)$ and $\|q\|_{\ell^1} = \sum_{j=0}^{d} |b_j|$. We set $2\mathbb{N} = \bigcup_{k \geq 1} \mathcal{A}_k$ where for any $k \geq 1$, $\mathcal{A}_k = \left\{2^k(2j-1); j \in \mathbb{N}\right\}$. Denote by $\mathcal{P}$ the countable set of polynomials with rational coefficients and let us also consider pairs (q, l) with $q \in \mathcal{P}$ and $l \in \mathbb{N}$ satisfying $\|q\|_{\ell^1} \leq l$, displayed as a single sequence (q_k, l_k). Clearly, (q_k) is a dense set in $H(\mathbb{D})$. Hence, for any $k \geq 1$, we set $d_k = \deg(q_k)$ and we have

$$\|q_k\|_{\ell^1} \leq l_k \text{ for every } k \geq 1.$$

For any $\alpha \in \mathbb{R}$, for any positive integer $k \geq 1$, we set $\tilde{q}_k(z) = \sum_{j=0}^{d_k} (j+1)^{\alpha} b_j^{(k)} z^j$.

Let α be a real number and $p \in (1, \infty]$. For all integer $n \geq 0$, we set $I_n = \{2^n, \ldots, 2^{n+1}-1\}$. Next, for $k \geq 1$, let us define the integers

$$\alpha_k = 1 + \left\lfloor \max\left(l_k^2(1+d_k)^{2\max(\alpha,0)}, d_k + \max(3, 3+\alpha) l_k^2 + \max(\alpha, 0) l_k \log(1+d_k)\right)\right\rfloor$$

and

$$\alpha_k^* = 1 + \left\lfloor \max\left(l_k^q(1+d_k)^{q\max(\alpha,0)}, d_k + \max(3, 3+\alpha) l_k^2 + \max(\alpha, 0) l_k \log(1+d_k)\right)\right\rfloor.$$

We set $f_\alpha = \sum_{n \geq 0} P_{n,\alpha}$ where the blocks $(P_{n,\alpha})$ are polynomials defined as follows, using Rudin–Shapiro polynomials given by Lemma 4.6,

$$P_{n,\alpha}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{n-1} < \alpha_k \\ z^{2^n} Q_n(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{n-1} \geq \alpha_k \end{cases} \tag{16}$$

with for $n \in \mathcal{A}_k$,

$$Q_n(z) = \sum_{j \in I_n} (j+1)^{-\alpha} c_{j-2^n}^{(k)} z^{j-2^n}$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^n(1-\gamma)}{\alpha_k} \rfloor}(z^{\alpha_k})\tilde{q}_k(z)$.

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We also set $f^*_\alpha = \sum_{n\geq 0} P^*_{n,\alpha}$ where the blocks $(P^*_{n,\alpha})$ are polynomials defined as follows, using the de la Vallée–Poussin polynomials given by Lemma 4.6,

$$P^*_{n,\alpha}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{n-1} < \alpha^*_k \\ z^{2^n} Q^*_n(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{n-1} \geq \alpha^*_k, \end{cases} \tag{17}$$

with, for $n \in \mathcal{A}_k$,

$$Q^*_n(z) = \sum_{j\in I_n} (j+1)^{-\alpha} c^{(k)}_{j-2^n} z^{j-2^n}$$

where the sequence $(c^{(k)}_j)$ denotes the sequence of the coefficients of the polynomial $p^*_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha^*_k} \rfloor}(z^{\alpha^*_k})\tilde{q}_k(z)$.

A combination of Lemma 4.7 below with the triangle inequality shows that the function f_α (resp. f^*_α) belongs to $H(\mathbb{D})$. Observe that, if we denote the polynomial $z \mapsto p_{\lfloor \frac{2n-1}{\alpha_k} \rfloor}(z^{\alpha_k})$ (resp. $z \mapsto p^*_{\lfloor \frac{2n-1}{\alpha_k} \rfloor}(z^{\alpha^*_k})$) by g_k (resp. g^*_k), we have, for all $1 \leq p \leq +\infty$, $\|g_k\|_p = \|p_{\lfloor \frac{2n-1}{\alpha_k} \rfloor}\|_p$ (resp. $\|g^*_k\|_p = \|p^*_{\lfloor \frac{2n-1}{\alpha^*_k} \rfloor}\|_p$). Finally for any integer n, let us denote $(\phi_n(k))$ the sequence defined as follows

$$\phi_n(k) = \begin{cases} (k+1)^{-\alpha} & \text{if } k \in I_n \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 4.7. *Let $\alpha \in \mathbb{R}$. The following estimates hold.*

(i) For any $2 \leq p \leq +\infty$, any $0 < r < 1$ and any $n \in \mathbb{N}$, we have

$$M_p(P_{n,\alpha}, r) \lesssim 2^{n(\frac{1-\gamma}{2} - \alpha)} r^{2^n}.$$

(ii) For any $1 < p < 2$, any $0 < r < 1$ and any $n \in \mathbb{N}$, we have

$$M_p(P^*_{n,\alpha}, r) \lesssim 2^{n(\frac{1-\gamma}{q} - \alpha)} r^{2^n}.$$

Proof.

(i) On one hand, we deal with the case $2 \leq p < +\infty$. Let n be a positive integer. Without loss of generality, we can assume that n belongs to the set $\mathcal{A}_k$ for some

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$k \geq 1$. Let r be in $(0, 1)$. Since $r \mapsto M_p(f, .)$ is increasing, we get

$$M_p(P_{n,\alpha}, r) \leq r^{2^n} \|Q_n\|_p.$$

Then, the polynomial Q_n can be viewed as a trigonometric polynomial obtained by an abstract convolution operator on $\mathbb{T}$, given by $(c_k)_{k\geq 0} \mapsto (\phi_n(j)c^{(k)}_{j-2^n})_{j\geq 0}$ (where $(c^{(k)}_j)$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^n-1}{\alpha_k} \rfloor}\tilde{q_k})$. Now, we are going to apply the Marcinkiewicz Multiplier Theorem [10, Theorem 8.2 p.148]. To do this, observe that we have, for any $l \geq 1$,

$$\sup_{j\in I_l} |\phi_n(j)| \leq \sup_{j\in I_n} |\phi_n(j)| \lesssim 2^{-n\alpha}$$

and

$$\sup_l \sum_{j\in I_l} |\phi_n(j + 1) - \phi_n(j)| \leq \sum_{j\in I_n} |\phi_n(j + 1) - \phi_n(j)| \lesssim 2^{-n\alpha}.$$

Hence, taking into account the choice of α_k and Lemma 4.6, we get

$$\begin{aligned} \|Q_n\|_p &\lesssim 2^{-n\alpha} \|p_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor}\|_p \|\tilde{q_k}\|_\infty \\ &\lesssim 2^{-n\alpha} \sqrt{\frac{2^{n(1-\gamma)}}{\alpha_k}}\; l_k (1 + d_k)^{\max(\alpha,0)} \\ &\lesssim 2^{n(\frac{1-\gamma}{2} - \alpha)}. \end{aligned}$$

Finally, we obtain the desired estimate

$$M_p(P_{n,\alpha}, r) \lesssim 2^{n(\frac{1-\gamma}{2} - \alpha)} r^{2^n}.$$

On the other hand, we deal with the case $p = \infty$. Let us recall that $P_{n,\alpha}(z) = 0$ or $z^{2^n} Q_n(z)$ with $Q_n(z) = \sum_{j\in I_n} (j+1)^{-\alpha} c^{(k)}_{j-2^n} z^{j-2^n}$ where $(c^{(k)}_j)$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor}(z^{\alpha_k})\tilde{q_k}(z)$. First, assume that $\alpha \leq 0$. We write

$$M_\infty(P_{n,\alpha}, r) \lesssim r^{2^n} \|Q_n\|_\infty.$$

Using the form of Q_n, as in the proof of Lemma 3.6 of [25], we apply a fractional Bernstein's inequality to obtain, taking into consideration Lemma 4.6,

$$M_\infty(P_{n,\alpha}, r) \lesssim r^{2^n} 2^{-n\alpha} \|Q_n\|_\infty \lesssim 2^{-n\alpha} \|p_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor}\|_\infty \|\tilde{q_k}\|_\infty \lesssim 2^{-n\alpha} \sqrt{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor} l_k.$$

Thanks to the choice of α_k, we have, for $\alpha \leq 0$,

$$M_\infty(P_{n,\alpha}, r) \lesssim 2^{n(\frac{1-\gamma}{2} - \alpha)}.$$

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To conclude, it suffices to mimic the induction of the proof of Lemma 3.7 of [25].

(ii) The proof is similar as that of the case $2 \leq p < +\infty$ by applying Lemma 4.6 for $1 < p < 2$.

□

Now we are ready to obtain the rate of growth of the aforementioned functions f_α and f_α^*. We refer to Lemmas 3.4, 3.5 and 3.8 of [25] with obvious modifications.

Lemma 4.8.

(1) Let $2 \leq p \leq +\infty$. For all $0 < r < 1$, the following estimates hold

$$M_p(f_\alpha, r) \lesssim \begin{cases} (1-r)^{\alpha - \frac{1-\gamma}{2}} & \text{if } \alpha < \frac{1-\gamma}{2}. \\ 1 & \text{if } \alpha > \frac{1-\gamma}{2} \end{cases}$$

(2) Let $1 < p < 2$. For all $0 < r < 1$, the following estimates hold

$$M_p(f_\alpha^*, r) \lesssim \begin{cases} (1-r)^{\alpha - \frac{1-\gamma}{q}} & \text{if } \alpha < \frac{1-\gamma}{q}. \\ 1 & \text{if } \alpha > \frac{1-\gamma}{q} \end{cases}$$

Now we are going to prove that the functions f_α and f_α^* are U_{β^γ}-frequently hypercyclic for T_α.

Proposition 4.9. *For $p \geq 2$ (resp. $1 < p < 2$), the function f_α (resp. f_α^*) is a U_{β^γ}-frequently hypercyclic vector for the operator T_α.*

Proof. We only prove that the vector f_α is U_{β^γ}-frequently hypercyclic for the operator T_α. We do not repeat the details for f_α^*: it will be enough to make the appropriate modifications.

Let k be a large enough integer. Let us consider $n \in \mathcal{A}_k$ such that $2^{n-1} \geq \alpha_k$. We consider $\mathcal{B}_n$ the set of s in I_n such that the coefficient of z^s in the polynomial $z^{2^n} p_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor}(z^{\alpha_k})$ is equal to 1 and we denote by $T_k = \left\{ s : s \in \mathcal{B}_n,\ n \in \mathcal{A}_k,\ 2^{n-1} \geq \alpha_k \right\}$.

Observe that $\max(\mathcal{B}_n) \leq 2^n + \lfloor 2^{n(1-\gamma)} \rfloor$ and since at least half of the coefficients of $p_{\lfloor \frac{2^{n(1-\gamma)}}{\alpha_k} \rfloor}$ being $+1$, we get

$$\frac{\sum_{\substack{j \leq \max(\mathcal{B}_n); \\ j \in T_k}} e^{j^\gamma}}{\sum_{j \leq \max(\mathcal{B}_n)} e^{j^\gamma}} \geq \frac{\sum_{j=2^n}^{2^n + 2^{-1}\lfloor 2^{n(1-\gamma)} \rfloor} e^{j^\gamma}}{\sum_{j=1}^{2^n + \lfloor 2^{n(1-\gamma)} \rfloor} e^{j^\gamma}} = \frac{\sum_{j=1}^{2^n + 2^{-1}\lfloor 2^{n(1-\gamma)} \rfloor} e^{j^\gamma}}{\sum_{j=1}^{2^n + \lfloor 2^{n(1-\gamma)} \rfloor} e^{j^\gamma}} - \frac{\sum_{j=1}^{2^n - 1} e^{j^\gamma}}{\sum_{j=1}^{2^n + \lfloor 2^{n(1-\gamma)} \rfloor} e^{j^\gamma}}. \tag{18}$$

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Clearly, we have

$$\left(2^n + \frac{\lfloor 2^{n(1-\gamma)} \rfloor}{2}\right)^\gamma - \left(2^n + \lfloor 2^{n(1-\gamma)} \rfloor\right)^\gamma \to -\frac{\gamma}{2}$$

and

$$(2^n - 1)^\gamma - \left(2^n + \lfloor 2^{n(1-\gamma)} \rfloor\right)^\gamma \to -\gamma,$$

which implies, using similar estimations as those of the proof of Lemma 4.4,

$$\frac{\sum_{j=1}^{2^n + 2^{-1}\lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{j^\gamma}}{\sum_{j=1}^{2^n + \lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{j^\gamma}} \to e^{-\gamma/2} \quad \text{and} \quad \frac{\sum_{j=1}^{2^n - 1} \mathrm{e}^{j^\gamma}}{\sum_{j=1}^{2^n + \lfloor 2^{n(1-\gamma)} \rfloor} \mathrm{e}^{j^\gamma}} \to \mathrm{e}^{-\gamma}, \quad \text{as } n \to +\infty.$$

Hence the inequality (18) ensures that

$$\overline{d}_{\beta^\gamma}(T_k) > 0.$$

Then let α be a real number and let $k \in \mathbb{N}$. Let us consider $s \in \mathcal{B}_n$ with $n \in \mathcal{A}_k$ satisfying $2^{n-1} \geq \alpha_k$. As in the proof of Lemma 3.9 of [25] with easy modifications, we can prove that

$$\sup_{|z| = 1 - \frac{1}{l_k}} |T_\alpha^s(f_\alpha)(z) - q_k(z)| \lesssim \frac{1}{l_k},$$

provided that k is chosen large enough. This allows to obtain U_{β^γ}-frequent hypercyclicity of f_α. □

In summary, Lemma 4.8 and Proposition 4.9 leads to the following result, which shows that the statement of Theorem 4.5 is optimal whenever α is not the critical exponent.

Theorem 4.10. *Let $0 < \gamma < 1$ and $1 < p \leq \infty$.*

(1) for $\alpha < \frac{1-\gamma}{\max(2,q)}$, there exists a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for the operator T_α such that

$$M_p(f, r) \lesssim (1 - r)^{\alpha - \frac{1-\gamma}{\max(2,q)}};$$

(2) for $\alpha > \frac{1-\gamma}{\max(2,q)}$, there exists a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for the operator T_α such that

$$M_p(f, r) \lesssim 1.$$

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Remark 4.11.

(1) For the critical value $\alpha = \frac{1-\gamma}{\max(2,q)}$, the optimality of the rate of growth in this theorem will be obtained in Theorem 5.11.
(2) Let $0 < \gamma < 1$. It seems important to note that the functions constructed for the proof of Theorem 4.10 are $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic for T_α but neither $\mathcal{U}_{\beta^{\gamma'}}$-frequently hypercyclic for $0 < \gamma' < \gamma$ nor $\mathcal{U}$-frequently hypercyclic, since they don't satisfy the estimates given by Theorem 4.5 or Theorem 3.3.

Finally let us say some words for the case $p = 1$. As in the proof of Theorem 3.5, observe that a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function is necessarily hypercyclic and a $\mathcal{U}$-frequently hypercyclic function is necessarily $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic. This leads to the following statement.

Theorem 4.12. *Let $0 < \gamma < 1$. Let f be a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for the operator T_α. Then, the following assertions hold*

$$\limsup_{r\to 1^-} \left((1-r)^{-\alpha} M_1(f,r)\right) = +\infty, \qquad \text{if } \alpha \leq 0,$$
$$\limsup_{r\to 1^-} M_1(f,r) > 0, \qquad \text{if } \alpha > 0.$$

These results are optimal in the following sense: for any positive integer $l \geq 1$, there exists a $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic function for the operator T_α such that for every $0 < r < 1$ sufficiently large

$$M_1(f,r) \lesssim \begin{cases} (1-r)^{\alpha} \log_l(-\log(1-r)) & \text{if } \alpha < 0 \\ 1 & \text{if } \alpha > 0. \end{cases}$$

Remark 4.13. For the critical value $\alpha = 0$, the optimality of the rate of growth in Theorem 4.12 will be obtained in Theorem 5.11 again.

5. Optimal estimates: the case of the critical exponent

In this section, we are going to show that the growth of $\mathcal{U}$-frequently or U_{β^γ}-frequently hypercyclic functions for T_α can be arbitrarily slow when α is the critical exponent. The situation will therefore be similar to the hypercyclic case for which for all $1 \leq p \leq \infty$ the critical exponent is $\alpha = 0$ and, according Theorem 2.2, the two following properties hold: for all hypercyclic function f for T_α, $\limsup_{r\to 1^-} M_p(f,r) = +\infty$ and for any function $\varphi : [0,1) \to \mathbb{R}_+$ tending to infinity as r tends to 1, there is a hypercyclic function f such that $M_p(f,r) \leq \varphi(r)$. For this, we are going to adapt the constructive method used in § 4.3. Before we start, we establish a lemma that will be useful in the following.

Lemma 5.1. *Let (w_n) be an increasing sequence of positive integers such that $\frac{w_{n+1}}{w_n} \to +\infty$ as n tends to infinity. Let (a_n) be a bounded sequence of positive real numbers such*

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that $\sum a_n = +\infty$. *If we denote by* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *an increasing function with, for all* $n \in \mathbb{N}$, $h(n) = w_n$, *the following estimate holds*

$$\sum_{n\geq 0} a_n \left(1 - \frac{1}{h(x)}\right)^{w_n} \sim \sum_{n\leq x} a_n, \quad \textit{as } x \to \infty.$$

Proof. Let $\varepsilon > 0$. Let $n \leq x < n+1$, $n \geq 1$ so big that $h(n) > 1$. Clearly, the following inequality holds

$$\left(1 - \frac{1}{h(n)}\right)^{h(k)} \leq \left(1 - \frac{1}{h(x)}\right)^{h(k)} \leq \left(1 - \frac{1}{h(n+1)}\right)^{h(k)}. \tag{19}$$

From this, we get for $k \leq n-1$

$$1 \geq \left(1 - \frac{1}{h(x)}\right)^{h(k)} \geq \left(1 - \frac{1}{h(n)}\right)^{h(n-1)} = \left(\left(1 - \frac{1}{w_n}\right)^{w_n}\right)^{\frac{w_{n-1}}{w_n}} \to 1$$

as $n \to +\infty$ since $\frac{w_{n+1}}{w_n} \to +\infty$. Thus, there is some $N > 0$ such that, whenever $N \leq k \leq n-1$ and $n \leq x < n+1$,

$$1 - \varepsilon \leq \left(1 - \frac{1}{h(x)}\right)^{h(k)} \leq 1. \tag{20}$$

Next let $k \geq n+2$. From 19, we get

$$\left(1 - \frac{1}{h(x)}\right)^{h(k)} \leq \left(1 - \frac{1}{h(n+1)}\right)^{h(k)} = \left(\left(1 - \frac{1}{w_{n+1}}\right)^{w_{n+1}}\right)^{\frac{w_k}{w_{n+1}}}.$$

Now there is some N' such that, for all $n \geq N'$,

$$\left(1 - \frac{1}{w_{n+1}}\right)^{w_{n+1}} \leq \frac{1}{2}$$

and, for all $k \geq n+2 \geq N'$,

$$\frac{w_k}{w_{n+1}} = \prod_{j=0}^{k-n+2} \frac{w_{k-j}}{w_{k-j-1}} \geq 2^{k-(n+1)} \geq k - (n+1),$$

so that, with $M = \sup_n a_n$,

$$\sum_{k=n+2}^{+\infty} a_k \left(1 - \frac{1}{h(x)}\right)^{h(k)} \leq M \sum_{k=n+2}^{+\infty} \frac{1}{2^{k-(n+1)}} = M. \tag{21}$$

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Thus, for $n \geq \max(N, N')$ and $n \leq x < n+1$, let us define

$$\begin{aligned}\rho_n &:= \frac{\sum\limits_{k\geq 0} a_k \left(1-\frac{1}{h(x)}\right)^{h(k)}}{\sum\limits_{k\leq x} a_k}\\ &= \frac{\sum\limits_{k=0}^{N-1} u_k(x) + \sum\limits_{k=N}^{n-1} u_k(x) + \sum\limits_{k=n}^{n+1} u_k(x) + \sum\limits_{k\geq n+2} u_k(x)}{\sum\limits_{k=0}^{N-1} a_k + \sum\limits_{k=N}^{n-1} a_k + a_n},\end{aligned}$$

with $u_k(x) = a_k \left(1-\frac{1}{h(x)}\right)^{h(k)}$. Since $(1-\varepsilon)\sum\limits_{k=N}^{n-1} a_k \leq \sum\limits_{k=N}^{n-1} u_k(x) \leq \sum\limits_{k=N}^{n-1} a_k$ thanks to (20), we deduce, taking into account (21) and the fact that (a_n) is a bounded sequence,

$$1-\varepsilon \leq \liminf_{n\to+\infty} \rho_n \leq \limsup_{n\to+\infty} \rho_n \leq 1,$$

which implies the claim. □

As a direct corollary of Lemma 5.1, we can state the following result.

Lemma 5.2. *Let (w_n) be an increasing sequence of positive integers such that $\frac{w_{n+1}}{w_n} \to +\infty$ as n tends to infinity. Let (a_n) be a bounded sequence of positive real numbers such that $\sum a_n = +\infty$. If we denote by $h : \mathbb{R}_+ \to \mathbb{R}_+$ a continuous and strictly increasing function with, for all $n \in \mathbb{N}$, $h(n) = w_n$, the following estimate holds*

$$\sum_{n\geq 0} a_n r^{w_n} \sim (\theta_a \circ h^{-1})\left(\frac{1}{1-r}\right) \qquad \text{as } r \to 1^-,$$

where for all $x \in \mathbb{R}_+$, $\theta_a(x) = \sum\limits_{n\leq x} a_n$.

5.1. The $\mathcal{U}$-frequently hypercyclic case

We keep the definitions and the notations of § 4.3. Let us also consider an increasing function $h : \mathbb{R}_+ \to \mathbb{R}_+$ tending to infinity such that, for any $n \in \mathbb{N}$, $h(n) := u_n \in \mathbb{N}$ and $u_{n+1} - u_n \to +\infty$ as n tends to infinity. Let α be a real number. For all integer $n \geq 0$, we set $I_n^{(u)} = \{2^{u_n}, \dots, 2^{u_{n+1}} - 1\}$. Next, for $k \geq 1$, we keep the definition of integers α_k and α_k^* given in § 4.3. We set $f_\alpha^{(u)} = \sum\limits_{n\geq 0} P_{n,\alpha}^{(u)}$ where the blocks $(P_{n,\alpha}^{(u)})$ are polynomials defined as follows, using Rudin–Shapiro polynomials given by Lemma 4.6,

$$P_{n,\alpha}^{(u)}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} < \alpha_k \\ z^{2^{u_n}} Q_n(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} \geq \alpha_k \end{cases} \tag{22}$$

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with for $n \in \mathcal{A}_k$,

$$Q_n^{(u)}(z) = \sum_{j \in I_n^{(u)}} (j+1)^{-\alpha} c_{j-2^{un}}^{(k)} z^{j-2^{un}}$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^{un}}{\alpha_k} \rfloor}(z^{\alpha_k})\tilde{q}_k(z)$. We also set $f_\alpha^{*(u)} = \sum_{n \geq 0} P_{n,\alpha}^{*(u)}$ where the blocks $(P_{n,\alpha}^{*(u)})$ are polynomials defined as follows, using polynomials given by Lemma 4.6,

$$P_{n,\alpha}^{*(u)}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{un-1} < \alpha_k^* \\ z^{2^{un}} Q_n^{*(u)}(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{un-1} \geq \alpha_k^*, \end{cases} \tag{23}$$

with, for $n \in \mathcal{A}_k$,

$$Q_n^{*(u)}(z) = \sum_{j \in I_n^{(u)}} (j+1)^{-\alpha} c_{j-2^{un}}^{(k)} z^{j-2^{un}}$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p^*_{\lfloor \frac{2^{un}}{\alpha_k^*} \rfloor}(z^{\alpha_k^*})\tilde{q}_k(z)$.

For $1 \leq p \leq \infty$, we denote by α_c the critical exponent $\alpha_c = \frac{1}{\max(2,q)}$.

Lemma 5.3. *We have, for any* $0 < r < 1$,

$$M_p(P_{n,\alpha_c}^{(u)}, r) \lesssim r^{2^{un}} \text{ if } 2 \leq p \leq \infty, \quad M_p(P_{n,\alpha_c}^{*(u)}, r) \lesssim r^{2^{un}} \text{ if } 1 < p < 2,$$
$$\text{and} \qquad M_1(P_{n,0}^{*(u)}, r) \lesssim r^{2^{un}} l_k.$$

Proof. For $p > 1$, it suffices to argue along the same lines as the proof of Lemma 4.7 replacing the sequence (2^n) by (2^{un}).

Now let us consider the case $p = 1$ (hence $\alpha_c = 0$). We can write, keeping in mind that $q_k = \tilde{q}_k$ for $\alpha = 0$,

$$\begin{aligned} M_1(P_{n,0}^{*(u)}, r) &\leq \frac{r^{2^{un}}}{2\pi} \int_0^{2\pi} \left| Q_n^{*(u)}(re^{it}) \right| dt \\ &\leq \frac{r^{2^{un}}}{2\pi} \int_0^{2\pi} \left| \sum_{j \in I_n^{(u)}} c_{j-2^{un}}^{(k)} (re^{it})^{j-2^{un}} \right| dt \\ &\leq \frac{r^{2^{un}}}{2\pi} \int_0^{2\pi} \left| p^*_{\lfloor \frac{2^{un}}{\alpha_k^*} \rfloor}((re^{it})^{\alpha_k^*})\tilde{q}_k(re^{it}) \right| dt \\ &\leq r^{2^{un}} \|p^*_{\lfloor \frac{2^{un}}{\alpha_k^*} \rfloor}\|_1 \|q_k\|_\infty \\ &\lesssim r^{2^{un}} l_k. \end{aligned}$$

□

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From Lemma 5.2 and 5.3, we deduce the rate of growth of the functions $f_{\alpha_c}^{(u)}$ and $f_{\alpha_c}^{*(u)}$. We begin by the case $p \neq 1$.

Lemma 5.4. *Let $1 < p \leq \infty$. Under the preceding definitions and assumptions, the following estimates hold: for all $0 < r < 1$,*

$$M_p(f_{\alpha_c}^{(u)}, r) \lesssim h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right) \text{ if } 2 \leq p \leq \infty$$
$$\text{and} \quad M_p(f_{\alpha_c}^{*(u)}, r) \lesssim h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right) \text{ if } 1 < p < 2.$$

Proof. Let $2 \leq p \leq \infty$. Combining Lemma 5.3 with triangle inequality, we get

$$M_p(f_{\alpha_c}^{(u)}, r) \lesssim \sum_{n\geq 0} r^{2^{u_n}}.$$

By hypothesis $2^{u_{n+1}-u_n} \to +\infty$ as n tends to infinity. We apply Lemma 5.2 with $w_n = 2^{u_n}$ and $a_n = 1$ and we obtain, for $0 < r < 1$,

$$M_p(f_{\alpha_c}^{(u)}, r) \lesssim h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right).$$

For the case $1 < p < 2$, the proof works along the same lines. □

Now we are interested in the specific case $p = 1$.

Lemma 5.5. *There is a function of the form $f_0^{*(u)}$ such that*

$$M_1(f_0^{*(u)}, r) \lesssim \left(h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right)\right)^2.$$

Proof. Without loss of generality we can assume that $\alpha_k^* > 1 + \lfloor m_k \rfloor$ where m_k is the least real number such that $g(\log(\alpha_k^*)/\log(2)) > l_k$. Observe that, for all $k \geq 1$, for any $n \in \mathcal{A}_k$ with $2^{u_{n-1}} \geq \alpha_k^*$, we have $h^{-1}(u_n) \geq h^{-1}(u_{n-1}) \geq h^{-1}(\log(\alpha_k^*)/\log(2))$. Taking into account Lemma 5.3 and the inequality $1 - t \leq e^{-t}$, we get, for any $1 - \frac{1}{2^{u_j}} \leq r < 1 - \frac{1}{2^{u_{j+1}}}$ $(j \geq 1)$,

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$$
\begin{aligned}
M_1(f_0^{*(u)}, r) &\leq \sum_{n\geq 1} M_1(P_{n,0}^{*(u)}, r) \\
&\lesssim \sum_k \sum_{n\in\mathcal{A}_k; 2^{u_n-1}\geq \alpha_k^*} \left(1-\frac{1}{2^{u_{j+1}}}\right)^{2^{u_n}} l_k \\
&\lesssim \sum_k \sum_{n\in\mathcal{A}_k; 2^{u_n-1}\geq \alpha_k^*} \mathrm{e}^{-2^{u_n-u_{j+1}}} l_k \frac{h^{-1}(u_n)}{h^{-1}(\log(\alpha_k^*)/\log(2))} \\
&\lesssim \sum_{n=1}^{j+1} \mathrm{e}^{-2^{u_n-u_{j+1}}} h^{-1}(u_n) \\
&\lesssim (j+1)h^{-1}(u_{j+1}) = (j+1)^2.
\end{aligned}
$$

Since $2^{u_j} \leq \frac{1}{1-r} < 2^{u_{j+1}}$, we find $j \leq h^{-1}(\frac{-\log(1-r)}{\log(2)})$ and we get

$$
M_1(f_0^{*(u)}, r) \lesssim \left(h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right)\right)^2.
$$

□

Now we are going to prove that the functions $f_{\alpha c}^{(u)}$ and $f_{\alpha c}^{*(u)}$ are $\mathcal{U}$-frequently hypercyclic for T_{α_c}.

Proposition 5.6. *For $p \geq 2$ (resp. $1 \leq p < 2$), the function $f_{\alpha c}^{(u)}$ (resp. $f_{\alpha c}^{*(u)}$) is a $\mathcal{U}$-frequently hypercyclic vector for the operator T_{α_c}.*

Proof. We only prove that the vector $f_{\alpha c}^{(u)}$ is frequently hypercyclic for the operator T_{α_c}. We do not repeat the details for $f_{\alpha c}^{*(u)}$: it will be enough to make the appropriate modifications.

Let k be a large enough integer. Let us consider $n \in \mathcal{A}_k$ such that $2^{u_n-1} \geq \alpha_k$. We consider $\mathcal{B}_n$ the set of s in $I_n^{(u)}$ such that the coefficient z^s in the polynomial $z^{2^{u_n}} p_{\lfloor \frac{2^{u_n}}{\alpha_k} \rfloor}(z^{\alpha_k})$ is equal to 1 and we denote by $T_k = \{s : s \in \mathcal{B}_n,\ n \in \mathcal{A}_k,\ 2^{u_n-1} \geq \alpha_k\}$.

Observe that $\max(\mathcal{B}_n) \leq 2^{u_n+1}$ and since at least half of the coefficients of $p_{\lfloor \frac{2^{u_n}}{\alpha_k} \rfloor}$ being $+1$, we get

$$
\frac{\#\{j \leq \max(\mathcal{B}_n); j \in T_k\}}{\max(\mathcal{B}_n)} \geq \frac{2^{u_n-1}}{2^{u_n+1}} = \frac{1}{4}, \tag{24}
$$

which implies

$$
\overline{d}(T_k) > 0.
$$

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Then let α be a real number and let k be in $\mathbb{N}$. Now let us consider $s \in \mathcal{B}_n$ with $n \in \mathcal{A}_k$ satisfying $2^{u_{n-1}} \geq \alpha_k$. As in Proposition 4.9, by construction we get

$$\sup_{|z|=1-\frac{1}{l_k}} |T^s_{\alpha_c}(f^{(u)}_{\alpha_c})(z) - q_k(z)| \lesssim \frac{1}{l_k},$$

provided that k is chosen large enough. This allows to obtain frequent hypercyclicity of $f^{(u)}_{\alpha_c}$. □

Combining Lemma 5.4 with Proposition 5.6, we obtain the following result.

Theorem 5.7. *Let $1 \leq p \leq \infty$ and $\alpha_c = \frac{1}{\max(2,q)}$. Then, for any function $\varphi : [0,1) \to \mathbb{R}_+$ with $\varphi(r) \to +\infty$ as $r \to 1^-$, there is a function f in $H(\mathbb{D})$ with*

$$M_p(f,r) \lesssim \varphi(r), \quad \textit{for } 0 < r < 1 \textit{ sufficiently close to } 1,$$

that is $\mathcal{U}$-frequently hypercyclic for T_{α_c}.

Proof. We begin by the case $p > 1$. Without loss of generality, we can assume that the function φ is a continuous increasing function that can be written, for all $0 < r < 1$, $\varphi(r) = \psi\left(\frac{1}{1-r}\right)$ where ψ is a continuous increasing function with, for all $n \in \mathbb{N}$, $u_n := \psi^{-1}(n) \in \mathbb{N}$ and $u_{n+1} - u_n \to +\infty$. Thus Lemma 5.4 and Proposition 5.6 ensure that, for all $1 < p \leq \infty$, there is a function f in $H(\mathbb{D})$ with

$$M_p(f,r) \lesssim \psi\left(\frac{-\log(1-r)}{\log(2)}\right) \lesssim \psi\left(\frac{1}{1-r}\right) = \varphi(r)$$

that is $\mathcal{U}$-frequently hypercyclic for T_{α_c}.

Now we deal with the case $p = 1$. Without loss of generality, we can assume that φ is a continuous increasing function that can be written, for all $0 < r < 1$, $\varphi(r) = \left(\psi\left(\frac{1}{1-r}\right)\right)^2$ where ψ is a continuous and increasing function such that, for all $n \in \mathbb{N}$, $u_n := \psi^{-1}(n) \in \mathbb{N}$ and $u_{n+1} - u_n \to +\infty$. Applying Lemma 5.5 and Proposition 5.6, we find a function $f \in H(\mathbb{D})$ with

$$M_1(f,r) \lesssim \left(\psi\left(\frac{-\log(1-r)}{\log(2)}\right)\right)^2 \lesssim \left(\psi\left(\frac{1}{1-r}\right)\right)^2 = \varphi(r)$$

that is $\mathcal{U}$-frequently hypercyclic for T_0.

The proof is complete. □

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The $\mathcal{U}_{\beta\gamma}$-frequently hypercyclic case

We keep the definitions and the notations of § 5.1. We modify the definitions of polynomials $P_{n,\alpha}^{(u)}$ and $P_{n,\alpha}^{*(u)}$ as follows:

$$P_{n,\alpha}^{(u)}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} < \alpha_k \\ z^{2^{u_n}} Q_n(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} \geq \alpha_k \end{cases} \tag{25}$$

with for $n \in \mathcal{A}_k$,

$$Q_n^{(u)}(z) = \sum_{j \in I_n^{(u)}} (j+1)^{-\alpha} c_{j-2^{u_n}}^{(k)} z^{j-2^{u_n}}$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p_{\lfloor \frac{2^{u_n}(1-\gamma)}{\alpha_k} \rfloor}(z^{\alpha_k})\tilde{q}_k(z)$.

$$P_{n,\alpha}^{*(u)}(z) = \begin{cases} 0 \text{ if } n \text{ is odd} \\ 0 \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} < \alpha_k^* \\ z^{2^{u_n}} Q_n^{*(u)}(z) \text{ if } n \in \mathcal{A}_k \text{ and } 2^{u_{n-1}} \geq \alpha_k^*, \end{cases} \tag{26}$$

with, for $n \in \mathcal{A}_k$,

$$Q_n^{*(u)}(z) = \sum_{j \in I_n^{(u)}} (j+1)^{-\alpha} c_{j-2^{u_n}}^{(k)} z^{j-2^{u_n}}$$

where the sequence $(c_j^{(k)})$ denotes the sequence of the coefficients of the polynomial $p^*_{\lfloor \frac{2^{u_n}(1-\gamma)}{\alpha_k^*} \rfloor}(z^{\alpha_k^*})\tilde{q}_k(z)$.

Let $1 < p \leq \infty$. Set $\alpha_c = \frac{1-\gamma}{\max(2,q)}$.

Lemma 5.8. *We have, for any $0 < r < 1$ and all $n \in \mathbb{N}$,*

$$M_p(P_{n,\alpha_c}^{(u)}, r) \lesssim r^{2^{u_n}} \text{ if } 2 \leq p \leq \infty \quad \text{and} \quad M_p(P_{n,\alpha_c}^{*(u)}, r) \lesssim r^{2^{u_n}} \text{ if } 1 < p < 2.$$

Proof. It suffices to argue along the same lines as the proof of Lemma 4.7 replacing the sequence (2^n) by (2^{u_n}). □

From Lemma 5.2 and 5.8, we deduce the rate of growth of the functions $f_{\alpha_c}^{(u)}$ and $f_{\alpha_c}^{*(u)}$.

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Lemma 5.9. *Let* $1 < p \leq \infty$ *and* $\alpha_c = \frac{1-\gamma}{\max(2,q)}$. *Then, for all* $0 < r < 1$,

$$M_p(f_{\alpha_c}^{(u)}, r) \lesssim h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right) \text{ if } 2 \leq p \leq \infty,$$

and

$$M_p(f_{\alpha_c}^{*(u)}, r) \lesssim h^{-1}\left(\frac{-\log(1-r)}{\log(2)}\right) \text{ if } 1 < p < 2.$$

Now we are going to prove that the functions $f_{\alpha_c}^{(u)}$ and $f_{\alpha_c}^{*(u)}$ are U_{β^γ}-frequently hypercyclic for T_{α_c}.

Proposition 5.10. *For* $p \geq 2$ *(resp.* $1 < p < 2$*), the function* $f_{\alpha_c}^{(u)}$ *(resp.* $f_{\alpha_c}^{*(u)}$*) is a* U_{β^γ}*-frequently hypercyclic vector for the operator* T_{α_c}.

Proof. We only prove that the vector $f_{\alpha_c}^{(u)}$ is frequently hypercyclic for the operator T_{α_c}. We do not repeat the details for $f_{\alpha_c}^{*(u)}$: it will be enough to make the appropriate modifications.

Let k be a large enough integer. Let us consider $n \in \mathcal{A}_k$ such that $2^{u_n - 1} \geq \alpha_k$. We consider $\mathcal{B}_n$ the set of s in $I_n^{(u)}$ such that the coefficient z^s in the polynomial $z^{2^{u_n}} p_{\lfloor \frac{2^{u_n(1-\gamma)}}{\alpha_k} \rfloor}(z^{\alpha_k})$ is equal to 1 and we denote by $T_k = \{s : s \in \mathcal{B}_n,\ n \in \mathcal{A}_k,\ 2^{u_n-1} \geq \alpha_k\}$.

Observe that $\max(\mathcal{B}_n) \leq 2^{u_n} + \lfloor 2^{u_n(1-\gamma)} \rfloor$ and since at least half of the coefficients of $p_{\lfloor \frac{2^{u_n(1-\gamma)}}{\alpha_k} \rfloor}$ being $+1$, we get

$$\frac{\sum\limits_{\substack{j \leq \max(\mathcal{B}_n);\\ j \in T_k}} e^{j^\gamma}}{\sum\limits_{j \leq \max(\mathcal{B}_n)} e^{j^\gamma}} \geq \frac{\sum\limits_{j=2^{u_n}}^{2^{u_n}+2^{-1}\lfloor 2^{u_n(1-\gamma)} \rfloor} e^{j^\gamma}}{\sum\limits_{j=1}^{2^{u_n}+\lfloor 2^{u_n(1-\gamma)} \rfloor} e^{j^\gamma}} = \frac{\sum\limits_{j=1}^{2^{u_n}+2^{-1}\lfloor 2^{u_n(1-\gamma)} \rfloor} e^{j^\gamma}}{\sum\limits_{j=1}^{2^{u_n}+\lfloor 2^{u_n(1-\gamma)} \rfloor} e^{j^\gamma}} - \frac{\sum\limits_{j=1}^{2^{u_n}-1} e^{j^\gamma}}{\sum\limits_{j=1}^{2^{u_n}+\lfloor 2^{u_n(1-\gamma)} \rfloor} e^{j^\gamma}}. \tag{27}$$

Clearly, we have

$$\left(2^{u_n} + \frac{\lfloor 2^{u_n(1-\gamma)} \rfloor}{2}\right)^\gamma - \left(2^{u_n} + \lfloor 2^{u_n(1-\gamma)} \rfloor\right)^\gamma \to -\frac{\gamma}{2},$$
$$(2^{u_n} - 1)^\gamma - \left(2^{u_n} + \lfloor 2^{u_n(1-\gamma)} \rfloor\right)^\gamma \to -\gamma,$$

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which implies, using similar estimations as those of the proof of Lemma 4.4,

$$\frac{\sum_{j=1}^{2^{u_n}+2^{-1}\lfloor 2^{u_n(1-\gamma)}\rfloor} e^{j^\gamma}}{\sum_{j=1}^{2^{u_n}+\lfloor 2^{u_n(1-\gamma)}\rfloor} e^{j^\gamma}} - \frac{\sum_{j=1}^{2^{u_n}-1} e^{j^\gamma}}{\sum_{j=1}^{2^{u_n}+\lfloor 2^{u_n(1-\gamma)}\rfloor} e^{j^\gamma}} \to e^{-\gamma/2}(1-e^{-\gamma/2}), \text{ as } n \to +\infty.$$

Hence the inequality (27) ensures that

$$\overline{d}_{\beta^\gamma}(T_k) > 0.$$

Then let α be a real number and let k be in $\mathbb{N}$. Now for $s \in \mathcal{B}_n$ with $n \in \mathcal{A}_k$ satisfying $2^{u_{n-1}} \geq \alpha_k$, as in Proposition 4.9, by construction we get

$$\sup_{|z|=1-\frac{1}{l_k}} |T^s_{\alpha_c}(f^{(u)}_{\alpha_c})(z) - q_k(z)| \lesssim \frac{1}{l_k},$$

provided that k is chosen large enough. This allows to obtain frequent hypercyclicity of $f^{(u)}_{\alpha_c}$. □

Combining Lemma 5.9 with Proposition 5.10 we obtain the following result.

Theorem 5.11. *Let $0 < \gamma < 1$. Let $1 \leq p \leq \infty$ and $\alpha_c = \frac{1-\gamma}{\max(2,q)}$. Then, for any function $\varphi : [0,1) \to \mathbb{R}_+$, with $\varphi(r) \to +\infty$ as $r \to 1^-$, there is a function f in $H(\mathbb{D})$ with*

$$M_p(f,r) \lesssim \varphi(r), \qquad \textit{for } 0 < r < 1 \textit{ sufficiently close to } 1,$$

that is $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic for T_{α_c}.

Proof. For $p = 1$, we have $\alpha_c = 0$ and the result is given by Theorem 5.7. Now let $p > 1$. Without loss of generality, we can assume that the φ is a continuous increasing function such that, for all $0 < r < 1$, $\varphi(r) = \psi\left(\frac{1}{1-r}\right)$ where ψ is continuous and increasing with, for all $n \in \mathbb{N}$, $u_n := \psi^{-1}(n) \in \mathbb{N}$ and $u_{n+1} - u_n \to +\infty$. Thus, Lemma 5.9 and Proposition 5.10 ensure that, for all $1 < p \leq \infty$, there is a function f in $H(\mathbb{D})$ with

$$M_p(f,r) \lesssim \psi\left(\frac{-\log(1-r)}{\log(2)}\right) \lesssim \varphi(r)$$

that is $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic for T_{α_c}. The proof is complete. □

From Theorems 2.2, 5.7 and 5.11, we can state the following result that unifies what happens in the critical case, given by the critical exponent, for the L^p growth of $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic functions for T_α, when γ belongs to $[0,1]$.

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Theorem 5.12. *Let $0 \leq \gamma \leq 1$ and $1 \leq p \leq \infty$. Then, for any function $\varphi : [0,1) \to \mathbb{R}_+$, with $\varphi(r) \to +\infty$ as $r \to 1^-$, there is a function f in $H(\mathbb{D})$ with $M_p(f,r) \lesssim \varphi(r)$ that is $\mathcal{U}_{\beta^\gamma}$-frequently hypercyclic for T_{α_c} where $\alpha_c = \frac{1-\gamma}{\max(2,q)}$.*

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