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ON FLATNESS COVERS OF CYCLIC ACTS OVER MONOIDS

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Abstract. The covers of cyclic acts over monoids were investigated by Mahmoudi and Renshaw (M. Mahmoudi and J. Renshaw, On covers of cyclic acts over monoids, *Semigroup Forum* **77** (2008), 325–338) and the authors posed some open problems. In the present paper, we give answers to their problems 1 and 5, and we also give a sufficient and necessary condition that a cyclic act has a weakly pullback flat cover.

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1. Introduction. Throughout this paper, S always stands for a monoid, and N for the set of natural numbers.

Over the past several decades, the covers of modules have been investigated by many authors and ample results have been obtained (see [**1**, **3**, **4**, **12**, **15**]). Covers of acts over monoids are studied in [**5**, **7**, **8**]. Further investigations about this field were laid dormant until the recent appearance of [**13**].

Let us recall results and definitions that we shall use below. We refer the reader to [**11**] for a detailed account of these.

A monoid S is said to be *right reversible* if for any $p, q \in S$ there exist $u, v \in S$ such that $up = vq$. A monoid S is said to be *weakly left collapsible* if for any $p, q, r \in S$ with $pr = qr$ there exists $u \in S$ such that $up = uq$.

In [**2**], the acts, now called *strongly flat*, were introduced: A right S-act A_S is strongly flat if the functor $A_S \bigotimes -$ preserves pullbacks and equalizers. In the same paper, strongly flat acts were characterized as the acts satisfying two interpolation conditions, later labelled as condition (P) and condition (E):

$$(P) \qquad (\forall\, a, a' \in A)(\forall\, s, t \in S)(as = a't \Rightarrow (\exists\, a'' \in A)(\exists\, u, v \in S)(a = a''u \wedge a' = a''v \wedge us = vt)),$$

$$(E) \qquad (\forall\, a \in A)(\forall\, s, s' \in S)(as = as' \Rightarrow (\exists\, a' \in A)(\exists\, u \in S)(a = a'u \wedge us = us')).$$

In [**14**] Laan called a right S-act A *weakly pullback flat* if it satisfies condition (P) and the following condition (E').

$$(E') \qquad (\forall\, a \in A)(\forall\, s, s', z \in S)(as = as' \wedge sz = s'z \Rightarrow (\exists\, a' \in A)(\exists\, u \in S)(a = a'u \wedge us = us')).$$

Definition 1.1 (Definition 2.1 in [**13**]). Let S be a monoid and A an S-act. An S-act C together with an S-epimorphism $f\colon C \to A$ is a cover of A if there is no proper subact B of C such that $f|B$ is onto. We shall usually refer to C as the cover.

Definition 1.2 (Definition 2.2 in [**13**]). Let S be a monoid and $f\colon C \to A$ be an S-epimorphism. We call f coessential if for each S-act B and each S-map $g\colon B \to C$, if fg is an epimorphism then g is an epimorphism.

The purpose of the present paper is to answer the open problem 1 and open problem 5 in [**13**]. We also show that a cyclic S-act S/ρ has a weakly pullback flat cover if and only if $[1]_\rho$ contains a right reversible and weakly left collapsible submonoid R such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.

2. On (*P*)-covers of cyclic acts.

In [**13**] the following open problem is posed.

Problem. Is there a monoid S and a cyclic S-act A that do not have a (P)-cover?

In the following, we will give an affirmative answer to this question.

Lemma 2.1 (Lemma 7 in [**9**]). *Let $P \subseteq S$ be a right reversible submonoid and let ρ be the relation on S defined by*

$$s\rho s' \Leftrightarrow (\exists p, q \in P)(ps = qs').$$

Then:

(1) *ρ is a right congruence.*
(2) *The right S-act S/ρ satisfies condition (P).*
(3) *If P is weakly left collapsible, then S/ρ is weakly pullback flat.*

Let X be a non-empty set, and let X^+ denote the free semigroup generated by X. If we adjoin an identity 1 to X^+, we obtain the free monoid on X and we denote this by X^*. For each w in X, by [**6**] the *content* $C(w)$ is defined as the (necessarily finite) set of elements of X appearing in w. Let R be a subsemigroup of a free semigroup X^+. We define the *content* $C(R)$ of R as

$$C(R) = \bigcup_{w \in R} C(w).$$

Lemma 2.2. *Let X be a non-empty set and let R be a subsemigroup of X^+. Then R is right reversible if and only if $|C(R)| = 1$.*

Proof. For the sufficiency, since $|C(R)| = 1$, we suppose $C(R) = \{x\}$. For any $s, t \in R$, there exist $m, n \in N$ such that $s = x^m$ and $t = x^n$. It is clear that $x^n \cdot x^m = x^m \cdot x^n$. Now R is right reversible.

For necessity if $|C(R)| > 1$, then there exist two distinct elements $x, x' \in X$ such that the elements $x_m \cdots x_s x x_{s-1} \cdots x_1$ and $x'_n \cdots x'_t x' x'_{t-1} \cdots x'_1$ belong to R, where $m, n \in N$ and $x_i, x'_j \in X$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Then by the property of the semigroup X^+, R is not right reversible. This is a contradiction. Hence, $|C(R)| = 1$.

Lemma 2.3 (Corollary 4.3 in [**13**]). *The* 1*-element S-act Θ has a (P)-cover if and only if there exists a right reversible submonoid R of S such that for all $u \in S$, there exists $s \in S$ with $us \in R$.*

Example 2.4. Let X be a set with more than 3 elements and $S = X^*$, the free monoid generated by X. Then the (cyclic) 1-element S-act Θ has no (P)-cover.

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Proof. Suppose R is a right reversible submonoid of X^*. By Lemma 2.2 $|C(R)| = 1$, and we suppose $C(R) = \{x\}$. There exists $u \in X^*$ such that for every $s \in X^*$, $us \notin R$. By Lemma 2.3 Θ has no (P)-cover. □

3. On strongly flat covers of cyclic acts. In [**13**] the following open problem is posed.

PROBLEM. Are strongly flat covers unique?

LEMMA 3.1 (Lemma 2.1 in [**10**]). *Let $P \subseteq S$ be a left collapsible submonoid and let ρ be the relation on S defined by*

$$s\rho s' \Leftrightarrow (\exists p, q \in P)(ps = qs').$$

Then:
(1) *ρ is a right congruence.*
(2) *S/ρ is strongly flat.*

LEMMA 3.2 (Theorem 3.2 in [**13**]). *Let S be a monoid. Then the cyclic S-act S/ρ has a strongly flat cover if and only if $[1]_\rho$ contains a left collapsible submonoid R such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.*

LEMMA 3.3 (Theorem 2.7 in [**13**]). *Let S be a monoid and S/ρ a cyclic S-act. The map $f : S/\sigma \to S/\rho$ given by $s\sigma \mapsto s\rho$ is a coessential epimorphism if and only if*

$$\sigma \subseteq \rho \text{ and for all } u \in [1]_\rho, \quad uS \cap [1]_\sigma \neq \emptyset.$$

EXAMPLE 3.4. Let

$$S = \langle a, b, c | ab = ba = ac = ca = a,\ bc = c^2, cb = b^2, a^4 = a^5, b^5 = b^6, c^6 = c^7\rangle \cup \{1\}.$$

Define an equivalence relation ρ on S by

$$s\rho t \Longleftrightarrow (s, t \in \langle a\rangle) \text{ or } (s, t \in (\langle b\rangle \cup \langle c\rangle \cup \{1\}).$$

It is easy to verify that ρ is a right congruence on S. Then $[1]_\rho = \langle b\rangle \cup \langle c\rangle \cup \{1\}$ and the strongly flat cover of S/ρ is not unique.

Proof. By the definition of ρ, it is a proper right congruence of S. Denote $R_1 = \langle b\rangle \cup \{1\}$ and $R_2 = \langle c\rangle \cup \{1\}$, then R_1 and R_2 are both left collapsible submonoids of $[1]_\rho$.

Define a right congruence σ_1 on S by

$$s\,\sigma_1 t \Longleftrightarrow (\exists p, q \in R_1)(ps = qt).$$

Define a right congruence σ_2 on S by

$$s\,\sigma_2 t \Longleftrightarrow (\exists p, q \in R_2)(ps = qt).$$

Hence, for every $u \in [1]_\rho$, $uS \cap [1]_{\sigma_i} \neq \emptyset$ $(i = 1, 2)$.

Then by Lemma 3.1, S/σ_1 and S/σ_2 are both strongly flat. But $\sigma_1 \neq \sigma_2$, since $(b, 1) \in \sigma_1$ but $(b, 1) \notin \sigma_2$, $(c, 1) \in \sigma_2$ but $(c, 1) \notin \sigma_1$.

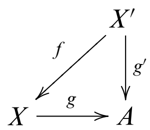
By Lemmas 3.2 and 3.3, S/σ_1 and S/σ_2 are both strongly flat covers of S/ρ.

Now we have the following. □

PROPOSITION 3.5. *Strongly flat covers of cyclic acts need not be unique.*

This proposition gives a negative answer to the previous question.

REMARK 3.6. Let S be a monoid, and A be an S-act. As in [**13**], let $\mathcal{X}$ be a class of acts that is closed under isomorphism. By a $\mathcal{X}$-precover of A we mean an S-map $g : X \to A$ from some $X \in \mathcal{X}$ such that for every S-map $g' : X' \to A$, for $X' \in \mathcal{X}$, there exists an S-map $f : X' \to X$ with $g' = gf$.



If in addition the precover satisfies the condition that each S-map $f : X \to X$ with $gf = g$ is an isomorphism, then we shall call it a $\mathcal{X}$-cover. It is clear that the $\mathcal{X}$-cover is unique up to isomorphism. If $\mathcal{SF}$ is the class strongly flat acts then by Proposition 3.5 we now know the strongly flat covers and $\mathcal{SF}$-covers do not coincide.

4. On weakly pullback flat covers of cyclic acts.

LEMMA 4.1 (Theorem 2.8 in [**13**]). *Let S be a monoid and S/ρ a cyclic S-act. If R is a submonoid of $[1]_\rho$ such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$, then there exists a right congruence σ on S such that $R \subseteq [1]_\sigma$ and S/σ is a cover of S/ρ. Moreover, $R = [1]_\sigma$ if and only if R is a left unitary submonoid of S.*

LEMMA 4.2 (Lemma 9 in [**9**]). *Let S be a monoid, σ a right congruence on S and let the cyclic S-act S/σ be weakly pullback flat. Then $R = [1]_\sigma$ is a right reversible and weakly left collapsible submonoid of S.*

THEOREM 4.3. *Let S be a monoid. Then the cyclic S-act S/ρ has a weakly pullback flat cover if and only if $[1]_\rho$ contains a right reversible and weakly left collapsible submonoid R such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.*

Proof. Suppose that S/ρ has a weakly pullback flat cover S/σ. Then by Lemma 3.3 we can assume that $R = [1]_\sigma \subseteq [1]_\rho$ and that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$. Moreover, R is right reversible and weakly left collapsible by Lemma 4.2.

Conversely, suppose that R is a right reversible and weakly left collapsible submonoid of $[1]_\rho$ such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$. Define a right congruence σ on S by

$$s \,\sigma\, t \Longleftrightarrow (\exists p, q \in R)(ps = qt).$$

Then by Lemma 2.1 S/σ is weakly pullback flat. Finally, by Lemma 3.3, S/σ is a weakly pullback flat cover of S/ρ. □

COROLLARY 4.4. *The 1-element S-act Θ has a weakly pullback flat cover if and only if there exists a right reversible and weakly left collapsible submonoid R of S such that for all $u \in S$, there exists $s \in S$ with $us \in R$.*

Now by Example 2.4 we also have the following.

REMARK 4.5. There exists a monoid S and a cyclic S-act A which does not have a weakly pullback flat cover.

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THEOREM 4.6. *If S is a monoid then every cyclic S-act has a weakly pullback flat cover if and only if every left unitary submonoid T of S contains a right reversible and weakly left collapsible submonoid R such that for all $u \in [1]_\rho$, $uS \cap R \neq \emptyset$.*

Since commutative monoids are necessarily right reversible and weakly left collapsible, we have the following.

THEOREM 4.7. *Let S be a commutative monoid. Then every cyclic S-act has a weakly pullback flat cover.*

COROLLARY 4.8 (Theorem 4.5 in [**13**]). *Let S be a commutative monoid. Then every cyclic S-act has a (P)-cover.*

COROLLARY 4.9. *Let S be a right cancellative monoid. The cyclic S-act S/ρ has a weakly pullback flat cover if and only if S/ρ has a (P)-cover.*

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