Can. J. Math., Vol. XXXIII, No. 6, 1981, pp. 1498–1525

CRITERIA FOR σ-SMOOTHNESS, τ-SMOOTHNESS, AND TIGHTNESS OF LATTICE REGULAR MEASURES, WITH APPLICATIONS

GEORGE BACHMAN AND PANAGIOTIS D. STRATIGOS

Introduction. Consider an arbitrary set X and an arbitrary disjunctive lattice of subsets of X, $\mathscr{L}$. The algebra of subsets of X generated by $\mathscr{L}$ is denoted by $\mathscr{A}(\mathscr{L})$, the set of all $\mathscr{L}$-regular measures on $\mathscr{A}(\mathscr{L})$, by $MR(\mathscr{L})$, and the associated Wallman space, a compact T_1 space, by $IR(\mathscr{L})$; assume X is embedded in $IR(\mathscr{L})$ (otherwise, consider the image of X in $IR(\mathscr{L})$).

In part of an earlier paper [**4**] the work of Knowles [**15**] and Gould and Mahowald [**11**] was generalized from the explicit topological setting of X, a Tychonoff space, with $\mathscr{L}$ the lattice of zero sets of X, to the above setting, with the added assumption that $\mathscr{L}$ was also δ and normal. This was done so that the important Alexandroff Representation Theorem [**1**] could be utilized in order to induce two associated measures $\hat{\mu}$ and $\tilde{\mu}$ defined on $\mathscr{A}(W(\mathscr{L}))$ and $\mathscr{A}(tW(\mathscr{L}))$, respectively, where $W(\mathscr{L})$ is the Wallman lattice in $IR(\mathscr{L})$. In terms of these measures, conditions were then given for the general element of $MR(\mathscr{L})$, μ, to be σ-smooth, τ-smooth, and tight, and applications were given. These conditions were expressed in terms of the measures $\hat{\mu}$ and $\tilde{\mu}$ and the remainder $IR(\mathscr{L}) - X$.

However, these results precluded a consideration of certain important lattices which are either not δ or not normal, such as the lattice of clopen sets in a T_2, 0-dimensional space or the lattice of closed sets in a T_1 topological space.

By utilizing regular measure-extension theorems, we can now generalize the above results, so that we need not assume $\mathscr{L}$ is δ and normal, but just disjunctive or at times separating. This has the advantage that we can systematically consider all the important topological lattices and can treat, for the first time, in a unified measure theoretical fashion, the particular remainders $\omega X - X$, $\beta X - X$, and $\beta_0 X - X$, where ωX is the Wallman compactification of X, [**22**], βX is the Stone-Čech compactification of X, [**10**], and $\beta_0 X$ is the Banachewski compactification of X [**6**].

Our techniques, in particular, lead to new measure-extension results for regular τ-smooth measures (Theorem 2.5), and for certain countably additive measures (Theorem 3.3). They also yield new criteria for lattice

---

Received July 29, 1980 and in revised form June 18, 1981.

countable compactness (Theorem 3.1), and for lattice repleteness (Theorems 3.2 and 3.5) having specific applications thence to α-completeness, realcompactness, N-compactness, etc.

Finally, the general results give new proofs and generalizations of various measure decomposition theorems, such as the Yosida-Hewitt Decomposition Theorem (Lemmas 4.1 and 4.2 and Theorem 4.1).

1. Terminology and notation. a). Consider any set X and any lattice of subsets of X, $\mathscr{L}$. We shall always assume, without loss of generality for our purposes, that $\emptyset, X \in \mathscr{L}$. $\mathscr{L}$ is said to be δ, if for every subset of $\mathscr{L}$, $\{L_\alpha; \alpha \in A\}$, if A is countable, then $\bigcap \{L_\alpha; \alpha \in A\} \in \mathscr{L}$. $\mathscr{L}$ is said to be *complemented*, if for every element of $\mathscr{L}$, L, $L' \in \mathscr{L}$. $\mathscr{L}$ is said to be T_2, if for any two elements of X, a, b, if $a \neq b$, then there exist two elements of $\mathscr{L}$, A, B, such that $a \in A'$ and $b \in B'$ and $A' \cap B' = \emptyset$. $\mathscr{L}$ is said to be *separating*, if for any two elements of X, a, b, if $a \neq b$, then there exists an element of $\mathscr{L}$, A, such that $a \in A$ and $b \notin A$. $\mathscr{L}$ is said to be *disjunctive*, if for every element of X, a, and for every element of $\mathscr{L}$, B, if $a \notin B$, then there exists an element of $\mathscr{L}$, A, such that $a \in A$ and $A \cap B = \emptyset$. $\mathscr{L}$ is said to be *regular*, if for every element of X, a, and for every element of $\mathscr{L}$, B, if $a \notin A$, then there exist two elements of $\mathscr{L}$, C, D, such that $a \in C'$ and $B \subset D'$ and $C' \cap D' = \emptyset$. $\mathscr{L}$ is said to be *normal* if for any two elements of $\mathscr{L}$, A, B, if $A \cap B = \emptyset$, then there exist two elements of $\mathscr{L}$, C, D, such that $A \subset C'$ and $B \subset D'$ and $C' \cap D' = \emptyset$. $\mathscr{L}$ is said to be *Lindelöf* if for every subset of $\mathscr{L}$, $\{L_\alpha; \alpha \in A\}$, if $\bigcap \{L_\alpha; \alpha \in A\} = \emptyset$, then there exists a subset of A, A^*, such that $\bigcap \{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is countable. $\mathscr{L}$ is said to be *compact* if for every subset of $\mathscr{L}$, $\{L_\alpha; \alpha \in A\}$, if $\bigcap \{L_\alpha; \alpha \in A\} = \emptyset$, then there exists a subset of A, A^*, such that $\bigcap \{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is finite. $\mathscr{L}$ is said to be *countably compact* if for every subset of $\mathscr{L}$, $\{L_\alpha; \alpha \in A\}$, if $\bigcap \{L_\alpha; \alpha \in A\} = \emptyset$ and A is countable, then there exists a subset of A, A^*, such that $\bigcap \{L_\alpha; \alpha \in A^*\} = \emptyset$ and A^* is finite. $\mathscr{L}$ is said to be *countably paracompact* if for every sequence in $\mathscr{L}$, $\langle A_n \rangle$, if $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$, then there exists a sequence in $\mathscr{L}$, $\langle B_n \rangle$, such that for every n, $A_n \subset B_n'$, and $\langle B_n' \rangle$ is decreasing and $\lim_n B_n' = \emptyset$.

Next, consider any two lattices of subsets of X, $\mathscr{L}_1$, $\mathscr{L}_2$. $\mathscr{L}_1$ is said to *separate* $\mathscr{L}_2$ if for any two elements of $\mathscr{L}_2$, L_2, $\tilde{L}_2$, if $L_2 \cap \tilde{L}_2 = \emptyset$, then there exist two elements of $\mathscr{L}_1$, L_1, $\tilde{L}_1$, such that $L_2 \subset L_1$ and $\tilde{L}_2 \subset \tilde{L}_1$ and $L_1 \cap \tilde{L}_1 = \emptyset$.

b). The set of natural numbers is denoted by $\mathbf{N}$. For an arbitrary function f, the domain of f is denoted by D_f. The set whose general element is the intersection of an arbitrary subset of $\mathscr{L}$ which is countable is denoted by $\delta\mathscr{L}$. The set whose general element is the intersection of an arbitrary subset of $\mathscr{L}$ is denoted by $t\mathscr{L}$. A function, f, from X to $\mathbf{R} \cup \{\pm\infty\}$ is said to be $\mathscr{L}$-*continuous* if for every closed subset of

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

$\mathbf{R} \cup \{\pm\infty\}$, $C, f^{-1}(C) \in \mathscr{L}$. The set whose general element is a function from X to $R \cup \{\pm\infty\}$ which is $\mathscr{L}$-continuous is denoted by $C(\mathscr{L})$. The set whose general element is an element of $C(\mathscr{L})$ which is bounded is denoted by $C_b(\mathscr{L})$. The set whose general element is a zero set of $\mathscr{L}$ is denoted by $\mathscr{Z}(\mathscr{L})$. The algebra of subsets of X generated by $\mathscr{L}$ is denoted by $\mathscr{A}(\mathscr{L})$. The σ-algebra of subsets of X generated by $\mathscr{L}$ is denoted by $\sigma(\mathscr{L})$. Next, consider any algebra of subsets of X, $\mathscr{A}$. A *measure* on $\mathscr{A}$ is defined to be a function, μ, from $\mathscr{A}$ to $\mathbf{R}$, such that μ is bounded and finitely additive. (See [**1**], p. 567.) The set whose general element is a measure on $\mathscr{A}(\mathscr{L})$ is denoted by $M(\mathscr{L})$. An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$*-regular* if for every element of $\mathscr{A}(\mathscr{L})$, E, for every positive number ϵ, there exists an element of $\mathscr{L}$, L, such that $L \subset E$ and $|\mu(E) - \mu(L)| < \epsilon$. The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$-regular is denoted by $MR(\mathscr{L})$. For the general element of $M(\mathscr{L})$, μ, the *support* of μ is defined to be $\cap \{L \in \mathscr{L} | \mu(L) = \mu(X)\}$ and is denoted by $S(\mu)$. An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$-(δ*-smooth*) if for every sequence in $\mathscr{A}(\mathscr{L})$, $\langle A_n \rangle$, if $\langle A_n \rangle$ is decreasing and $\lim_n A_n = \emptyset$, then $\lim_n \mu(A_n) = 0$. (See [**21**].) The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$- (σ-smooth) is denoted by $M(\sigma, \mathscr{L})$. An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$- (τ*-smooth*) if for every net in $\mathscr{L}$, $\langle L_\alpha \rangle$, if $\langle L_\alpha \rangle$ is decreasing and $\lim_\alpha L_\alpha = \emptyset$, then $\lim_\alpha \mu(L_\alpha) = 0$. (See [**21**].) The set whose general element is an element of $M(\mathscr{L})$, μ, which is $\mathscr{L}$-(τ-smooth) is denoted by $M(\epsilon, \mathscr{L})$. An element of $M(\mathscr{L})$, μ, is said to be $\mathscr{L}$*-tight* if $\mu \in M(\sigma, \mathscr{L})$ and for every positive number ϵ, there exists an $\mathscr{L}$-compact set, K, such that $\mu_*(K') < \epsilon$. (See [**21**].) The set whose general element is an element of $M(\mathscr{L})$ which is $\mathscr{L}$-tight is denoted by $M(t, \mathscr{L})$. The set whose general element is an element of $M(\mathscr{L})$, μ, such that for every element of $C(\mathscr{L})$, f, $\int f \, d\mu \in \mathbf{R}$ is denoted by $MI(\mathscr{L})$. The set whose general element is an element of $M(\mathscr{L})$, μ, such that $\mu(\mathscr{A}(\mathscr{L})) = \{0, 1\}$ is denoted by $I(\mathscr{L})$. $\mathscr{L}$ is said to be *replete* if for every element of $IR(\sigma, \mathscr{L})$, μ, $S(\mu) \neq \emptyset$.

Since every element of $M(\mathscr{L})$ is equal to the difference of nonnegative elements of $M(\mathscr{L})$, in the sequel we shall work, exclusively, with nonnegative elements of $M(\mathscr{L})$, without loss of generality.

2. In this section we work with an arbitrary set X and a fairly arbitrary lattice of subsets of X, $\mathscr{L}$; with this pair we associate the general Wallman space $IR(\mathscr{L})$ (see below) and for the general element of $MR(\mathscr{L})$ we investigate how the properties of σ-smoothness, τ-smoothness, and tightness reflect over to $IR(\mathscr{L})$ and conversely.

Preliminaries. (i). Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating and disjunctive. It is known that the

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

topological space $\langle IR(\mathscr{L}), tW(\mathscr{L})\rangle$ is compact and T_1; it is T_2, if and only if $\mathscr{L}$ is normal. (See e.g. [**2**] and [**18**].) Consider the function ϕ which is such that $D_\phi = X$ and for every element of X, x, $\phi(x) = \mu_x$. Then, ϕ is a $\langle t\mathscr{L}, tW(\mathscr{L})\rangle$-homeomorphism. For this reason, $\phi(X)$ is identifiable with X. Moreover, $\phi(X)$ is dense in $IR(\mathscr{L})$. Consequently, $IR(\mathscr{L})$ is a compactification of X; it is known as the general Wallman compactification of X. In case $\phi(X)$ is identified with X, then X is said to be embedded in $IR(\mathscr{L})$.

Denote the general element of $\mathscr{A}(\mathscr{L})$ by A. Then, $\{\mu \in IR(\mathscr{L}) | \mu(A) = 1\}$ is denoted by $W(A)$. The following statements are true:

1. If $A \in \mathscr{A}(\mathscr{L})$, then $W(A)' = W(A')$.

2. If $A, B \in \mathscr{A}(\mathscr{L})$, then α) $W(A \cup B) = W(A) \cup W(B)$; β)$W(A \cap B) = W(A) \cap W(B)$; γ) If $A \supset B$, then $W(A) \supset W(B)$; δ) If $W(A) \supset W(B)$, then $A \supset B$; ϵ) $A = B$, if and only if $W(A) = W(B)$.

3. $\mathscr{A}(W(\mathscr{L})) = W(\mathscr{A}(\mathscr{L}))$.

(Proofs are omitted. Note all these statements are true, if $\mathscr{L}$ is simply disjunctive.)

Next, consider any element of $M(\mathscr{L})$, μ, and the function $\hat{\mu}$ which is such that $D_{\hat{\mu}} = \mathscr{A}(W(\mathscr{L}))$ and for every element of $\mathscr{A}(W(\mathscr{L}))$, $W(A)$, $\hat{\mu}(W(A)) = \mu(A)$. Then, $\hat{\mu} \in M(W(\mathscr{L}))$ and, if $\mu \in MR(\mathscr{L})$, then $\hat{\mu} \in MR(W(\mathscr{L}))$. Conversely, consider any element of $M(W(\mathscr{L}))$, ν, and the function μ which is such that $D_\mu = \mathscr{A}(\mathscr{L})$ and for every element of $\mathscr{A}(\mathscr{L})$, A, $\mu(A) = \nu(W(A))$. Then, $\mu \in M(\mathscr{L})$ and $\nu = \hat{\mu}$, and, if $\nu \in MR(W(\mathscr{L}))$, then $\mu \in MR(\mathscr{L})$. Note since $W(\mathscr{L})$ is compact,

$$MR(W(\mathscr{L})) = MR(\sigma, W(\mathscr{L})) = MR(\tau, W(\mathscr{L})) = MR(t, W(\mathscr{L})).$$

Next, consider any element of $MR(\mathscr{L})$, μ. Then,

$$\hat{\mu} \in MR(W(\mathscr{L})) = MR(\sigma, W(\mathscr{L})).$$

Hence, $\hat{\mu}$ is extendible to the σ-algebra of $\hat{\mu}^*$-measurable sets, uniquely, and the extension is $\delta W(\mathscr{L})$-regular. Continue to use $\hat{\mu}$ for this extension.

(ii). The following statement is true:

$$\mathscr{A}(W_\sigma(\mathscr{L})) = W_\sigma(\mathscr{A}(\mathscr{L})).$$

(Proof omitted.) Next, consider any element of $M(\mathscr{L})$, μ, and the function μ' which is such that $D_{\mu'} = \mathscr{A}(W_\sigma(\mathscr{L}))$ and for every element of $\mathscr{A}(W_\sigma(\mathscr{L}))$, $W_\sigma(B)$, $\mu'(W_\sigma(B)) = \mu(B)$. Then, $\mu' \in M(W_\sigma(\mathscr{L}))$ and, if $\mu \in MR(\mathscr{L})$, then $\mu' \in MR(W_\sigma(\mathscr{L}))$. Conversely, consider any element of $M(W_\sigma(\mathscr{L}))$, ρ, and the function μ which is such that $D_\mu = \mathscr{A}(\mathscr{L})$ and for every element of $\mathscr{A}(\mathscr{L})$, B, $\mu(B) = \rho(W_\sigma(B))$. Then, $\mu \in M(\mathscr{L})$ and $\rho = \mu'$, and, if $\rho \in MR(W_\sigma(\mathscr{L}))$, then $\mu \in MR(\mathscr{L})$. The following statement is true: If $\mu \in MR(\mathscr{L})$, then $\mu \in MR(\sigma, \mathscr{L})$ if and only if $\mu' \in MR(\sigma, W_\sigma(\mathscr{L}))$. (Proof omitted.)

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

Observation. Note $\mu \in IR(\sigma, \mathscr{L})$ if and only if $\mu' \in IR(\sigma, W_\sigma(\mathscr{L}))$. Next, for the general element of $IR(\sigma, W_\sigma(\mathscr{L}))$, μ', note

$$S(\mu') = \cap \{W_\sigma(L) | L \in \mathscr{L} \text{ and } \mu'(W_\sigma(L)) = 1\}.$$

Further, note for every element of $\mathscr{L}$, L, if $\mu'(W_\sigma(L)) = 1$, then $\mu(L) = 1$, by the definition of μ', and, consequently, $\mu \in W_\sigma(L)$. Hence, $\mu \in S(\mu')$. Hence, $S(\mu') \neq \emptyset$. Consequently, $IR(\sigma, \mathscr{L})$ is $W_\sigma(\mathscr{L})$-replete.

Part I. (On σ-smoothness.)

THEOREM 2.1. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is (separating) and disjunctive. If $\mu \in MR(\mathscr{L})$, then the following statements are equivalent:*

1. $\mu \in MR(\sigma, \mathscr{L})$.
2. *If $\langle L_i; i \in \mathbf{N}\rangle$ is in $\mathscr{L}$ and $\langle L_i \rangle$ is decreasing and*

$$\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X,$$

then $\hat{\mu}(\bigcap_i W(L_i)) = 0$.

3. *If $\langle L_i; i \in \mathbf{N}\rangle$ is in $\mathscr{L}$ and $\langle L_i \rangle$ is decreasing and*

$$\bigcap_i W(L_i) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}),$$

then $\hat{\mu}(\bigcap_i W(L_i)) = 0$.

4. $\hat{\mu}^*(X) = \hat{\mu}(IR(\mathscr{L}))$.
5. $\hat{\mu}^*(IR(\sigma, \mathscr{L})) = \hat{\mu}(IR(\mathscr{L}))$.

Proof. α). Show 1 and 2 are equivalent. Assume 1, and show 2. Consider any sequence in $\mathscr{L}$, $\langle L_i \rangle$, such that $\langle L_i \rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$ and show $\hat{\mu}(\bigcap_i W(L_i)) = 0$. Note

$$\hat{\mu}(\bigcap_i W(L_i)) = \lim_i \hat{\mu}(W(L_i)),$$

since $\langle W(L_i) \rangle$ is decreasing (because $\langle L_i \rangle$ is decreasing), and

$$\hat{\mu} \in M(\sigma, W(\mathscr{L})) = \lim_i \mu(L_i) = \mu(\bigcap_i L_i),$$

since $\langle L_i \rangle$ is decreasing, and $\mu \in M(\sigma, \mathscr{L})$, by the assumption. Since $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, $\bigcap_i L_i = \emptyset$. Consequently,

$$\hat{\mu}(\bigcap_i W(L_i)) = 0.$$

Hence, 2 is true. Conversely, assume 2, and show 1. Consider any sequence in $\mathscr{L}$, $\langle L_i \rangle$, such that $\langle L_i \rangle$ is decreasing and $\lim_i L_i = \emptyset$, and show $\lim_i \mu(L_i) = 0$. Note

$$\lim_i \mu(L_i) = \lim_i \hat{\mu}(W(L_i)) = \hat{\mu}(\bigcap_i W(L_i)).$$

Show $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$. Assume

$$\bigcap_i W(L_i) \not\subset IR(\mathscr{L}) - X.$$

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

Then, there exists an element of X, x, such that $\mu_x \in \bigcap_i W(L_i)$. Consider any such x. Since $\mu_x \in \bigcap_i W(L_i)$, for every i, $\mu_x(L_i) = 1$. Hence, $\lim_i \mu_x(L_i) = 1$. Since $\mu_x \in I(\sigma, \mathscr{L})$, a contradiction has arisen. Hence, the assumption is wrong. Hence, $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$. Hence, since 2 is true, $\hat{\mu}(\bigcap_i W(L_i)) = 0$. Consequently, $\lim_i \mu(L_i) = 0$. Hence, $\mu \in MR(\sigma, \mathscr{L})$, since $\mu \in MR(\mathscr{L})$, i.e., 1 is true.

β). Show 1 and 3 are equivalent. Assume 1, and show 3. (Proof omitted.)

Conversely, assume 3, and show 1. Consider any sequence in $\mathscr{L}$, $\langle L_i \rangle$, such that $\langle L_i \rangle$ is decreasing and $\lim_i L_i = \emptyset$, and show $\lim_i \mu(L_i) = 0$. Note

$$\lim_i \mu(L_i) = \lim_i \hat{\mu}(W(L_i)) = \hat{\mu}(\textstyle\bigcap_i W(L_i)).$$

Show $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$. Assume

$$\textstyle\bigcap_i W(L_i) \not\subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}).$$

Then, there exists an element of $IR(\mathscr{L})$, ν, such that $\nu \in \bigcap_i W(L_i)$ and $\nu \in IR(\sigma, \mathscr{L})$. Consider any such ν. Since $\nu \in \bigcap_i W(L_i)$, for every i, $\nu(L_i) = 1$. Hence, $\lim_i \nu(L_i) = 1$. Since $\nu \in I(\sigma, \mathscr{L})$, a contradiction has arisen. Hence, the assumption is wrong. Hence,

$$\textstyle\bigcap_i W(L_i) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}).$$

Hence, since 3 is true,

$$\hat{\mu}(\textstyle\bigcap_i W(L_i)) = 0.$$

Consequently, $\lim_i \mu(L_i) = 0$. Hence, $\mu \in MR(\sigma, \mathscr{L})$, since $\mu \in MR(\mathscr{L})$, i.e., 1 is true.

γ). Show 2 and 4 are equivalent. Note

$$\hat{\mu}^*(X) + \hat{\mu}_*(IR(\mathscr{L}) - X) = \hat{\mu}(IR(\mathscr{L}))$$

and, since $\hat{\mu}$ is $W(\mathscr{L})$-regular,

$$\hat{\mu}_*(IR(\mathscr{L}) - X) = \sup \{\hat{\mu}(K) | K \in \delta W(\mathscr{L}) \text{ and } K \subset IR(\mathscr{L}) - X\}.$$

Hence, $\hat{\mu}^*(X) = \hat{\mu}(IR(\mathscr{L}))$, if and only if $\hat{\mu}_*(IR(\mathscr{L}) - X) = 0$, if and only if whenever $K \in \delta W(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - X$, then $\hat{\mu}(K) = 0$, if and only if whenever $\langle L_i; i \in \mathbf{N} \rangle$ is in $\mathscr{L}$ and $\langle L_i \rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, then $\hat{\mu}(\bigcap_i W(L_i)) = 0$. Hence, 2 and 4 are equivalent.

δ). Show 3 and 5 are equivalent. (Use the same method as for γ).)

Thus, the theorem is proved.

Remark. The part of the assumption "$\mathscr{L}$ is separating" is not needed, in case $\phi(X)$ is not identified with X. Whenever we wish to indicate this in a theorem, we shall enclose the word "separating" (e.g., in the hypothesis), in parentheses.

Observation 1. Note the statement $\hat{\mu}_*(IR(\mathscr{L}) - X) = 0$ is equivalent to the statement "X is $\hat{\mu}$-thick". (See [**13**], pp. 74, 75.) Consequently, 4 is equivalent to the statement "X is $\hat{\mu}$-thick". Next, assume $\mu \in MR(\sigma, \mathscr{L})$. Then, by the theorem, X is $\hat{\mu}$-thick. Hence, since

$$\mathscr{A}(W(\mathscr{L})) \cap X = \mathscr{A}(W(\mathscr{L}) \cap X) = \mathscr{A}(\mathscr{L}),$$

the projection of $\hat{\mu}$ on X is defined. Denote the projection of $\hat{\mu}$ on X by $\hat{\mu}_1$. Then, for every element of $\mathscr{A}(\mathscr{L})$, A,

$$\hat{\mu}_1(A) = \hat{\mu}_1(W(A) \cap X) = \hat{\mu}(W(A)),$$

by the definition of the projection, $= \mu(A)$. Hence, $\hat{\mu}_1 = \mu$.

Observation 2. Note the statement

$$\hat{\mu}_*(IR(\mathscr{L}) - IR(\sigma, \mathscr{L})) = 0$$

is equivalent to the statement "$IR(\sigma, \mathscr{L})$ is $\hat{\mu}$-thick". Consequently, 5 is equivalent to the statement "$IR(\sigma, \mathscr{L})$ is $\hat{\mu}$-thick". Next, assume $\mu \in MR(\sigma, \mathscr{L})$. Then, by the theorem, $IR(\sigma, \mathscr{L})$ is $\hat{\mu}$-thick. Hence, since

$$\mathscr{A}(W(\mathscr{L})) \cap IR(\sigma, \mathscr{L}) = \mathscr{A}(W(\mathscr{L}) \cap IR(\sigma, \mathscr{L})) = \mathscr{A}(W_\sigma(\mathscr{L})),$$

the projection of $\hat{\mu}$ on $IR(\sigma, \mathscr{L})$ is defined. Denote the projection of $\hat{\mu}$ on $IR(\sigma, \mathscr{L})$ by $\hat{\mu}_2$. Then, for every element of $\mathscr{A}(W_\sigma(\mathscr{L}))$,

$$W_\sigma(B), \hat{\mu}_2(W_\sigma(B)) = \hat{\mu}_2(W(B) \cap IR(\sigma, \mathscr{L})) = \hat{\mu}(W(B)),$$

by the definition of the projection, $= \mu(B) = \mu'(W_\sigma(B))$. Hence, $\hat{\mu}_2 = \mu'$.

Examples. (Note if $L \in \mathscr{L}$, since $\mathscr{L}$ is (separating) and disjunctive, $W(L) = \bar{L}$.)

(1). Consider any topological space X such that X is T_1, and denote its collection of closed sets by $\mathscr{F}$. Then, $IR(\mathscr{F})$ is known as the Wallman compactification of X and is denoted by ωX. (See [**22**].) If $\mu \in MR(\mathscr{F})$, then the following statements are equivalent:

1. $\mu \in MR(\sigma, \mathscr{F})$.
2. If $\langle F_i; i \in \mathbf{N}\rangle$ is in $\mathscr{F}$ and $\langle F_i\rangle$ is decreasing and $\bigcap_i \bar{F}_i \subset \omega X - X$, then $\hat{\mu}(\bigcap_i \bar{F}_i) = 0$.
3. If $\langle F_i; i \in \mathbf{N}\rangle$ is in $\mathscr{F}$ and $\langle F_i\rangle$ is decreasing and $\bigcap_i \bar{F}_i \subset \omega X - IR(\sigma, \mathscr{F})$, then $\hat{\mu}(\bigcap_i \bar{F}_i) = 0$.
4. $\hat{\mu}^*(X) = \hat{\mu}(\omega X)$.
5. $\hat{\mu}^*(IR(\sigma, \mathscr{F})) = \hat{\mu}(\omega X)$.

(2). Consider any topological space X, such that X is $T_{3\frac{1}{2}}$, and denote its collection of zero sets by $\mathscr{Z}$. Then, $IR(\mathscr{Z})$ is known as the Stone-Čech compactification of X and is denoted by βX; $IR(\sigma, \mathscr{Z})$ is known as the Realcompactification of X and is denoted by υX. (See [**10**].)

If $\mu \in MR(\mathscr{Z})$, then the following statements are equivalent:

1. $\mu \in MR(\sigma, \mathscr{Z})$.
2. If $\langle Z_i; i \in \mathbf{N}\rangle$ is in $\mathscr{Z}$ and $\langle Z_i\rangle$ is decreasing and $\bigcap_i \bar{Z}_i \subset \beta X - X$, then $\hat{\mu}(\bigcap_i \bar{Z}_i) = 0$.
3. If $\langle Z_i; i \in \mathbf{N}\rangle$ is in $\mathscr{Z}$ and $\langle Z_i\rangle$ is decreasing and $\bigcap_i \bar{Z}_i \subset \beta X - \upsilon X$, then $\hat{\mu}(\bigcap_i \bar{Z}_i) = 0$.
4. $\hat{\mu}^*(X) = \hat{\mu}(\beta X)$.
5. $\hat{\mu}^*(\upsilon X) = \hat{\mu}(\beta X)$.

(3). Consider any topological space X such that X is T_1 and 0-dimensional, and denote its collection of clopen sets by $\mathscr{C}$. Then, $IR(\mathscr{C})$ is known as the Banaschewski compactification of X and is denoted by $\beta_0 X$ (see [**6**]), and $IR(\sigma, \mathscr{C})$ is known as the N-compactification of X and is denoted by $\upsilon_0 X$. (See [**14**].) Since $\mathscr{C}$ is an algebra, $MR(\mathscr{C}) = M(\mathscr{C})$.

If $\mu \in M(\mathscr{C})$, then the following statements are equivalent:

1. $\mu \in M(\sigma, \mathscr{C})$.
2. If $\langle C_i; i \in \mathbf{N}\rangle$ is in $\mathscr{C}$ and $\langle C_i\rangle$ is decreasing and $\bigcap_i \bar{C}_i \subset \beta_0 X - X$, then $\hat{\mu}(\bigcap_i \bar{C}_i) = 0$.
3. If $\langle C_i; i \in \mathbf{N}\rangle$ is in $\mathscr{C}$ and $\langle C_i\rangle$ is decreasing and $\bigcap_i \bar{C}_i \subset \beta_0 X - \upsilon_0 X$, then $\hat{\mu}(\bigcap_i \bar{C}_i) = 0$.
4. $\hat{\mu}^*(X) = \hat{\mu}(\beta_0 X)$.
5. $\hat{\mu}^*(\upsilon_0 X) = \hat{\mu}(\beta_0 X)$.

(4). Consider any topological space X such that X is T_1, and denote its collection of Borel sets by $\mathscr{B}$. Since $\mathscr{B}$ is an algebra, $MR(\mathscr{B}) = M(\mathscr{B})$.

If $\mu \in M(\mathscr{B})$, then the following statements are equivalent:

1. $\mu \in M(\sigma, \mathscr{B})$.
2. If $\langle B_i; i \in \mathbf{N}\rangle$ is in $\mathscr{B}$ and $\langle B_i\rangle$ is decreasing and $\bigcap_i \bar{B}_i \subset I(\mathscr{B}) - X$, then $\hat{\mu}(\bigcap_i \bar{B}_i) = 0$.
3. If $\langle B_i; i \in \mathbf{N}\rangle$ is in $\mathscr{B}$ and $\langle B_i\rangle$ is decreasing and $\bigcap_i \bar{B}_i \subset I(\mathscr{B}) - I(\sigma, \mathscr{B})$, then $\hat{\mu}(\bigcap_i \bar{B}_i) = 0$.
4. $\hat{\mu}^*(X) = \hat{\mu}(I(\mathscr{B}))$.
5. $\hat{\mu}^*(I(\sigma, \mathscr{B})) = \hat{\mu}(I(\mathscr{B}))$.

THEOREM 2.2. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating and disjunctive. The following statements are true:*

1. *If $\mathscr{L}$ is δ and normal, then*

$$\mathscr{Z}(tW(\mathscr{L})) \subset \delta W(\mathscr{Z}(\mathscr{L})).$$

2. *If $\mathscr{L}$ is countably paracompact and normal, then if $\langle L_i; i \in \mathbf{N}\rangle$ is in $\mathscr{L}$ and $\langle L_i\rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, then there exists an element of $\mathscr{Z}(tW(\mathscr{L}))$, K_0, such that*

$$\bigcap_i W(L_i) \subset K_0 \subset IR(\mathscr{L}) - X.$$

Proof. 1. Assume $\mathscr{L}$ is δ and normal. Consider any element of

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

$\mathscr{Z}(tW(\mathscr{L}))$, K_0. It is known that, since $\mathscr{L}$ is separating, disjunctive, δ, and normal, the function which maps the general element of $C_b(\mathscr{L})$, f, into the element of $C(tW(\mathscr{L}))$, $\hat{f}$, which is such that for every element of $IR(\mathscr{L})$, μ, $\hat{f}(\mu) = \int f\, d\mu$, is surjective; (it is even a congruence between $C_b(\mathscr{L})$ and $C(tW(\mathscr{L}))$. (See [3] and [18].) Consequently, there exists an element of $C_b(\mathscr{L})$, f, such that $K_0 = \hat{f}^{-1}(\{0\})$. Consider any such f. Then

$$K_0 = \bigcap_{n=1}^{\infty} \{\mu \in IR(\mathscr{L}) \mid |\hat{f}(\mu)| \leqq 1/n\}.$$

Note since $\hat{f} \in C(tW(\mathscr{L}))$, for every n,

$$\{\mu \in IR(\mathscr{L}) \mid |\hat{f}(\mu)| \leqq 1/n\} \in \mathscr{Z}(tW(\mathscr{L})).$$

Denote $\{\mu \in IR(\mathscr{L}) \mid |\hat{f}(\mu)| \leqq 1/n\}$ by K_n. Then

$$K_n \cap X = \{x \in X \mid |f(x)| \leqq 1/n\}.$$

Note since $f \in C_b(\mathscr{L})$,

$$\{x \in X \mid |f(x)| \leqq 1/n\} \in \mathscr{Z}(\mathscr{L}).$$

Denote $\{x \in X \mid |f(x)| \leqq 1/n\}$ by L_n.

Show $K_0 = \bigcap_n W(L_n)$. α). Show $K_0 \supset \bigcap_n W(L_n)$. Note for every n, $K_n \supset L_n$. Hence, since K_n is closed, $K_n \supset \bar{L}_n$. Hence, since $\bar{L}_n = W(L_n)$, $K_n \supset W(L_n)$. Consequently, $K_0 \supset \bigcap_n W(L_n)$.

β). Show $K_0 \subset \bigcap_n W(L_n)$. Assume $K_0 \neq \emptyset$. Consider any element of K_0, μ. Then, since X is dense in $IR(\mathscr{L})$, there exists a net in X, $\langle \mu_{x_\alpha} \rangle$, such that $\lim_\alpha \mu_{x_\alpha} = \mu$. Consider any such $\langle \mu_{x_\alpha} \rangle$. Then, since $\hat{f}$ is continuous,

$$\lim_\alpha \hat{f}(\mu_{x_\alpha}) = \hat{f}(\mu).$$

Since $\mu \in K_0$ and $K_0 = \hat{f}^{-1}(\{0\})$, $\hat{f}(\mu) = 0$. Consequently,

$$\lim_\alpha \hat{f}(\mu_{x_\alpha}) = 0.$$

Hence, for every n, there exists a value of α, α_0, such that if $\alpha \geqq \alpha_0$, then $|\hat{f}(\mu_{x_\alpha})| < 1/n$. Consider any such α_0. Then, if $\alpha \geqq \alpha_0$, then

$$\mu_{x_\alpha} \in L_n = W(L_n) \cap X \subset W(L_n).$$

Hence, since $\lim_\alpha \mu_{x_\alpha} = \mu$, $\mu \in W(L_n)$. Hence, $\mu \in \bigcap_n W(L_n)$. Hence, $K_0 \subset \bigcap_n W(L_n)$.

γ). Consequently, $K_0 = \bigcap_n W(L_n)$, and for every n, $L_n \in \mathscr{Z}(\mathscr{L})$. Hence, $K_0 \in \delta W(\mathscr{Z}(\mathscr{L}))$. Hence, $\mathscr{Z}(tW(\mathscr{L})) \subset \delta W(\mathscr{Z}(\mathscr{L}))$.

2. Assume $\mathscr{L}$ is countably paracompact and normal. Consider any sequence in $\mathscr{L}$, $\langle L_i \rangle$, such that $\langle L_i \rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, and show there exists an element of $\mathscr{Z}(tW(\mathscr{L}))$, K_0, such that

$$\bigcap_i W(L_i) \subset K_0 \subset IR(\mathscr{L}) - X.$$

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

Since $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, $\bigcap_i L_i = \emptyset$. Consequently, $\lim_i L_i = \emptyset$. Hence, since $\mathscr{L}$ is countably paracompact, there exists a sequence in $\mathscr{L}$, $\langle \tilde{L}_i \rangle$, such that for every i, $L_i \subset \tilde{L}_i'$, and $\langle \tilde{L}_i' \rangle$ is decreasing and $\lim_i \tilde{L}_i' = \emptyset$. Consider any such $\langle \tilde{L}_i \rangle$. Then, for every n, since $L_n \subset \tilde{L}_n'$, $W(L_n) \subset W(\tilde{L}_n') = W(\tilde{L}_n)'$. Hence, $\bigcap_i W(L_i) \subset W(\tilde{L}_n)'$. Note $\bigcap_i W(L_i)$ is compact and $W(\tilde{L}_n)'$ is open. Since $\mathscr{L}$ is normal, $IR(\mathscr{L})$ is T_2. Consequently, $IR(\mathscr{L})$ is locally compact and T_2. Hence, by the Baire Sandwich Theorem (see [**13**]), there exists a compact G_δ-set, K_n, such that $\bigcap_i W(L_i) \subset K_n \subset W(\tilde{L}_n)'$. Consider any such K_n. Then,

$$\bigcap_i W(L_i) \subset \bigcap_n K_n \subset \bigcap_n W(\tilde{L}_n)'.$$

Note $\bigcap_n K_n$ is a compact G_δ-set. Hence, since $IR(\mathscr{L})$ is T_2 and normal, $\bigcap_n K_n \in \mathscr{Z}(tW(\mathscr{L}))$. Denote $\bigcap_n K_n$ by K_0. Then,

$$\bigcap_i W(L_i) \subset K_0 \subset \bigcap_i W(\tilde{L}_i)'.$$

Since $\langle \tilde{L}_i' \rangle$ is decreasing and $\lim_i \tilde{L}_i' = \emptyset$, $\bigcap_i \tilde{L}_i' = \emptyset$. Hence,

$$\bigcap_i W(\tilde{L}_i') \subset IR(\mathscr{L}) - X.$$

Consequently, $\bigcap_i W(L_i) \subset K_0 \subset IR(\mathscr{L}) - X$.

Thus, the theorem is proved.

The following theorem generalizes part of [**4**], which was itself a generalization of the work of Knowles [**15**].

THEOREM 2.3. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating, disjunctive, δ, normal, and countably paracompact. If $\mu \in MR(\mathscr{L})$, then the following statements are equivalent:*

1. $\mu \in MR(\sigma, \mathscr{L})$.
2. *If $K_0 \in \mathscr{Z}(tW(\mathscr{L}))$ and $K_0 \subset IR(\mathscr{L}) - X$, then $\hat{\mu}(K_0) = 0$.*

Proof. (Note since $\mathscr{L}$ is separating, disjunctive, δ, and normal, $\mathscr{Z}(tW(\mathscr{L})) \subset \delta W(\mathscr{Z}(\mathscr{L}))$, by Theorem 2.2, Part 1, $\subset \delta W(\mathscr{L}) \subset D_{\hat{\mu}}$. Note, in general, for an arbitrary lattice of subsets of X, $\mathscr{L}$, for every element of $\mathscr{Z}(tW(\mathscr{L}))$, Z, there exists a sequence in $\mathscr{L}$, $\langle L_n \rangle$, such that $Z = \bigcap_n W(L_n)'$. Consequently, $\mathscr{Z}(tW(\mathscr{L})) \subset \sigma(W(\mathscr{L})) \subset D_{\hat{\mu}}$.)

Assume 1, and show 2. Consider any element of $\mathscr{Z}(tW(\mathscr{L}))$, K_0, such that $K_0 \subset IR(\mathscr{L}) - X$, and show $\hat{\mu}(K_0) = 0$. Since $\mathscr{L}$ is separating, disjunctive, δ, and normal, $K_0 \in \delta W(\mathscr{L})$. Hence, since $\mu \in MR(\sigma, \mathscr{L})$, by assumption, by Theorem 2.1, $\hat{\mu}(K_0) = 0$. Hence, 2 is true. Conversely, assume 2, and show 1. Use Theorem 2.1, namely, show if $\langle L_i; i \in \mathbf{N} \rangle$ is in $\mathscr{L}$ and $\langle L_i \rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, then

$$\hat{\mu}(\bigcap_i W(L_i)) = 0.$$

Consider any sequence in $\mathscr{L}$, $\langle L_i \rangle$, such that $\langle L_i \rangle$ is decreasing and $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$. Then, since $\mathscr{L}$ is countably paracompact and normal, by Theorem 2.2, Part 2, there exists an element of

$\mathscr{Z}(tW(\mathscr{L}))$, K_0, such that

$$\bigcap_i W(L_i) \subset K_0 \subset IR(\mathscr{L}) - X.$$

Consider any such K_0. Then, $\hat{\mu}(K_0) = 0$, by the assumption. Consequently, $\hat{\mu}(\bigcap_i W(L_i)) = 0$. Hence, by Theorem 2.1, $\mu \in MR(\sigma, \mathscr{L})$, i.e., 1 is true.

Thus, the theorem is proved.

Examples. (We use the notation introduced earlier in this section.)

(1). Consider any topological space X such that X is T_1, normal, and countably paracompact. If $\mu \in MR(\mathscr{F})$, then $\mu \in MR(\sigma, \mathscr{F})$, if and only if whenever K_0 is a zero set of ωX and $K_0 \subset \omega X - X$, then $\hat{\mu}(K_0) = 0$.

(2). Consider any topological space X such that X is $T_{3\frac{1}{2}}$. If $\mu \in MR(\mathscr{Z})$, then $\mu \in MR(\sigma, \mathscr{Z})$, if and only if whenever K_0 is a zero set of βX and $K_0 \subset \beta X - X$, then $\hat{\mu}(K_0) = 0$. (This result is due to Knowles [**15**].)

(3). Consider any topological space X such that X is T_1. If $\mu \in M(\mathscr{B})$, then $\mu \in M(\sigma, \mathscr{B})$, if and only if whenever K_0 is a zero set of $I(\mathscr{B})$ and $K_0 \subset I(\mathscr{B}) - X$, then $\hat{\mu}(K_0) = 0$.

Part II. (On τ-smoothness.)

LEMMA 2.1. *Consider any set X and any two lattices of subsets of X, $\mathscr{L}_1$, $\mathscr{L}_2$, such that $\mathscr{L}_1 \subset \mathscr{L}_2$. If $\mu_1 \in MR(\mathscr{L}_1)$, then there exists an element of $MR(\mathscr{L}_2)$, μ_2, such that $\mu_2|_{\mathscr{A}(\mathscr{L}_1)} = \mu_1$ and, if $\mathscr{L}_1$ separates $\mathscr{L}_2$, then μ_2 is unique.* (*See* [**5**] *and* [**16**].)

Next, consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is disjunctive. Consider any element of $MR(\mathscr{L})$, μ. Then, $\hat{\mu} \in MR(W(\mathscr{L}))$. Hence, by the lemma, there exists an element of $MR(tW(\mathscr{L}))$, $\tilde{\mu}$, such that $\tilde{\mu}|_{\mathscr{A}(W(\mathscr{L}))} = \hat{\mu}$ and, since $W(\mathscr{L})$ separates $tW(\mathscr{L})$ (because $W(\mathscr{L})$ is compact), $\tilde{\mu}$ is unique.

Note since $tW(\mathscr{L})$ is compact,

$$MR(tW(\mathscr{L})) = MR(\sigma, tW(\mathscr{L})) = MR(\tau, tW(\mathscr{L})) = MR(t, tW(\mathscr{L})).$$

Consequently, $\tilde{\mu} \in MR(\sigma, tW(\mathscr{L}))$. Hence, $\tilde{\mu}$ is extensible to the σ-algebra of $\tilde{\mu}^*$-measurable sets, uniquely, and the extension is $tW(\mathscr{L})$-regular. Continue to use $\tilde{\mu}$ for this extension.

LEMMA 2.2. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is δ. The following statements are equivalent:*

1. $\mu \in MR(\tau, \mathscr{L})$.
2. *If* $\langle L_\alpha; \alpha \in A\rangle$ *(net) is in* $\mathscr{L}$ *and* $\langle L_\alpha\rangle$ *is decreasing, then*

$$\mu^*(\bigcap_\alpha L_\alpha) = \inf_\alpha \mu(L_\alpha).$$

3. *If* $\{L_\alpha; \alpha \in A\} \subset \mathscr{L}$ *and* $\{L_\alpha; \alpha \in A\}$ *is a filter base, then*

$$\mu^*(\bigcap_\alpha L_\alpha) = \inf_\alpha \mu(L_\alpha).$$

(*See* [**19**].)

THEOREM 2.4. *Consider any set* X *and any lattice of subsets of* X, $\mathscr{L}$, *such that* $\mathscr{L}$ *is (separating) and disjunctive. If* $\mu \in MR(\mathscr{L})$, *then the following statements are equivalent:*

1. $\mu \in MR(\tau, \mathscr{L})$.
2. *If* $\langle L_\alpha; \alpha \in A\rangle$ *(net) is in* $\mathscr{L}$ *and* $\langle L_\alpha\rangle$ *is decreasing and* $\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - X$, *then*

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0.$$

3. $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$.

Proof. α). Show 1 and 2 are equivalent. Assume 1, and show 2. Consider any net in $\mathscr{L}$, $\langle L_\alpha\rangle$, such that $\langle L_\alpha\rangle$ is decreasing and $\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - X$, and show

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0.$$

Since $\langle L_\alpha\rangle$ is decreasing, $\langle W(L_\alpha)\rangle$ is decreasing. Hence, since $\tilde{\mu} \in MR(\tau, \mathscr{L})$ and $tW(\mathscr{L})$ is δ, by Lemma 2.2,

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = \lim_\alpha \tilde{\mu}(W(L_\alpha)).$$

Consequently,

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = \lim_\alpha \tilde{\mu}(W(L_\alpha)) = \lim_\alpha \hat{\mu}(W(L_\alpha)) = \lim_\alpha \mu(L_\alpha).$$

Since $\langle L_\alpha\rangle$ is decreasing, $\lim_\alpha L_\alpha = \bigcap_\alpha L_\alpha$. Since $\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - X$, $\bigcap_\alpha L_\alpha = \emptyset$. Consequently, $\lim_\alpha L_\alpha = \emptyset$. Hence, since $\mu \in M(\sigma, \mathscr{L})$, by the assumption,

$$\lim_\alpha \mu(L_\alpha) = 0.$$

Consequently,

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0.$$

Hence, 2 is true. Conversely, assume 2, and show 1. Consider any net in $\mathscr{L}$, $\langle L_\alpha\rangle$, such that $\langle L_\alpha\rangle$ is decreasing and $\lim_\alpha L_\alpha = \emptyset$, and show $\lim_\alpha \mu(L_\alpha) = 0$. Note

$$\lim_\alpha \mu(L_\alpha) = \lim_\alpha \hat{\mu}(W(L_\alpha)) = \lim_\alpha \tilde{\mu}(W(L_\alpha)).$$

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

Since $\langle L_\alpha \rangle$ is decreasing, $\langle W(L_\alpha)\rangle$ is decreasing. Consequently,

$$\lim_\alpha \tilde{\mu}(W(L_\alpha)) = \tilde{\mu}(\bigcap_\alpha W(L_\alpha)).$$

Since $\bigcap_\alpha L_\alpha = \emptyset$, $\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - X$. Hence, since 2 is true,

$$\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0.$$

Consequently, $\lim_\alpha \mu(L_\alpha) = 0$. Hence, $\mu \in MR(\tau, \mathscr{L})$, i.e., 1 is true.

β). Show 2 and 3 are equivalent.

Remark. The method of proof of this statement is the same as that of the statement "2 and 4 are equivalent" in Theorem 2.1, and, for this reason, it is omitted.

Thus, the theorem is proved.

Observation. Statement 2 is equivalent to the statement: If $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{C}) - X$, then $\tilde{\mu}(K) = 0$.

Examples. (1). If $\mu \in MR(\mathscr{F})$, then $\mu \in MR(\tau, \mathscr{F})$ if and only if $\tilde{\mu}$ vanishes on every closed subset of ωX, contained in $\omega X - X$.

(2). If $\mu \in MR(\mathscr{Z})$, then $\mu \in MR(\sigma, \mathscr{Z})$ if and only if $\tilde{\mu}$ vanishes on every closed subset of βX, contained in $\beta X - X$.

(3). If $\mu \in M(\mathscr{C})$, then $\mu \in M(\tau, \mathscr{C})$ if and only if $\tilde{\mu}$ vanishes on every closed subset of $\beta_0 X$, contained in $\beta_0 X - X$.

(4). If $\mu \in M(\mathscr{B})$, then $\mu \in M(\tau, \mathscr{B})$ if and only if $\tilde{\mu}$ vanishes on every closed subset of $I(\mathscr{B})$, contained in $I(\mathscr{B}) - X$.

THEOREM 2.5. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating and disjunctive. If $\mu \in MR(\tau, \mathscr{L})$ then there exists an element of $MR(\tau, t\mathscr{L})$, ν, such that $\nu|_{\mathscr{A}(\mathscr{L})} = \mu$ and ν is unique in the sense that if $\rho \in MR(\tau, t\mathscr{L})$ and $\rho|_{\mathscr{A}(\mathscr{L})} = \mu$, then $\rho = \nu$; moreover, ν is $\mathscr{L}$-regular on $(t\mathscr{L})'$.*

Proof. (i). Existence. Since $\mu \in MR(\tau, \mathscr{L})$, by Theorem 2.4, $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$. Hence, X is $\tilde{\mu}$-thick. Hence, since

$$\mathscr{A}(tW(\mathscr{C})) \cap X = \mathscr{A}(tW(\mathscr{L}) \cap X) = \mathscr{A}(t\mathscr{L}),$$

the projection of $\tilde{\mu}$ on X is defined. Denote the projection of $\tilde{\mu}$ on X by ν. Denote the general element of $\mathscr{A}(t\mathscr{L})$ by A. Then, there exists an element of $\mathscr{A}(tW(\mathscr{L}))$, A^*, such that $A = A^* \cap X$. Consider any such A^*. Then, $\nu(A) = \tilde{\mu}(A^*)$, by the definition of the projection.

α). Show $\nu|_{\mathscr{A}(\mathscr{L})} = \mu$. Note if $A \in \mathscr{A}(\mathscr{L})$, then

$$\nu(A) = \nu(W(A) \cap X) = \tilde{\mu}(W(A)) = \hat{\mu}(W(A)) = \mu(A).$$

Hence, $\nu_{\mathscr{A}(\mathscr{L})} = \mu$.

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

β). Show ν is $t\mathscr{L}$-regular. Note

$$\begin{aligned}\nu(A) &= \tilde{\mu}(A^*) = \sup\{\tilde{\mu}(K)|K \in tW(\mathscr{L}) \text{ and } K \subset A^*\}\\ &= \sup\{\nu(K \cap X)|K \in tW(\mathscr{L}) \text{ and } K \subset A^*\}\\ &\leqq \sup\{\nu(K \cap X)|K \cap X \in t\mathscr{L} \text{ and } K \cap X \subset A^* \cap X\}\\ &= \sup\{\nu(F)|F \in t\mathscr{L} \text{ and } F \subset A\} \leqq \nu(A).\end{aligned}$$

Hence,

$$\nu(A) = \sup\{\nu(F)|F \in t\mathscr{L} \text{ and } F \subset A\}.$$

Hence, ν is $t\mathscr{L}$-regular.

γ). Show $\nu \in M(\tau, t\mathscr{L})$. Consider any net in $t\mathscr{L}$, $\langle F_\alpha; \alpha \in \Lambda\rangle$, such that $\langle F_\alpha; \alpha \in \Lambda\rangle$ is decreasing and $\lim_\alpha F_\alpha = \emptyset$, and show $\lim_\alpha \nu(F_\alpha) = 0$. Consider any positive number ϵ. For every α, consider the set whose general element is an element of $\mathscr{L}$, L, such that $F_\alpha \subset L$, and denote it by $\{L_{\alpha,\beta_\alpha}; \beta_\alpha \in \Lambda_\alpha\}$. Then, since $F_\alpha \in t\mathscr{L}$,

$$F_\alpha = \cap\{L_{\alpha,\beta_\alpha}; \beta_\alpha \in \Lambda_\alpha\}.$$

Since $\langle F_\alpha\rangle$ is decreasing and $\lim_\alpha F_\alpha = \emptyset$, $\cap_\alpha F_\alpha = \emptyset$. Consequently,

$$\emptyset = \cap_\alpha F_\alpha = \cap\{L_{\alpha,\beta_\alpha}; \alpha \in \Lambda, \beta_\alpha \in \Lambda_\alpha\}.$$

Consider $\{L_{\alpha,\beta_\alpha}; \alpha \in \Lambda, \beta_\alpha \in \Lambda_\alpha\}$, and denote it by $\{L_\gamma; \gamma \in \Gamma\}$. Consider the partial ordering $\geqq$, of Γ, which is such that whenever $\gamma_1, \gamma_2 \in \Gamma$, then $\gamma_1 \geqq \gamma_2$ if and only if $L_{\gamma_1} \subset L_{\gamma_2}$. Then, Γ is directed by $\geqq$ and $\langle L_\gamma; \gamma \in \Gamma\rangle$ is decreasing and $\lim_\gamma L_\gamma = \emptyset$. Hence, since $\nu|_{\mathscr{A}(\mathscr{L})} = \mu$, and $\mu \in M(\tau, \mathscr{L})$, by the assumption, $\lim_\gamma \nu(L_\gamma) = 0$.

Hence, there exists a value of γ, γ_0, such that $\nu(L_{\gamma_0}) < \epsilon$. Consider any such γ_0. Note there exists a value of α, α_0, such that $F_{\alpha_0} \subset L_{\gamma_0}$. Consider any such α_0. Then, since $\langle F_\alpha\rangle$ is decreasing, if $\alpha \geqq \alpha_0$, then $F_\alpha \subset F_{\alpha_0}$. Consequently, if $\alpha \geqq \alpha_0$, then

$$\nu(F_\alpha) \leqq \nu(F_{\alpha_0}) \leqq \nu(L_{\gamma_0}) < \epsilon.$$

Hence, $\lim_\alpha \nu(F_\alpha) = 0$. Hence, $\nu \in M(\tau, t\mathscr{L})$.

δ). Consequently, $\nu \in MR(\tau, t\mathscr{L})$.

(ii). Uniqueness. (Proof omitted.)

(iii). Show ν is $\mathscr{L}$-regular on $(t\mathscr{L})'$. Consider any element of $(t\mathscr{L})'$, B, and show

$$\nu(B) = \sup\{\nu(L)|L \in \mathscr{L} \text{ and } L \subset B\}.$$

Consider any positive number ϵ. Since $B \in (t\mathscr{L})'$ and the relativization of $tW(\mathscr{L})$ to X is $t\mathscr{L}$, there exists an element of $(tW(\mathscr{L}))'$, G, such that $B = G \cap X$. Consider any such G. Then, since $\tilde{\mu}$ is $tW(\mathscr{L})$-regular, there exists an element of $tW(\mathscr{L})$, K, such that $K \subset G$ and $\tilde{\mu}(G - K) < \epsilon$. Consider any such K. Then, consider the set whose general element is

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

an element of $W(\mathscr{L})$, $W(L)$, such that $K \subset W(L)$, and denote it by $\{W(L_\alpha); \alpha \in A\}$. Then, since $K \in tW(\mathscr{L})$,

$$K = \bigcap \{W(L_\alpha); \alpha \in A\}.$$

Then, since $K \subset G$,

$$\bigcap \{W(L_\alpha); \alpha \in A\} \cap G' = \emptyset.$$

Hence, since $W(\mathscr{L})$ is compact, there exists a subset of A, A^*, such that

$$\bigcap \{W(L_\alpha); \alpha \in A^*\} \cap G' = \emptyset$$

and A^* is finite. Consider any such A^*. Then,

$$\bigcap \{W(L_\alpha); \alpha \in A^*\} = W(\bigcap \{L_\alpha; \alpha \in A^*\}).$$

Note $\bigcap \{L_\alpha; \alpha \in A^*\} \in \mathscr{L}$. Denote $\bigcap \{L_\alpha; \alpha \in A^*\}$ by $\tilde{L}$. Then, $K \subset W(\tilde{L}) \subset G$. Hence,

$$W(\tilde{L}) \cap X \subset G \cap X.$$

Consequently, $\tilde{L} \subset B$. Consequently, $\tilde{L} \in \mathscr{L}$ and $\tilde{L} \subset B$ and

$$\nu(B - \tilde{L}) = \nu((G - W(\tilde{L})) \cap X)$$
$$= \tilde{\mu}(G - W(\tilde{L})) \leqq \tilde{\mu}(G - K) < \epsilon.$$

Hence,

$$\nu(B) = \sup \{\nu(L) | L \in \mathscr{L} \text{ and } L \subset B\}.$$

Hence, ν is $\mathscr{L}$-regular on $(t\mathscr{L})'$.

Thus, the theorem is proved.

Remark. For a related type of extension involving content see [**20**].

Examples. (1). If $\mu \in MR(\tau, \mathscr{Z})$, then there exists an element of $MR(\tau, t\mathscr{Z}) = MR(\tau, \mathscr{F})$, ν, such that $\nu|_{\mathscr{A}(\mathscr{Z})} = \mu$ and ν is unique; moreover, ν is $\mathscr{Z}$-regular on $\mathscr{F}'$.

(2). If $\mu \in M(\tau, \mathscr{C})$, then there exists an element of $MR(\tau, t\mathscr{C}) = M(\tau, \mathscr{F})$, ν, such that $\nu|_{\mathscr{C}} = \mu$ and ν is unique; moreover, ν is $\mathscr{C}$-regular on $\mathscr{F}'$.

(3). If $\mu \in M(\tau, \mathscr{B})$, then there exists an element of $MR(\tau, t\mathscr{B}) = M(\tau, \mathscr{P}(X))$, ν, such that $\nu|_{\mathscr{B}} = \mu$ and ν is unique; moreover, ν is $\mathscr{B}$-regular on $\mathscr{P}(X)$.

THEOREM 2.6. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is disjunctive. If $\mu \in MR(\mathscr{L})$, then the following statements are equivalent:*

1. $\mu' \in MR(\tau, W_\sigma(\mathscr{L}))$.

2. *If* $\langle L_\alpha; \alpha \in A\rangle$ *(net) is in* $\mathscr{L}$ *and* $\langle L_\alpha\rangle$ *is decreasing and*

$$\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}),$$

then $\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0.$

3. $\tilde{\mu}^*(IR(\sigma, \mathscr{L})) = \tilde{\mu}(IR(\mathscr{L}))$.

Proof. α). Show 1 and 2 are equivalent. Assume 1, and show 2. Consider any net in $\mathscr{L}$, $\langle L_\alpha\rangle$, such that $\langle L_\alpha\rangle$ is decreasing and

$$\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}),$$

and show $\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0$. Since $\langle L_\alpha\rangle$ is decreasing, $\langle W(L_\alpha)\rangle$ is decreasing. Consequently,

$$\begin{aligned}\tilde{\mu}(\textstyle\bigcap_\alpha W(L_\alpha)) &= \lim_\alpha \tilde{\mu}(W(L_\alpha)) = \lim_\alpha \hat{\mu}(W(L_\alpha))\\ &= \lim_\alpha \mu(L_\alpha) = \lim_\alpha \mu'(W_\sigma(L_\alpha)).\end{aligned}$$

Since $\langle W(L_\alpha)\rangle$ is decreasing, $\langle W_\sigma(L_\alpha)\rangle$ is decreasing. Show $\lim_\alpha W_\sigma(L_\alpha) = \emptyset$. Since $\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$,

$$\bigcap_\alpha W(L_\alpha) \cap IR(\sigma, \mathscr{L}) = \emptyset.$$

Hence, $\bigcap_\alpha W_\sigma(L_\alpha) = \emptyset$. Consequently, $\lim_\alpha W_\sigma(L_\alpha) = \emptyset$. Hence, since $\mu' \in M(\tau, W_\sigma(\mathscr{L}))$, by the assumption,

$$\lim_\alpha \mu'(W_\sigma(L_\alpha)) = 0.$$

Consequently, $\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0$. Hence, 2 is true. Conversely, assume 2, and show 1. Consider any net in $W_\sigma(\mathscr{L})$, $\langle W_\sigma(L_\alpha)\rangle$, such that $\langle W_\sigma(L_\alpha)\rangle$ is decreasing and

$$\lim_\alpha W_\sigma(L_\alpha) = \emptyset,$$

and show

$$\lim_\alpha \mu'(W_\sigma(L_\alpha)) = 0.$$

Note

$$\begin{aligned}\lim_\alpha \mu'(W_\sigma(L_\alpha)) &= \lim_\alpha \mu(L_\alpha) = \lim_\alpha \hat{\mu}(W(L_\alpha))\\ &= \lim_\alpha \tilde{\mu}(W(L_\alpha)) = \tilde{\mu}(\textstyle\bigcap_\alpha W(L_\alpha)).\end{aligned}$$

Show $\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0$. Since $\langle W_\sigma(L_\alpha)\rangle$ is decreasing and

$$\lim_\alpha W_\sigma(L_\alpha) = \emptyset, \ \bigcap_\alpha W_\sigma(L_\alpha) = \emptyset.$$

Hence,

$$\bigcap_\alpha W(L_\alpha) \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L}).$$

Hence, since 2 is true, $\tilde{\mu}(\bigcap_\alpha W(L_\alpha)) = 0$. Consequently,

$$\lim_\alpha \mu'(W_\sigma(L_\alpha)) = 0.$$

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

Hence, $\mu' \in MR(\tau, W_\sigma(\mathscr{L}))$, i.e., 1 is true.

β). Show 2 and 3 are equivalent. (Proof omitted.)

Thus, the theorem is proved.

Observation. Statement 2 is equivalent to the statement: If $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, then $\tilde{\mu}(K) = 0$.

Examples. (1). If $\mu \in MR(\mathscr{F})$, then $\mu' \in MR(\tau, W_\sigma(\mathscr{F}))$, if and only if $\tilde{\mu}$ vanishes on every closed subset of ωX, contained in $\omega X - IR(\sigma, \mathscr{F})$.

(2). If $\mu \in MR(\mathscr{Z})$, then $\mu' \in MR(\tau, W_\sigma(\mathscr{Z}))$, if and only if $\tilde{\mu}$ vanishes on every closed subset of βX, contained in $\beta X - \upsilon X$. (Note $W_\sigma(\mathscr{Z})$ is just the collection of zero sets of $IR(\sigma, \mathscr{Z}) = \upsilon X$.)

(3). If $\mu \in M(\mathscr{C})$, then $\mu' \in M(\tau, W_\sigma(\mathscr{C}))$, if and only if $\tilde{\mu}$ vanishes on every closed subset of $\beta_0 X$, contained in $\beta_0 X - \upsilon_0 X$.

Part III. (On tightness.)

THEOREM 2.7. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is separating, disjunctive, and normal, (or T_2). If $\mu \in MR(\mathscr{L})$, then the following statements are equivalent:*

1. $\mu \in MR(t, \mathscr{L})$.
2. $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$ *and X is $\tilde{\mu}^*$-measurable.*

Proof. Assume 1, and show 2. Note it suffices to show

$$\tilde{\mu}^*(IR(\mathscr{L}) - X) = 0.$$

Consider any positive number ϵ. Then, since $\mu \in MR(t, \mathscr{L})$, by assumption, there exists an $\mathscr{L}$-compact set, K, such that $\mu_*(K') < \epsilon$. Consider any such K. Since $\mu \in MR(t, \mathscr{L})$ and $MR(t, \mathscr{L}) \subset MR(\tau, \mathscr{L})$, $\mu \in MR(\tau, \mathscr{L})$. Hence, by Theorem 2.5, there exists an element of $MR(\tau, t\mathscr{L})$, ν, such that $\nu|_{\mathscr{A}(\mathscr{L})} = \mu$ and ν is unique; moreover, ν is $\mathscr{L}$-regular on $(t\mathscr{L})'$. Since K is $\mathscr{L}$-compact, and $\mathscr{L}$ is separating, disjunctive, and normal, (or T_2), $K \in t\mathscr{L}$. Hence, $K' \in (t\mathscr{L})'$. Consider the extension of μ to $\sigma(\mathscr{L})$ and denote it by the same symbol; also, consider the extension of ν to $\sigma(t\mathscr{L})$ and denote it by the same symbol. Then,

$$\begin{aligned}\nu(K') &= \sup\{\nu(L) | L \in \mathscr{L} \text{ and } L \subset K'\} \\ &= \sup\{\mu(L) | L \in \mathscr{L} \text{ and } L \subset K'\} \\ &\leqq \sup\{\mu(E) | E \in \sigma(\mathscr{L}) \text{ and } E \subset K'\} \\ &= \sup\{\nu(E) | E \in \sigma(\mathscr{L}) \text{ and } E \subset K'\},\end{aligned}$$

since $\nu|_{\sigma(\mathscr{L})} = \mu$,

$$\leqq \sup\{\nu(E) | E \in \sigma(t\mathscr{L}) \text{ and } E \subset K'\} = \nu(K').$$

Hence,

$$\nu(K') = \sup\{\mu(E)|E \in \sigma(\mathscr{L}) \text{ and } E \subset K'\} = \mu_*(K').$$

Note $K' = X - K = (IR(\mathscr{L}) - K) \cap X$. Also, since K is $\mathscr{L}$-compact and $tW(\mathscr{L})$ is T_2 (because $\mathscr{L}$ is normal), $K \in tW(\mathscr{L})$. Consequently,

$$\tilde{\mu}^*(IR(\mathscr{L}) - X) \leqq \tilde{\mu}(IR(\mathscr{L}) - K)$$
$$= \nu((IR(\mathscr{L}) - K) \cap X) = \nu(K') = \mu_*(K') < \epsilon.$$

Hence, $\tilde{\mu}^*(IR(\mathscr{L}) - X) = 0$. Hence, $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$ and X is $\tilde{\mu}^*$-measurable. Hence, 2 is true.

Conversely, assume 2, and show 1. Since $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$ by assumption, by Theorem 2.4, $\mu \in MR(\tau, \mathscr{L})$. Consequently, $\mu \in MR(\sigma, \mathscr{L})$. Now, consider any positive number ϵ, and show there exists an $\mathscr{L}$-compact set, K, such that $\mu_*(K') < \epsilon$. Since X is $\tilde{\mu}^*$-measurable, by assumption, and $\tilde{\mu}$ is $tW(\mathscr{L})$-regular on the σ-algebra of $\tilde{\mu}^*$-measurable sets,

$$\tilde{\mu}^*(X) = \sup\{\tilde{\mu}(K)|K \in tW(\mathscr{L}) \text{ and } K \subset X\}.$$

Consequently, there exists an element of $tW(\mathscr{L})$, K, such that $K \subset X$ and $\tilde{\mu}(K) > \tilde{\mu}^*(X) - \epsilon$. Consider any such K. Note K is $\mathscr{L}$-compact. Hence, since $\mathscr{L}$ is separating, disjunctive, and normal, (or T_2), $K \in t\mathscr{L}$, and $\nu(K') = \mu_*(K')$ (as above). Consequently, $\nu(K) = \mu^*(K)$. Also,

$$\nu(K) = \nu(K \cap X) = \tilde{\mu}(K).$$

Consequently, $\mu^*(K) > \tilde{\mu}^*(X) - \epsilon$. Hence, since $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$, by assumption,

$$\mu^*(K) > \tilde{\mu}(IR(\mathscr{L})) - \epsilon.$$

Hence, since $\tilde{\mu}(IR(\mathscr{L})) = \mu(X)$, $\mu^*(K) > \mu(X) - \epsilon$. Consequently, $\mu_*(K') < \epsilon$. Hence, $\mu \in MR(t, \mathscr{L})$, i.e., 1 is true.

Thus, the theorem is proved.

Remark. $\mathscr{L}$ is said to be strongly measure replete if $MR(\sigma, \mathscr{L}) = MR(t, \mathscr{L})$. The following statement is true: If $\mathscr{L}$ is separating, disjunctive, δ, and normal, then $\mathscr{L}$ is strongly measure replete, if and only if for every element of $MR(\sigma, \mathscr{L})$, μ, there exists an $\mathscr{L}$-compact set, K, such that $\mu^*(K) > 0$. (Proof omitted.) (This generalizes a result of [**17**].)

Examples. (1). Consider any topological space X such that X is T_4. If $\mu \in MR(\mathscr{F})$, then $\mu \in MR(t, \mathscr{F})$ if and only if $\tilde{\mu}^*(X) = \tilde{\mu}(\omega X)$ and X is $\tilde{\mu}^*$-measurable. (Note that since X is normal, $\omega X = \beta X$.)

(2). If $\mu \in MR(\mathscr{Z})$, then $\mu \in MR(t, \mathscr{Z})$ if and only if $\tilde{\mu}^*(X) = \tilde{\mu}(\beta X)$ and X is $\tilde{\mu}^*$-measurable.

(3). If $\mu \in M(\mathscr{B})$, then $\mu \in M(t, \mathscr{B})$ if and only if $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{B}))$ and X is $\tilde{\mu}^*$-measurable.

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

(4). $\mathscr{L}$ is said to be Čech-complete if and only if $IR(\mathscr{L}) - X$ is an $\mathscr{F}_\sigma$-set of $tW(\mathscr{L})$. (See [**9**], p. 142.)

THEOREM. *If $\mathscr{L}$ is also normal, Čech-complete, and Lindelöf, then* $MR(\sigma,\mathscr{L}) = MR(\tau,\mathscr{L}) = MR(t,\mathscr{L})$.

Proof. Since $\mathscr{L}$ is Lindelöf, $MR(\sigma,\mathscr{L}) = MR(\tau,\mathscr{L})$. Next, show $MR(\tau,\mathscr{L}) \subset MR(t,\mathscr{L})$. Consider any element of $MR(\tau,\mathscr{L})$, μ. Then, by Theorem 2.4, $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$. Also, since X is Čech-complete, $IR(\mathscr{L}) - X$ is an $\mathscr{F}_\sigma$-set of $tW(\mathscr{L})$. Hence, $IR(\mathscr{L}) - X \in \sigma(tW(\mathscr{L}))$. Hence, $X \in \sigma(tW(\mathscr{L}))$. Consequently, X is $\tilde{\mu}^*$-measurable. Consequently, $\tilde{\mu}^*(X) = \tilde{\mu}(IR(\mathscr{L}))$ and X is $\tilde{\mu}^*$-measurable. Hence, by Theorem 2.7, $\mu \in MR(t,\mathscr{L})$. Hence, $MR(\tau,\mathscr{L}) \subset MR(t,\mathscr{L})$. Consequently, $MR(\sigma,\mathscr{L}) = MR(\tau,\mathscr{L}) = MR(t,\mathscr{L})$.

APPLICATION 1. *Consider any topological space X such that X is complete, separable, and metrizable. Then*

$$M(\sigma,\mathscr{F}) = MR(\sigma,\mathscr{F}) = MR(\tau,\mathscr{F}) = MR(t,\mathscr{F}).$$

Proof. Since X is metrizable, $\mathscr{Z} = \mathscr{F}$; also, $\mathscr{Z}$ is δ and $\sigma(\mathscr{Z}) \subset s(\mathscr{Z})$. Consequently, $M(\sigma,\mathscr{F}) = MR(\sigma,\mathscr{F})$. (See [**3**].) Since X is metrizable, it is separating and disjunctive. Since X is metrizable and separable, it is Lindelöf. Since X is metrizable and complete, it is Čech-complete. (See [**9**], p. 105.) Consequently,

$$M(\sigma,\mathscr{F}) = MR(\sigma,\mathscr{F}) = MR(\tau,\mathscr{F}) = MR(t,\mathscr{F}).$$

APPLICATION 2. *Consider any topological space X such that X is locally compact, T_2, and Lindelöf. Then,*

$$MR(\sigma,\mathscr{F}) = MR(\tau,\mathscr{F}) = MR(t,\mathscr{F}).$$

Proof. Since X is T_2, $\mathscr{F}$ is separating and disjunctive. Since X is locally compact and T_2, $\mathscr{F}$ is regular. Consequently, $\mathscr{F}$ is δ, regular, and Lindelöf. Hence, $\mathscr{F}$ is normal. Since X is locally compact, it is Čech-compact. (See [**9**], pp. 142, 143). Consequently,

$$MR(\sigma,\mathscr{F}) = MR(\tau,\mathscr{F}) = MR(t,\mathscr{F}).$$

APPLICATION 2′. *Consider any topological space X such that X is locally compact, T_2, and paracompact and separable. Then*

$$MR(\sigma,\mathscr{F}) = MR(\tau,\mathscr{F}) = MR(t,\mathscr{F}).$$

Proof. Since X is paracompact and separable, it is Lindelöf. (See [**7**].) Now, see Application 2.

3. In this section we give certain further applications of the theory developed in Section 2.

Part I. (On countable compactness.)

THEOREM 3.1. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is (separating) and disjunctive. The following statements are equivalent:*

1. *$\mathscr{L}$ is countably compact.*
2. *$IR(\mathscr{L}) - X$ does not contain any nonempty element of $\delta W(\mathscr{L})$.*

Proof. Assume 1, and show 2. Assume 2 is false. Then, $IR(\mathscr{L}) - X$ does contain a nonempty element of $\delta W(\mathscr{L})$. Consider any such element of $\delta W(\mathscr{L})$, $\bigcap_i W(L_i)$. Since $\bigcap_i W(L_i) \neq \emptyset$, consider any element of $\bigcap_i W(L_i)$, μ. Then, $\mu \in IR(\mathscr{L})$ and for every i, $\mu(L_i) = 1$. Also, since $\bigcap_i W(L_i) \subset IR(\mathscr{L}) - X$, $\bigcap_i L_i = \emptyset$. Consequently, $\mu \notin IR(\sigma, \mathscr{L})$. Hence, $IR(\mathscr{L}) \not\subset IR(\sigma, \mathscr{L})$. Hence, $\mathscr{L}$ is not countably compact. Since this statement is false, the assumption is wrong. Hence, 2 is true.

Conversely, assume 2, and show 1. (Proof omitted.)

Thus, the theorem is proved.

Examples. (1). Consider any topological space X such that X is T_1. Then, X is countably compact if and only if $\omega X - X$ does not contain any nonempty closed set of the form $\bigcap_i \bar{F}_i$, with $F_i \in \mathscr{F}$, for every i.

(2). Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then, X is pseudocompact if and only if $\beta X - X$ does not contain any nonempty closed set which is a G_δ.

(3). Consider any topological space X such that X is T_4. Then, X is countably compact if and only if $\omega X - X$ does not contain any nonempty zero set.

(4). Consider any topological space X such that X is T_1 and 0-dimensional. Then, X is clopen-countably compact (i.e., mildly countably compact) if and only if $\beta_0 X - X$ does not contain any nonempty closed set of the form $\bigcap_i \bar{C}_i$, with $C_i \in \mathscr{C}$, for every i.

Part II. (The sets $\hat{M}R(\mathscr{L})$ and $\tilde{M}R(\mathscr{L})$.)

The set $\hat{M}R(\mathscr{L})$. Preliminaries. Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is disjunctive. Then, the set whose general element is an element of $MR(\mathscr{L})$, μ, such that $\mu' \in MR(\tau, W_\sigma(\mathscr{L}))$ is denoted by $\hat{M}R(\mathscr{L})$.

THEOREM 3.2. (On $\hat{M}R(\mathscr{L})$.) *The following statements are true:*

1. *$\hat{M}R(\mathscr{L}) \subset MR(\sigma, \mathscr{L})$.*
2. *$IR(\sigma, \mathscr{L}) \subset \hat{M}R(\mathscr{L})$.*
3. *$\mathscr{L}$ is replete if and only if $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$.*

Proof. 1. Consider any element of $\hat{M}R(\mathscr{L})$, μ. Then $\mu' \in MR(\tau, W_\sigma(\mathscr{L}))$. Hence, $\mu' \in MR(\sigma, W_\sigma(\mathscr{L}))$. Hence, $\mu \in MR(\sigma, \mathscr{L})$. Hence, 1 is true.

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

2. Consider any element of $IR(\sigma, \mathscr{L})$, μ. Then $S(\tilde{\mu}) = \{\mu\}$. (Proof omitted.) Hence, whenever $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, then $\tilde{\mu}(K) = 0$. Hence, by Theorem 2.6,

$$\mu' \in MR(\tau, W_\sigma(\mathscr{L})).$$

Consequently, $\mu \in \hat{M}R(\mathscr{L})$. Hence, 2 is true.

3. Assume $\mathscr{L}$ is replete, and show $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$. Consider any element of $\hat{M}R(\mathscr{L})$, μ. Then, $\mu' \in MR(\tau, W_\sigma(\mathscr{L}))$. Hence, by Theorem 2.6, whenever $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, then $\tilde{\mu}(K) = 0$. Since $\mathscr{L}$ is replete, $IR(\sigma, \mathscr{L}) = X$. Consequently, whenever $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - X$, then $\tilde{\mu}(K) = 0$. Hence, by Theorem 2.4, $\mu \in MR(\tau, \mathscr{L})$. Hence, $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$. Conversely, assume $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$, and show $\mathscr{L}$ is replete. Show $IR(\sigma, \mathscr{L}) = X$. Assume $IR(\sigma, \mathscr{L}) \neq X$. Then, $IR(\sigma, \mathscr{L}) - X \neq \emptyset$. Consider any element of $IR(\sigma, \mathscr{L}) - X$, μ. Since 2 is true, $\mu \in \hat{M}R(\mathscr{L})$. Hence, since $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$, $\mu \in MR(\tau, \mathscr{L})$. Hence, by Theorem 2.4, whenever $K \in tW(\mathscr{L})$ and $K \subset IR(\mathscr{L}) - X$, then $\tilde{\mu}(K) = 0$. Hence, since $\{\mu\} \in tW(\mathscr{L})$ and $\{\mu\} \subset IR(\mathscr{L}) - X$, $\tilde{\mu}(\{\mu\}) = 0$. Since $S(\tilde{\mu}) = \{\mu\}$, a contradiction has arisen. Hence, the assumption is wrong. Hence, $IR(\sigma, \mathscr{L}) = X$. Hence, $\mathscr{L}$ is replete. Consequently, 3 is true.

Thus, the theorem is proved.

Examples. (1). X is α-complete, if and only if $\hat{M}R(\mathscr{F}) \subset MR(\tau, \mathscr{F})$ [**8**].

(2). X is realcompact if and only if $\hat{M}R(\mathscr{Z}) \subset MR(\tau, \mathscr{Z})$ [**10**].

(3). X is N-compact if and only if $\hat{M}(\mathscr{C}) \subset M(\tau, \mathscr{C})$ [**14**].

(4). X is Borel-complete if and only if $\hat{M}(\mathscr{B}) \subset M(\tau, \mathscr{B})$ [**12**].

The following theorem gives a useful condition on extension of certain countably additive measures to countably additive measures.

THEOREM 3.3. (On $\hat{M}R(\mathscr{L})$.) *Consider any set X and any two lattices of subsets of X, $\mathscr{L}_1, \mathscr{L}_2$, such that $\mathscr{L}_1$ is separating and disjunctive, $\mathscr{L}_2$ is disjunctive and δ, and $\mathscr{L}_1 \subset \mathscr{L}_2$. Assume there exists a function from $IR(\sigma, \mathscr{L}_1)$ to $IR(\sigma, \mathscr{L}_2)$, ψ, such that ψ is a homeomorphism and ψ leaves X fixed, pointwise. If $\mu \in \hat{M}R(\mathscr{L}_1)$, then there exists an element of $MR(\sigma, \mathscr{L}_2)$, ϵ, such that $\epsilon|_{\mathscr{A}(\mathscr{L}_1)} = \mu$.*

Outline of proof. Consider any such ψ and any element of $\hat{M}R(\mathscr{L})$, μ. Then, $\mu' \in MR(\tau, W_\sigma(\mathscr{L}_1))$. Hence, by Theorem 2.5, there exists an element of $MR(\tau, tW_\sigma(\mathscr{L}_1))$, γ, such that

$$\gamma|_{\mathscr{A}(W_\sigma(\mathscr{L}_1))} = \mu'$$

and γ is unique. Next, consider the element of $M(tW_\sigma(\mathscr{L}_2))$, ρ, which is such that for every element of $\mathscr{A}(tW_\sigma(\mathscr{L}_2))$, E_2,

$$\rho(E_2) = \gamma(\psi^{-1}(E_2)).$$

Then,

$$\rho \in MR(\tau, tW_\sigma(\mathscr{L}_2)).$$

Consider $\rho|_{\mathscr{A}(W\sigma(\mathscr{L}_2))}$, and denote it by ν. Then,

$$\nu \in MR(\sigma, W_\sigma(\mathscr{L}_2)).$$

Next, consider the element of $MR(\sigma, \mathscr{L}_2)$, ϵ, which is such that $\epsilon' = \nu$. Then, $\epsilon|_{\mathscr{A}(\mathscr{L}_1)} = \mu$.

Remark. This theorem generalizes a result of [**4**].

The set $\tilde{M}R(\mathscr{L})$. *Preliminaries*. The set whose general element is an element of $MR(\mathscr{L})$, μ, such that whenever $\rho \in IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, then there exists an element of $(tW(\mathscr{L}))'$, O, such that $\rho \in O$ and $\tilde{\mu}(O) = 0$, is denoted by $\tilde{M}R(\mathscr{L})$. Then, $\tilde{M}R(\mathscr{L}) \subset \hat{M}R(\mathscr{L})$. (For a proof of this statement use a compactness argument.)

THEOREM 3.4. (On $\tilde{M}R(\mathscr{L})$.) *If* $\mathscr{L}$ *is also* δ, *normal*, *and countably paracompact*, *then* $\tilde{M}R(\mathscr{L}) = MRI(\mathscr{L})$.

Proof. α). Show $\tilde{M}R(\mathscr{L}) \subset MRI(\mathscr{L})$. Consider any element of $\tilde{M}R(\mathscr{L})$, μ. Then, consider any element of $C(\mathscr{L})$, f, and show

$$\left| \int f d\mu \right| < +\infty.$$

Consider the function θ which is such that $D_\theta = [-\infty, +\infty]$, and for every element of $(-\infty, +\infty)$, r, $\theta(r) = r/(1 + |r|)$, and $\theta(-\infty) = -1$ and $\theta(+\infty) = 1$. Then, $\theta([-\infty, +\infty]) = [-1, 1]$ and θ is a homeomorphism. Next, consider the function f^* which is such that $f^* = \theta^{-1} \circ (\theta \circ f)\hat{}$. (See Section 2 for the notation related to $(\theta \circ f)\hat{}$.) Then, f^* maps $IR(\mathscr{L})$ into $[-\infty, +\infty]$ and is $tW(\mathscr{L})$-continuous. Also,

$$\int f d\mu = \int f^* d\tilde{\mu}.$$

(See [**4**], p. 283.) Next, consider

$$\{\rho \in IR(\mathscr{L}) - IR(\sigma, \mathscr{L}) | f^*(\rho) = +\infty\}.$$

Note f^* is finite on $IR(\sigma, \mathscr{L})$. Hence, since f^* is $tW(\mathscr{L})$-continuous,

$$\{\rho \in IR(\mathscr{L}) - IR(\sigma, \mathscr{L}) | f^*(\rho) = +\infty\} \in tW(\mathscr{L}).$$

Denote $\{\rho \in IR(\mathscr{L}) - IR(\sigma, \mathscr{L}) | f^*(\rho) = +\infty\}$ by K. Then, since $\mu \in \tilde{M}R(\mathscr{L})$ and K is compact, there exists an element of $(tW(\mathscr{L}))'$, O,

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

such that $K \subset O$ and $\tilde{\mu}(O) = 0$. Consider any such O. Then,

$$\left|\int f d\mu\right| = \left|\int f^* d\tilde{\mu}\right| = \left|\int_O f^* d\tilde{\mu} + \int_{O'} f^* d\tilde{\mu}\right| = \left|\int_{O'} f^* d\tilde{\mu}\right| .$$

Note that f^* is finite on O'. Hence, since f^* is continuous on O' and O' is compact,

$$\left|\int_{O'} f^* d\tilde{\mu}\right| < +\infty .$$

Consequently,

$$\left|\int f d\mu\right| < +\infty .$$

Hence, $\mu \in MRI(\mathscr{L})$. Hence, $\tilde{M}R(\mathscr{L}) \subset MRI(\mathscr{L})$.

β). Show $MRI(\mathscr{L}) \subset \tilde{M}R(\mathscr{L})$. (See [**4**].)

Examples. (1). Consider any topological space X such that X is T_4 and countably paracompact. Then, $\mu \in \tilde{M}R(\mathscr{F})$ if and only if μ integrates all continuous functions.

(2). Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then, $\mu \in \tilde{M}R(\mathscr{L})$ if and only if μ integrates all continuous functions.

(3). Consider any topological space X such that X is T_1. Then, $\mu \in \tilde{M}(\mathscr{B})$ if and only if μ integrates all Borel measurable functions.

THEOREM 3.5. (On $\tilde{M}R(\mathscr{L})$.) *The following statements are true:*

1. $IR(\sigma, \mathscr{L}) \subset \tilde{M}R(\mathscr{L})$.
2. *If* $\mu \in MR(\mathscr{L})$, *then* $\mu \in \tilde{M}R(\mathscr{L})$ *if and only if* $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$.
3. $\mathscr{L}$ *is replete if and only if whenever* $\mu \in \tilde{M}R(\mathscr{L})$, *then* $S(\tilde{\mu}) \subset X$.
4. $\mathscr{L}$ *is replete if and only if* $\tilde{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$.

Proof. 1. Consider any element of $IR(\sigma, \mathscr{L})$, μ. Next, consider any element of $IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, ρ. Then, $\rho \neq \mu$. Hence, since $tW(\mathscr{L})$ is T_1, there exists an element of $(tW(\mathscr{L}))'$, O, such that $\rho \in O$ and $\mu \notin O$. Consider any such O. Since $\mu \in IR(\mathscr{L})$, $S(\tilde{\mu}) = \{\mu\}$. Consequently, $\tilde{\mu}(O) = 0$. Consequently, $\mu \in \tilde{M}R(\mathscr{L})$. Hence, 1 is true.

2. Consider any element of $MR(\mathscr{L})$, μ. Assume $\mu \in \tilde{M}R(\mathscr{L})$, and show $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$. Assume $S(\tilde{\mu}) \not\subset IR(\sigma, \mathscr{L})$. Then there exists an element of $IR(\mathscr{L})$, ρ, such that $\rho \in S(\tilde{\mu})$ and $\rho \notin IR(\sigma, \mathscr{L})$. Consider any such ρ. Then, since $\mu \in \tilde{M}R(\mathscr{L})$, there exists an element of $(tW(\mathscr{L}))'$, O, such that $\rho \in O$ and $\tilde{\mu}(O) = 0$. Consider any such O. Then $O' \in tW(\mathscr{L})$ and $\tilde{\mu}(O') = 1$. Hence, since $\rho \in S(\tilde{\mu})$, $\rho \in O'$. This is a contradiction. Hence, $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$.

Conversely, assume $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$, and show $\mu \in \tilde{M}R(\mathscr{L})$. Consider any element of $IR(\mathscr{L}) - IR(\sigma, \mathscr{L})$, ρ. Since $\rho \notin IR(\sigma, \mathscr{L})$ and $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$, $\rho \notin S(\tilde{\mu})$. Hence, there exists an element of $(tW(\mathscr{L}))'$,

O, such that $\tilde{\mu}(O') = \tilde{\mu}(IR(\mathscr{L}))$ and $\rho \notin O'$. Consider any such O. Then, $\rho \in O$ and $\tilde{\mu}(O) = 0$. Hence, $\mu \in \tilde{M}R(\mathscr{L})$. Consequently, 2 is true.

3. Assume $\mathscr{L}$ is replete, and show whenever $\mu \in \tilde{M}R(\mathscr{L})$, then $S(\tilde{\mu}) \subset X$. Consider any element of $\tilde{M}R(\mathscr{L})$, μ. Then, since 2 is true, $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{L})$. Since $\mathscr{L}$ is replete, $IR(\sigma, \mathscr{L}) = X$. Consequently, $S(\tilde{\mu}) \subset X$.

Conversely, assume whenever $\mu \in \tilde{M}R(\mathscr{L})$, then $S(\tilde{\mu}) \subset X$, and show $\mathscr{L}$ is replete. Show $IR(\sigma, \mathscr{L}) = X$. Assume $IR(\sigma, \mathscr{L}) \neq X$. Then, $IR(\sigma, \mathscr{L}) - X \neq \emptyset$. Consider any element of $IR(\sigma, \mathscr{L}) - X$, ρ. Then, since 1 is true, $\rho \in \tilde{M}R(\mathscr{L})$. Hence, by assumption, $S(\tilde{\rho}) \subset X$. Since $\rho \in IR(\mathscr{L})$, $S(\tilde{\rho}) = \{\rho\}$. Consequently, $\rho \in X$, a contradiction. Hence, $IR(\sigma, \mathscr{L}) = X$. Hence, $\mathscr{L}$ is replete. Consequently, 3 is true.

4. Assume $\mathscr{L}$ is replete, and show $\tilde{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$. Since $\mathscr{L}$ is replete, by Theorem 3.2, Part 3, $\hat{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$. Hence, since $\tilde{M}R(\mathscr{L}) \subset \hat{M}R(\mathscr{L})$, $\tilde{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$.

Conversely, assume $\tilde{M}R(\mathscr{L}) \subset MR(\tau, \mathscr{L})$, and show $\mathscr{L}$ is replete. (Proof omitted.) Consequently, 4 is true.

Thus, the theorem is proved.

Examples. (1). α). If $\mu \in MR(\mathscr{F})$, then $\mu \in \tilde{M}R(\mathscr{F})$ if and only if $S(\tilde{\mu}) \subset IR(\sigma, \mathscr{F})$.

β). X is α-complete if and only if whenever $\mu \in \tilde{M}R(\mathscr{F})$, then $S(\tilde{\mu}) \subset X$.

γ). X is α-complete if and only if $\tilde{M}R(\mathscr{F}) \subset MR(\tau, \mathscr{F})$.

(2). α). If $\mu \in MR(\mathscr{Z})$, then $\mu \in \tilde{M}R(\mathscr{Z})$ if and only if $S(\tilde{\mu}) \subset \upsilon X$.

β). X is realcompact if and only if whenever $\mu \in \tilde{M}R(\mathscr{Z})$, then $S(\tilde{\mu}) \subset X$.

γ). X is realcompact if and only if $\tilde{M}R(\mathscr{Z}) \subset MR(\tau, \mathscr{Z})$.

(3). α). If $\mu \in M(\mathscr{C})$, then $\mu \in \tilde{M}R(\mathscr{C})$ if and only if $S(\tilde{\mu}) \subset \upsilon_0 X$.

β). X is N-compact if and only if whenever $\mu \in \tilde{M}(\mathscr{C})$, then $S(\tilde{\mu}) \subset X$.

γ). X is N-compact if and only if $\tilde{M}(\mathscr{C}) \subset M(\tau, \mathscr{C})$.

(4). α). If $\mu \in M(\mathscr{B})$, then $\mu \in \tilde{M}(\mathscr{B})$ if and only if $S(\tilde{\mu}) \subset I(\sigma, \mathscr{B})$.

β). X is Borel complete if and only if whenever $\mu \in \tilde{M}(\mathscr{B})$, then $S(\tilde{\mu}) \subset X$.

γ). X is Borel complete if and only if $\tilde{M}(\mathscr{B}) \subset M(\tau, \mathscr{B})$.

4. In this section, as a result of our previous development, we give a different proof of the well-known Yosida-Hewitt Decomposition Theorem.

Preliminaries. Consider any set X and any lattice of subsets of X, $\mathscr{L}$. An element of $MR(\mathscr{L})$, μ, (such that $\mu \geqq 0$), is said to be purely finitely additive (p.f.a.), if whenever $\gamma \in M(\mathscr{L})$, $0 \leqq \gamma \leqq \mu$, and $\gamma \in M(\sigma, \mathscr{L})$, then $\gamma = 0$.

https://doi.org/10.4153/CJM-1981-115-4 Published online by Cambridge University Press

LEMMA 4.1. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$. Consider any element of $MR(\mathscr{L})$, μ (such that $\mu \geqq 0$), and the measures $\hat{\mu}$ on $\sigma(W(\mathscr{L}))$* and $\tilde{\mu}$ on $\sigma(tW(\mathscr{L}))$. (Recall *$\hat{\mu}$ is $\delta W(\mathscr{L})$-regular and $\tilde{\mu}$ is $tW(\mathscr{L})$-regular.) Next, consider any subset of X, H. Then,*

Case 1: *There exists a countably additive measure on $\sigma(W(\mathscr{L}))$, ρ, such that $0 \leqq \rho \leqq \tilde{\mu}$, ρ is $\delta W(\mathscr{L})$-regular, and $\rho^*(H) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(H)$.*

Case 2: *There exists a countably additive measure on $\sigma(tW(\mathscr{L}))$, ρ, such that $0 \leqq \rho \leqq \tilde{\mu}$, ρ is $tW(\mathscr{L})$-regular, and*

$$\rho^*(H) = \rho(IR(\mathscr{L})) = \tilde{\mu}^*(H).$$

Proof. (For Case 1.) Since $\hat{\mu}$ is $W(\mathscr{L})$-regular,

$$\hat{\mu}^*(H) = \inf \{\hat{\mu}(A) | A \in (\delta W(\mathscr{L}))' \text{ and } A \supset H\}.$$

Hence, there exists a sequence in $(\delta W(\mathscr{L}))'$, $\langle A_n \rangle$, such that for every n, $A_n \supset H$, and $\langle A_n \rangle$ is decreasing and

$$\lim_n \hat{\mu}(A_n) = \hat{\mu}^*(H).$$

Consider any such $\langle A_n \rangle$. Then, $\bigcap_n A_n \in \sigma(W(\mathscr{L}))$. Denote $\bigcap_n A_n$ by A. Next, consider the function ρ which is such that $D_\rho = \sigma(W(\mathscr{L}))$ and for every element of $\sigma(W(\mathscr{L}))$, E, $\rho(E) = \hat{\mu}(E \cap A)$. Since $\hat{\mu}$ is a countably additive measure on $\sigma(W(\mathscr{L}))$, ρ is a countably additive measure on $\sigma(W(\mathscr{L}))$. Note that $0 \leqq \rho \leqq \hat{\mu}$. Also, since $\hat{\mu}$ is $\delta W(\mathscr{L})$-regular, ρ is $\delta W(\mathscr{L})$-regular.

Next, show $\rho^*(H) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(H)$. α). Note for every n,

$$\rho^*(H) \leqq \rho(A_n) = \hat{\mu}(A_n \cap A) = \hat{\mu}(A) = \lim_n \hat{\mu}(A_n) = \hat{\mu}^*(H).$$

Hence,

$$\rho^*(H) \leqq \hat{\mu}(A) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(H).$$

β). Show $\rho^*(H) \geqq \hat{\mu}^*(H)$. Since ρ is $\delta W(\mathscr{L})$-regular,

$$\rho^*(H) = \inf \{\rho(G) | G \in (\delta W(\mathscr{L}))' \text{ and } G \supset H\}.$$

Consider any element of $(\delta W(\mathscr{L}))'$, G, such that $G \supset H$. Then,

$$\rho(G) = \hat{\mu}(G \cap A) = \hat{\mu}(G \cap (\textstyle\bigcap_n A_n))$$
$$= \hat{\mu}(\textstyle\bigcap_n (G \cap A_n)) = \lim_n \hat{\mu}(G \cap A_n).$$

Note for every n, since $G \cap A_n \supset H$, $\hat{\mu}(G \cap A_n) \geqq \hat{\mu}^*(H)$. Hence,

$$\lim_n \hat{\mu}(G \cap A_n) \geqq \hat{\mu}^*(H).$$

Consequently, $\rho(G) \geqq \hat{\mu}^*(H)$. Consequently, $\rho^*(H) \geqq \hat{\mu}^*(H)$.

γ). Consequently, $\rho^*(H) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(H)$. (Similarly, for Case 2.)

Thus, the lemma is proved.

Observation. $\rho|_{\mathscr{A}(\delta W(\mathscr{L}))} \in MR(\sigma, \delta W(\mathscr{L}))$. Hence, since $W(\mathscr{L})$ separates $\delta W(\mathscr{L})$ (because $W(\mathscr{L})$ is compact),

$$\rho|_{\mathscr{A}(W(\mathscr{L}))} \in MR(\sigma, W(\mathscr{L})).$$

Continue to use ρ for $\rho|_{\mathscr{A}(W(\mathscr{L}))}$.

Remark. This lemma generalizes a result of Knowles ([**15**], p. 143).

LEMMA 4.2. *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is complemented, i.e., $\mathscr{L}$ is an algebra. Then $\mathscr{A}(\mathscr{L}) = \mathscr{L}$. Hence, $MR(\mathscr{L}) = M(\mathscr{L})$ and $IR(\mathscr{L}) = I(\mathscr{L})$. If $\mu \in M(\mathscr{L})$ (and $\mu \geqq 0$), then μ is* p.f.a. *if and only if $\hat{\mu}^*(X) = 0$.*

Proof. Assume μ is p.f.a., and show $\hat{\mu}^*(X) = 0$. Assume $\hat{\mu}^*(X) \neq 0$. By Lemma 4.1, there exists an element of $MR(\sigma, W(\mathscr{L}))$, ρ, such that $0 \leqq \rho \leqq \hat{\mu}$ and

$$\rho^*(X) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(X).$$

Consider any such ρ. Next, consider the element of $M(\mathscr{L})$, ν, which is such that $\rho = \hat{\nu}$. Then, since $0 \leqq \rho \leqq \hat{\mu}$, $0 \leqq \hat{\nu} \leqq \hat{\mu}$. Hence, $0 \leqq \nu \leqq \mu$. Also, since $\mathscr{A}(\mathscr{L}) = \mathscr{L}$, $\nu \in MR(\mathscr{L})$, and, since $\rho^*(X) = \rho(IR(\mathscr{L}))$, $\hat{\nu}^*(X) = \hat{\nu}(IR(\mathscr{L}))$. Hence, by Theorem 2.1, $\nu \in MR(\sigma, \mathscr{L}) = M(\sigma, \mathscr{L})$. Hence, since μ is p.f.a., by assumption, $\nu = 0$. Moreover, since $\rho(IR(\mathscr{L})) = \hat{\mu}^*(X)$,

$$\nu(X) = \hat{\nu}(IR(\mathscr{L})) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(X) \neq 0,$$

by assumption, a contradiction. Hence, $\hat{\mu}^*(X) = 0$.

Conversely, assume $\hat{\mu}^*(X) = 0$, and show μ is p.f.a. Consider any element of $M(\mathscr{L})$, ν, such that $0 \leqq \nu \leqq \mu$ and $\nu \in M(\sigma, \mathscr{L})$, and show $\nu = 0$. Note that $\nu \in MR(\sigma, \mathscr{L})$. Hence, by Theorem 2.1,

$$\hat{\nu}^*(X) = \hat{\nu}(IR(\mathscr{L})).$$

Also, since $0 \leqq \nu \leqq \mu$, $0 \leqq \hat{\nu} \leqq \hat{\mu}$. Hence, $0 \leqq \nu^* \leqq \hat{\mu}^*$. (Proof omitted.) Hence, $0 \leqq \hat{\nu}^*(X) \leqq \hat{\mu}^*(X)$. Hence, since $\hat{\mu}^*(X) = 0$, by assumption, $\nu^*(X) = 0$. Consequently, $\hat{\nu}(IR(\mathscr{L})) = 0$. Consequently, $\nu = 0$. Hence, μ is p.f.a.

Thus, the lemma is proved.

THEOREM 4.1. (The Yosida-Hewitt Decomposition Theorem.) *Consider any set X and any lattice of subsets of X, $\mathscr{L}$, such that $\mathscr{L}$ is complemented, i.e., $\mathscr{L}$ is an algebra. If $\mu \in M(\mathscr{L})$ (and $\mu \geqq 0$), then there exist two elements of $M(\mathscr{L})$, λ, ν, such that $\mu = \lambda + \nu$, and λ is* p.f.a. *and $\nu \in M(\sigma, \mathscr{L})$; moreover, such a representation of μ is unique.*

Proof. Existence. Note that $\mu \in MR(\mathscr{L})$. Consider $\hat{\mu}$. Then, by Lemma 4.1, there exists an element of $MR(\sigma, W(\mathscr{L}))$, ρ, such that $0 \leqq \rho \leqq \hat{\mu}$

and

$$\rho^*(X) = \rho(IR(\mathscr{L})) = \hat{\mu}^*(X).$$

Consider any such ρ.

Next, consider the element of $M(\mathscr{L})$, ν, which is such that $\rho = \nu$. Then, since $0 \leqq \rho \leqq \hat{\mu}$, $0 \leqq \nu \leqq \hat{\mu}$. Hence, $0 \leqq \nu \leqq \mu$. Also, since $\mathscr{A}(\mathscr{L}) = \mathscr{L}$, $\nu \in MR(\mathscr{L})$, and, since $\rho^*(X) = \rho(IR(\mathscr{L}))$, $\hat{\nu}^*(X) = \hat{\nu}(IR(\mathscr{L}))$. Hence, by Theorem 2.1,

$$\nu \in MR(\sigma, \mathscr{L}) = M(\sigma, \mathscr{L}).$$

Next, consider $\mu - \nu$, and denote it by λ. Since $\nu \leqq \mu$, $\lambda \geqq 0$. Since $\lambda = \mu - \nu$, $\mu = \lambda + \nu$. Hence, $\hat{\mu} = \hat{\lambda} + \hat{\nu}$. Hence, $\hat{\mu}^* = \hat{\lambda}^* + \hat{\nu}^*$. (See [**24**], p. 33.) Hence, $\hat{\lambda}^* = \hat{\mu}^* - \hat{\nu}^*$. Hence, $\hat{\lambda}^*(X) = \hat{\mu}^*(X) - \hat{\nu}^*(X)$. Since $\rho^*(X) = \hat{\mu}^*(X)$, $\hat{\nu}^*(X) = \hat{\mu}^*(X)$. Consequently, $\hat{\lambda}^*(X) = 0$. Hence, by Lemma 4.2, λ is p.f.a. Consequently, $\mu = \lambda + \nu$, and λ is p.f.a. and $\nu \in M(\sigma, \mathscr{L})$.

Uniqueness. Consider any two elements of $M(\mathscr{L})$, λ_1, ν_1, such that $\mu = \lambda_1 + \nu_1$, and λ_1 is p.f.a. and $\nu_1 \in M(\sigma, \mathscr{L})$, and show $\lambda_1 = \lambda$ and $\nu_1 = \nu$. Note that $\lambda_1 + \nu_1 = \lambda + \nu$. Hence, $\nu_1 - \nu = \lambda - \lambda_1$. Hence, since $\lambda_1 \geqq 0$, $\nu_1 - \nu \leqq \lambda$. Hence, since $\lambda \geqq 0$,

$$0 \leqq (\nu_1 - \nu)^+ \leqq \lambda \text{ and } 0 \leqq -(\nu_1 - \nu)^- \leqq \lambda.$$

Hence, since $(\nu_1 - \nu)^+ \in M(\sigma, \mathscr{L})$ and $-(\nu_1 - \nu)^- \in M(\sigma, \mathscr{L})$ and λ is p.f.a., $(\nu_1 - \nu)^+ = 0$ and $-(\nu_1 - \nu)^- = 0$. Hence, $\nu_1 - \nu = 0$. Consequently, $\lambda_1 = \lambda$ and $\nu_1 = \nu$.

Thus, the theorem is proved.

Remark 1. Although there is nothing new in the uniqueness proof, we have included it for completeness.

Remark 2. Using the techniques developed in this paper, it is possible to extend the Yosida-Hewitt Decomposition Theorem to more general lattices than the complemented ones (i.e., algebras) considered, and to even obtain further refinements, but we will not pursue these matters here any further. The previous applications should already give an indication of the scope of the techniques developed.

References

1. A. D. Alexandroff, *Additive set functions in abstract spaces*, Mat. Sb. (N.S.), 9 *51* (1941), 563–628.

2. G. Bachman and P. Hsu, *Extensions of lattice continuous maps to generalized Wallman spaces*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. *62* (1977), 107–114.

3. G. Bachman and A. Sultan, *Extensions of regular lattice measures with topological applications*, J. Math. Anal. and Appl. *57* (1977), 539–559.

4. ——— *Representations of linear functionals on spaces of continuous functions, repletions, and general measure extensions*, J. Math. Anal. and Appl. *67* (1979), 277–293.
5. ——— *On regular extensions of measures*, Pacific J. Math. *86* (1980), 389–395.
6. B. Banaschewski, *Über nulldimensionale Raume*, Math. Nachr. *13* (1955), 129–140.
7. J. Dugundji, *Topology* (Allyn and Bacon, Boston, 1966).
8. N. Dykes, *Generalizations of realcompact spaces*, Pacific J. Math. *33* (1970), 571–581.
9. R. Engelking, *Outline of general topology* (North-Holland Publishing Co., Amsterdam, New York, 1968).
10. L. Gillman and M. Jerrison, *Rings of continuous functions* (Van Nostrand, Princeton, N.J., 1960).
11. G. G. Gould and M. Mahowald, *Measures on completely regular spaces*, J. London Math. Soc. *37* (1962), 103–111.
12. A. Hager, G. Reynolds and M. Rice, *Borel complete topological spaces*, Fund. Math. *75* (1972), 135–143.
13. P. Halmos, *Measure theory* (Van Nostrand, Toronto, New York, London, 1950).
14. H. Herrlich, *E-Kompacte Raume*, Math. Zeit. *96* (1967), 228–255.
15. J. Knowles, *Measures on topological spaces*, Proc. London Math. Soc. *17* (1967), 139–156.
16. J. Lembcke, *Konservative Abbildungen und Fortsetzung regulärer Masse*, Z. Wahr. *15* (1970), 57–96.
17. W. Moran, *Measures and mappings on topological spaces*, Proc. London Math. Soc. *19* (1969), 493–508.
18. A. Sultan, *Measure compactification and representation*, Can. J. Math. *30* (1978), 54–65.
19. M. Szeto, *Measure repleteness and mapping preservations*, (to appear in J. Indian Math. Soc.).
20. F. Topsøe, *Topology and measure*, Springer Lecture Notes *133* (Springer-Verlag, Berlin, Heidelberg, New York, 1970).
21. V. Varadarajan, *Measures on topological spaces*, Amer. Math. Soc. Trans., Ser. 2 *48* (1965), 161–228.
22. H. Wallman, *Lattices and topological spaces*, Ann. of Math. *42* (1938), 687–697.
23. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. *72* (1952), 46–66.
24. A. Zaanen, *An Introduction to the theory of integration* (North-Holland Publishing Co., Amsterdam, New York, 1958).

Polytechnic Institute of New York,
Brooklyn, New York;
Long Island University,
Brooklyn, New York