18

The Kobayashi–Maskawa matrix

In Chapter 14, in the theory of the weak interaction of quarks, there appeared the Kobayashi–Maskawa matrix:

$$\mathbf{V} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \tag{18.1}$$

and its parameterisation:

$$\mathbf{V} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\mathrm{e}^{-\mathrm{i}\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & -\mathrm{c}_{12}s_{23} - s_{12}c_{23}s_{13}\mathrm{e}^{\mathrm{i}\delta} & c_{23}c_{13} \end{pmatrix} \tag{18.2}$$

where $\mathrm{c}_{12} = \cos\theta_{12} > 0$, $s_{12} = \sin\theta_{12} > 0$, etc. The KM matrix couples quark fields of different flavours. It contains four physically significant parameters, which can be taken to be the three rotation angles θ_{12}, θ_{13}, θ_{23}, each lying in the first quadrant, and the phase angle δ.

There is no theory relating these parameters, just as there is no theory relating quark masses. Indeed, the quark sector of the Standard Model may appear to the reader to be lacking in aesthetic appeal. The parameters of the KM matrix must be determined from experiment, and in this chapter we indicate how experimental information has been obtained.

18.1 Leptonic weak decays of hadrons

We have seen in Section 15.3 two unitarity sum rules that support the validity of the Standard Model, and there are many independent measurements that both test for consistency and given consistency determine the parameters. So far no definitive inconsistencies have been established, and a large body of data is well described

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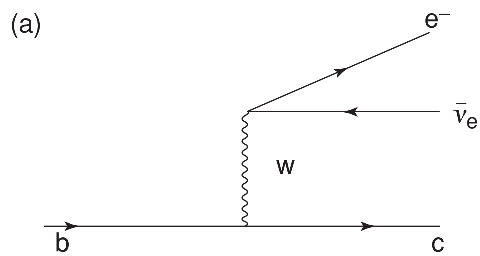
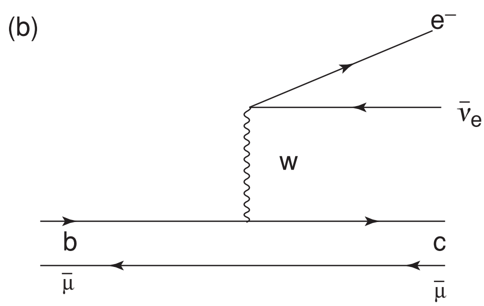


Figure 18.1(a) A Feynman diagram for the leptonic decay $b \to c + e^- + \bar{\nu}_e$



(b) A quark model diagram for the decay $B^- \to$ *charmed hadron system* $+ e^- + \bar{\nu}_e$

with the parameter values $s_{12} = 0.2243 \pm 0.0016$, $s_{23} = 0.0413 \pm 0.0015$, $s_{13} = 0.0037 \pm 0.0005$ and $\delta = 57^\circ \pm 14^\circ$.

A suitable starting point for the consideration of hadronic weak decays is first-order perturbation theory in the effective Lagrangian density of equation (14.21): $L = -2\sqrt{2}G_F j^\dagger_\mu j^\mu$, where j^μ is given by (14.20). Leptonic decays are the most simple for theoretical analysis because the leptonic parts of a transition matrix element can be calculated with some confidence. If quarks were available as isolated particles, the three rotation angles of the KM matrix could be determined by the measurement of the decay rates of leptonic decays such as

$$b \to c + e + \bar{\nu}_e.$$

In lowest order perturbation theory (see Fig. 18.1a) the decay rate for this process is given by

$$\frac{1}{\tau(b \to c)} = \frac{G_F^2 m_b^5}{192\pi^3} |V_{cb}|^2 f\left(\frac{m_c}{m_b}\right) \tag{18.3}$$

where $f(x) = 1 - 8x^2 + 8x^6 - x^8 - 24x^4 \ln(x)$ is a factor associated with the available phase space. This programme cannot be carried out directly since the b and c quarks are accompanied by other spectator quarks and gluons (see the quark

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model diagram of Fig. 18.1b), which involve the calculation of strong interaction matrix elements. To the extent that the hadronic matrix elements can be calculated, a measurement of the decay rate will determine $|V_{\mathrm{cb}}|^2$.

18.2 $|V_{\mathrm{ud}}|$ and nuclear β decay

Isospin symmetry (see Section 16.6) is important for the determination of the hadronic matrix elements of all nuclear β decays. Such decays involved the quark current

$$j_q^{\mu} = d_{\mathrm{L}}^{\dagger}\widetilde{\sigma}^{\mu}u_{\mathrm{L}} = \bar{d}\gamma^{\mu}(1/2)(1-\gamma^5)u. \tag{18.4}$$

Here we have expressed the current in terms of the Dirac four-component spinors u and d, with the help of the projection operator $(1/2)(1-\gamma^5)$ introduced in (5.32) and noting $\bar{d} = d^{\dagger}\gamma^{\mathrm{o}}$.

As in Chapter 16, we now take the u and d quarks together in an isotopic doublet:

$$\mathbf{D}(x) = \begin{pmatrix} u(x) \\ d(x) \end{pmatrix}.$$

The isospin operator $(1/2)(\tau^1 - i\tau^2)$ has the property

$$\frac{1}{2}\left(\tau^1 - i\tau^2\right)\begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ u \end{pmatrix},$$

so that we may write (see (16.31))

$$\begin{aligned} j_q^{u} &= (1/4)\overline{\mathbf{D}}(x)\gamma^{\mu}(1-\gamma^5)(\tau^1 - \mathrm{i}\tau^2)\mathbf{D}(x) \\ &= (1/2)\left[\nu^{\mu}(x) - a^{\mu}(x)\right]. \end{aligned} \tag{18.5}$$

We have split the current into the part $\nu^{\mu}(x)$, which transforms like a vector under space inversion and the part $a^{\mu}(x)$, which transforms like an axial vector (see Section 5.5):

$$\nu^{\mu}(x) = (1/2)\overline{\mathbf{D}}\gamma^{\mu}(\tau^1 - \mathrm{i}\tau^2)\mathbf{D}, \tag{18.6}$$

$$a^{\mu}(x) = (1/2)\overline{\mathbf{D}}\gamma^{\mu}\gamma^5(\tau^1 - \mathrm{i}\tau^2)\mathbf{D}. \tag{18.7}$$

We saw in Section 16.6 that exact isospin symmetry leads to conserved currents:

$$\nu_i^{\mu} = (1/2)\overline{\mathbf{D}}\gamma^{\mu}\tau^{\mathrm{i}}\mathbf{D}, \tag{18.8}$$

so that the vector part of the β decay current of the u and d quarks is a conserved isospin current.

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In the case of nucleons, we denote the isospin doublet of the effective Dirac fields $p(x)$ and $n(x)$ of the proton and neutron by

$$\mathbf{D}_{\mathrm{N}}(x) = \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}. \tag{18.9}$$

An effective Lagrangian density that at the low energies of nuclear physics describes the β decay of a nucleon is

$$L_{\mathrm{eff}} = -2\sqrt{2}G_{\mathrm{F}}C|j_{\mathrm{e}}^{\dagger} j_{\mathrm{N}}^{\mu} + j_{\mathrm{e}}^{\mu\dagger} j_{\mathrm{e}\mu}|, \tag{18.10}$$

with

$$j_{\mathrm{N}}^{\mu} = \frac{1}{4}\overline{\mathbf{D}}_{\mathrm{N}}\gamma^{\mu}(1 - g_{\mathrm{A}}\gamma^{5})(\tau_1 - \mathrm{i}\tau^2)\mathbf{D}_{\mathrm{N}}. \tag{18.11}$$

Experimentally, it is found from a range of nuclear data that

$$C = 0.9713 \pm 0.0013 \quad \text{and} \quad g_{\mathrm{A}} = 1.2739 \pm 0.0019.$$

(See Particle Data Group.)

The vector part of the current j_{N}^{μ} is the conserved isospin current of nuclear physics and corresponds to the more fundamental conserved isospin current at the quark level. Exact isospin symmetry would require that the contribution of the conserved nucleon isospin current to the effective interaction (18.8, 18.9) be the same as that of the quarks in (18.5, 18.6), so that we identify $C = V_{\mathrm{ud}} = 0.9713 \pm 0.0013$.

18.3 More leptonic decays

The most precise estimates of $|V_{\mathrm{us}}|$ have come from observations of leptonic K decays, for example $\mathrm{K}^-(\mathrm{s\bar{u}}) \to \pi^{\mathrm{o}}(\mathrm{u\bar{u}} - \mathrm{d\bar{d}})/\sqrt{2} + \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$. Analyses of these decays by lattice QCD, quark model calculations, and calculations based on chiral symmetry (see Section 16.7) all converge on the value $|V_{\mathrm{us}}| = 0.224 \pm 0.003$.

Estimates of $|V_{\mathrm{cs}}|$ and $|V_{\mathrm{cd}}|$ can be extracted from D decays, for example $\mathrm{D}^-(\mathrm{\bar{c}d}) \to \mathrm{K}^{\mathrm{o}}(\mathrm{\bar{s}d}) + \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$ or $\mathrm{D}^-(\mathrm{\bar{c}d}) \to \pi^{\mathrm{o}}(\mathrm{u\bar{u}} - \mathrm{d\bar{d}})/\sqrt{2} + \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$. These decay rates are proportional to $|V_{\mathrm{cs}}|^2$ and $|V_{\mathrm{cd}}|^2$ respectively.

More experimental information on $|V_{\mathrm{cd}}|^2$ comes from the deep inelastic scattering of neutrinos by atomic nuclei through processes such as

$$\nu_{\mu} + \mathrm{d} \to \mu^- + \mathrm{c}. \qquad \text{(See Appendix D.)}$$

Atomic nuclei provide an abundant source of d quark targets. The cross-section for producing a c quark rather than a u quark can be inferred by identifying those c quarks that decay as $\mathrm{c} \to \mathrm{d} + \mu^+ + \nu_{\mu}$. Overall, a characteristic $\mu^+\mu^-$ pair is produced.

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The conclusions, after much work along the lines indicated, and without imposing the unitarity condition, are

$$|V_{\mathrm{cd}}| = 0.224 \pm 0.014, \ |V_{\mathrm{cs}}| = 1.04 \pm 0.16.$$

Leptonic decays of B mesons ($\mathrm{b\bar{u}}$, $\mathrm{b\bar{d}}$, $\mathrm{\bar{b}u}$ and $\mathrm{\bar{b}d}$) provide the best data on $|V_{\mathrm{cb}}|$ and $|V_{\mathrm{ub}}|$, Three experimental facilities have been constructed to measure B decays: in the USA at Cornell (Cleo) and Stanford (Babar), and in Japan (Belle). At these ‘B meson factories’ many million B mesons have been produced for analysis.

In the case of $|V_{\mathrm{cb}}|$, the hadronic matrix elements for decays like $\mathrm{B}^- \to \mathrm{D}^\circ + \mathrm{e}^- + \bar{\nu}_{\mathrm{e}}$ can be calculated taking the heavy b quark in the $\mathrm{B}^-(\mathrm{b}, \bar{\mathrm{u}})$ meson as static in first approximation. Analysis of the data gives

$$|V_{\mathrm{cb}}| = 0.0413 \pm 0.0015, \quad |V_{\mathrm{ub}}| = 0.00367 \pm 0.00047.$$

The remaining three elements of the KM matrix involve the top quark. The mean life of the top quark is so short it is likely to decay before it has time to settle into a top quark hadron. The methods described above are unavailable for $|V_{ti}|$ (i = d, s or b).

18.4 *CP* symmetry violation in neutral kaon decays

In Section 14.4 we obtained the important result that the quark sector of the Standard Model is not invariant under the charge conjugation, parity, operation unless all the elements of the KM matrix can be made real. With the parameterisation (18.2), this requires that the phase angle $\delta = 0$.

CP violation was first observed in 1964 in the decay of neutral K mesons. The states of definite quark number are the $\mathrm{K}^\circ(\mathrm{d\bar{s}})$ and $\bar{\mathrm{K}}^\circ(\mathrm{\bar{d}s})$. These mesons are readily produced in strong interactions, for example $\pi^-(\mathrm{\bar{u}d}) + \mathrm{p(uud)} \to \mathrm{K^o(d\bar{s})} + \Lambda(\mathrm{uds})$. Without the weak interaction the K° and $\bar{\mathrm{K}}^\circ$ would have equal mass and be stable. The weak interaction is responsible for their instability and *CP* violation would be manifest if for example it were seen that the decay rates $\mathrm{K^o} \to \pi^+\pi^-$ and $\bar{\mathrm{K}}^\circ \to \pi^+\pi^-$ were different. Such a difference can occur in second-order perturbation theory in the weak interaction (first order in G_{F}. See (14.21)). This is known as direct *CP* violation.

The weak interaction also gives rise to the phenomenon of mixing (Appendix E, Fig. E1). Although mixing occurs only at second order in G_{F} it has the dramatic effect of splitting the mass degeneracy: it results in two mixed states of different mass. If *CP* were conserved the mixed states would be

$$|\mathrm{K}_1^{\mathrm{o}}\rangle = \left(1/\sqrt{2}\right)\left(|\mathrm{K^o}\rangle + |\bar{\mathrm{K}}^{\mathrm{o}}\rangle\right) \quad \text{and} \quad |\mathrm{K}_2^{\mathrm{o}}\rangle = \left(1/\sqrt{2}\right)\left(|\mathrm{K^o}\rangle - |\bar{\mathrm{K}}^{\mathrm{o}}\rangle\right).$$

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Acting on K^o and $\bar{K}^o$, the *CP* operator may be taken to give

$$CP\left|K^o\right\rangle = \left|\bar{K}^o\right\rangle \quad \text{and} \quad CP\left|\bar{K}^o\right\rangle = \left|K^o\right\rangle.$$

Then $\left|K_1^o\right\rangle$ and $\left|K_2^o\right\rangle$ are eigenstates of *CP* with eigenvalues $+1$ and -1 respectively. Experimentally two states with a mass difference 3.5×10^{-12} MeV are indeed observed; they also have very different mean lives

$$\tau_s = 8.9 \times 10^{-11}\text{s}, \; \tau_L = 5.17 \times 10^{-8}\text{s}.$$

The K_s^o decays predominantly into two pions, $\pi^+\pi^-$ or $\pi^o\pi^o$. Each of these two-pion states is an eigenstate of CP, with eigenvalue $+1$ (Problem 18.2). In its mesonic decay modes, the K_L^o decays predominantly into $\pi^o\pi^o\pi^o$, and these three-pion states are eigenstates of *CP* with eigenvalue -1 (Problem 18.3). However, in about three decays in a thousand K_L^o decays into two pions, with *CP* eigenvalue $+1$. If CP were conserved K_L^o would be either K_1^o or K_2^o and could not have both two pion and three pion decay modes. *CP* violation is also seen in leptonic K decays. These show that direct *CP* violation is not responsible for the anomalous K_L^o decays but they are predominantly due to *CP* violation in mixing.

It is shown in Appendix E that neither $\left|K_s^o\right\rangle$ nor $\left|K_L^o\right\rangle$ is an eigenstate of *CP*, but each can be written in terms of $|K^o\rangle$ and $\left|\bar{K}^o\right\rangle$:

$$\begin{aligned}\left|K_s^o\right\rangle &= N\left[p\left|K^o\right\rangle + q\left|\bar{K}^o\right\rangle\right],\\ \left|K_L^o\right\rangle &= N\left[p\left|K^o\right\rangle - q\left|\bar{K}^o\right\rangle\right].\end{aligned} \tag{18.12}$$

N is the normalisation factor: $(|p|^2 + |q|^2)^{-1/2}$. Note that q is not equal to p. In Appendix E we indicate how p and q can be calculated in the Standard Model.

We can similarly express $\left|K_s^o\right\rangle$ and $\left|K_L^o\right\rangle$ in terms of $\left|K_1^o\right\rangle$ and $\left|K_2^o\right\rangle$:

$$\begin{aligned}\left|K_s^o\right\rangle &= \left(N/\sqrt{2}\right)\left[(p+q)\left|K_1^o\right\rangle + (p-q)\left|K_2^o\right\rangle\right],\\ \left|K_L^o\right\rangle &= \left(N/\sqrt{2}\right)\left[(p-q)\left|K_1^o\right\rangle + (p+q)\left|K_2^o\right\rangle\right].\end{aligned} \tag{18.13}$$

Neglecting direct *CP* violation only K_1^o can decay into $\pi\pi$ so that the ratio of the decay rates

$$\frac{\Gamma(K_L) \to \pi\pi}{\Gamma(K_S) \to \pi\pi} = \frac{|p/q - 1|^2}{|p/q + 1|^2} = (5.25 \pm 0.05) \times 10^{-6} \text{ (from experiment).}$$

Defining $p/q = 1 + 2\varepsilon_K$ we infer that $|\varepsilon_K| = 2.3 \times 10^{-3}$; ε_K is a measure of *CP* violation.

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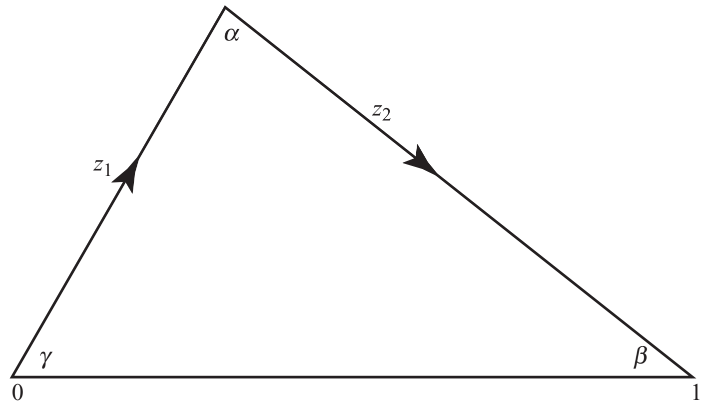

Figure 18.2 The unitarity triangle.

18.5 B meson decays and B^o, $\bar{B}^o$ mixing

At the B meson factories the 4s ($b\bar{b}$) meson is copiously produced by e^+e^- collisions with beam energies turned to the meson mass. The meson decays almost exclusively into B^+, B^- or B^o, $\bar{B}^o$ pairs and so provides a rich source of B mesons. With a mass of 5.28 GeV, B mesons decay into many different final states and many exhibit *CP* violation. An indication of why this is so can be seen by a consideration of the unitarity condition

$$V_{ud}V_{ub}^* + V_{cd}V_{cb}^* + V_{td}V_{tb}^* = 0,$$

which can be written as

$$z_1 + z_2 = 1 \tag{18.14}$$

where we have defined $z_1 = -\dfrac{V_{ud}V_{ub}^*}{V_{cd}V_{cb}^*}$ and $z_2 = -\dfrac{V_{td}V_{tb}^*}{V_{cd}V_{cb}^*}$.

z_1 and z_2 are complex numbers that, in the complex plane form a triangle, the unitarity triangle illustrated in Fig. 18.2. Also it can be seen from the parameters given in Section 18.1 that $V_{cd}V_{cb}^*$ is almost real and negative. Neglecting its very small imaginary part, the angle $\gamma = \delta$, the phase of V_{ub}^*, and β is the phase of V_{td}^*. Of all the unitarity triangles, this is the only one with direct access to the two KM matrix elements with large phases; it also involves the b quark and hence B mesons.

Of particular importance has been the measurement of the angle α through both charged and neutral decays $B \to \pi\pi$, $B \to \pi\rho$ and $B \to \rho\rho$ and of the angle β through B^o, $\bar{B}^0$ mixing. As one example it is shown in Appendix E how $\sin(2\beta)$ is measured at the B factories.

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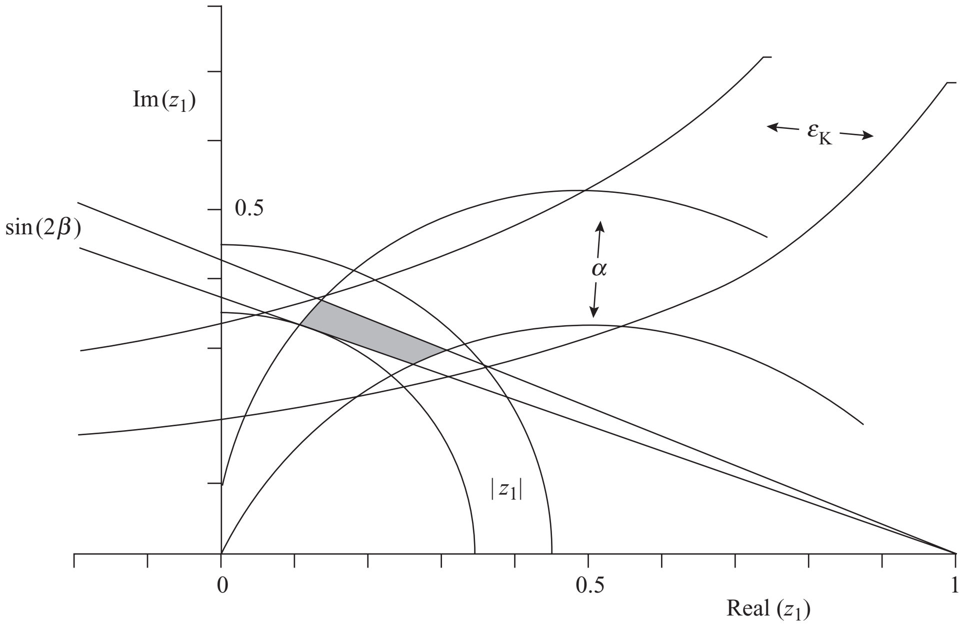

Figure 18.3 The apex of the unitarity triangle is in, or near, the shaded region of the plot.

The unitary triangle is specified by the position of its apex. This requires two parameters, say the real and imaginary parts of z_1. A single parameter defines a line on the complex plane and a parameter with errors defines a band. Four such bands inferred from experiment are shown in Fig. 18.3. The most important point illustrated by the figure is the consistency between four independent measurements. There is no indication of the Standard Model failing. The KM phase δ ($\approx\gamma$) can be seen to be in the region $\delta = 57^{\circ} \pm 14^{\circ}$. The apex of the unitarity triangle is in, or near, the shaded region of the figure.

18.6 The *CPT* theorem

We denote by T the operation of time reversal, $t \to t' = -t$. The *CPT theorem* states that, under very general conditions, a Lorentz invariant quantum field theory is invariant under the combined operations of charge conjugation, space inversion, and time reversal. The theorem was discovered by Pauli in 1955.

For the Standard Model, the *CPT* theorem implies that, since *CP* is not a symmetry of the Model, then neither is time reversal *T*. One may contemplate the implications for the 'Arrow of Time'.

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Problems

18.1 Draw quark model diagrams for the decays

$$\pi^- \to \mu^- + \bar{\nu}_\mu, \quad K^- \to \mu^- + \bar{\nu}_\mu.$$

Show that the decay amplitudes are proportional to V_{ud} and V_{us} respectively, and $V_{us}/V_{ud} = \tan\theta_{12}$.

Neglecting the effects of the different quark masses, the ratio α_K/α_π calculated in Problem 9.10 would equal V_{us}/V_{ud}. Use this observation to estimate $\sin\theta_{12}$.

18.2 A π^o meson is even under the charge conjugation operation C, i.e. $C|\pi^0\rangle = |\pi^0\rangle$. Also, $C|\pi^+\rangle = |\pi^-\rangle$ and $C|\pi^-\rangle = |\pi^+\rangle$.

Show that two pions $|\pi^o, \pi^o\rangle$ or $|\pi^+, \pi^-\rangle$ in a relative S state and with their centre of mass at rest satisfy $CP|\pi, \pi\rangle = |\pi, \pi\rangle$.

18.3 Show that a state of three π^o mesons $|\pi^o, \pi^o, \pi^o\rangle$ with angular momentum zero and centre of mass at rest satisfies $CP|\pi^o, \pi^o, \pi^o\rangle = -|\pi^o, \pi^o, \pi^o\rangle$. (See Problem 18.2.)

18.4 Show that the area of the unitary triangle of Fig. 18.5 is $J/2$.

18.5 Show that if the quark fields are subject to a change of phase

$$d \to e^{i\theta_d} d, \quad b \to e^{i\theta_b} b,$$

then the unitary triangle of Fig. 18.5 is rotated through an angle $(\theta_d - \theta_b)$.

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