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107.31 Obtaining a more general result from a functional equation by not differentiating

Introduction

Relying too much on tools with which we are familiar is a human trait that can cause us to overlook details or features that might be interesting. This is captured in *Maslow's Law* or *The Law of the Instrument*:

To a person with a hammer, everything looks like a nail [1].

That occurred in our Theorem 8 of [2, p. 429]:

> A sufficient condition for the twice differentiable function $y(x)$ to be a quadratic polynomial (parabola) is that any three distinct points (x_i, y_i) $i = 1, 2, 3$, that satisfy $y = y(x)$ with $x_1 < x_2 < x_3$, form an inscribed non-degenerate triangle and the formula for the area of the triangle with vertices at the points is
> $$C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1)$$
> for a single value of C for the curve.

The condition of twice differentiability is an unnecessary assumption that is instead a consequence of the conclusion. The *hammer* is differentiation and knowing how to solve a simple differential equation. The *nail* is the remainder of the theorem.

The requirement concerning the area of the inscribed triangle can be expressed as

$$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1). \tag{1}$$

Expanding the determinant on its first row and multiplying by 2 yields

$$x_1(y_2 - y_3) - y_1(x_2 - x_3) + (x_2y_3 - x_3y_2) = 2C(x_3 - x_2)(x_3 - x_1)(x_2 - x_1). \tag{2}$$

The order of the values of x is immaterial, because all orders yield an equivalent equation. This can be seen as follows. Consider switching between $x_2 < x_3$ and $x_3 < x_2$. On the left-hand side of (1), this interchanges rows 2 and 3 of the determinant, which introduces a minus sign [3, p. 3]. On the right-hand side, switching solely introduces a minus sign, as well. These minus signs cancel. Similarly, switching between any two x-values either just introduces a minus sign on both sides or does not.

A proof using differentiation

This proof resembles the proof in [2]. Because x_i are independently selected values for the single function y, select one of them, say x_1, and apply $\dfrac{\partial}{\partial x_1}$ to (2), remembering that the derivatives of x_2 and x_3 and of y evaluated at x_2 and x_3 are zero, to obtain

$$y_2 - y_3 - y_1'(x_2 - x_3) = 2C(x_3 - x_2)(-1)(x_2 - x_1) + 2C(x_3 - x_2)(x_3 - x_1)(-1). \quad (3)$$

Applying $\dfrac{\partial}{\partial x_1}$ to (3) gives the second order differential equation $y_1'' = 4C$. Dropping the subscript 1 and solving the differential equation results in the quadratic-polynomial solution

$$y = 2Cx^2 + C_1 x + C_2. \quad (4)$$

From (4),

$$y(0) = C_2 \text{ and } y(1) = 2C + C_1 + C_2,$$

$$C_2 = y(1) - y(0) - 2C \text{ and } C_2 = y(0),$$

and (4) is

$$y = 2Cx^2 + \big(y(1) - y(0) - 2C\big)x + y(0). \quad (5)$$

The proof resulting in (5) unnecessarily relies on the condition of twice differentiability, as well as the use of partial differentiation and solving a differential equation. As a result, the similar proof given in [2, p. 429] is more complicated than necessary. Instead, we can drop the condition of twice differentiability and view (1) and (2) as equivalent functional equations, whose solutions imply differentiability.

A proof using a functional equation approach

Equation (1) is a functional equation in three variables $(x_1, x_2 \text{ and } x_3)$ for one function $(y(x))$ [4, p. 25]. The indeterminacy allows two of the variables to be chosen as fixed numbers. For simplicity, use the values 0 and 1. Thus, without loss of generality, select

$$x_1 = x, \qquad x_2 = 0 \qquad \text{and} \qquad x_3 = 1,$$

and write

$$y_1 = y(x) = y, \qquad y_2 = y(x_2) = y(0) \qquad \text{and} \qquad y_3 = y(x_3) = y(1).$$

Equation (2) becomes

$$y = 2Cx^2 + \big(y(1) - y(0) - 2C\big)x + y(0),$$

which is (5).

The final step is to check that no spurious solutions have been introduced [4, p. 26]. Substituting (5) into (1) gives an identity, which shows that there are none.

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107.32 A new inductive proof of the AM - GM inequality

In what follows, we denote by A_n and G_n the arithmetic and geometric means of n non-negative real numbers $a_1, a_2, \ldots, a_n$ ($n \geqslant 1$), that is,

$$A_n = \frac{a_1 + a_2 + \ldots + a_n}{n} \text{ and } G_n = \sqrt[n]{a_1 a_2 \ldots a_n}.$$

Then the famous arithmetic mean - geometric mean inequality (see, e.g., [1, Subsection 2.1] and [2, Section 5]) states that

$$A_n \geqslant G_n, \tag{1}$$

where equality holds if, and only if, $a_1 = a_2 = \ldots = a_n$.

Several proofs of the arithmetic mean-geometric mean inequality are known in the literature (see, e.g., [3]). It was used as the inductive hypothesis in [4, 5, 6, 7, 8].