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The two membranes problem in a regular tree

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In this article, we study the two membranes problem for operators given in terms of a mean value formula on a regular tree. We show existence of solutions under adequate conditions on the boundary data and the involved source terms. We also show that, when the boundary data are strictly separated, the coincidence set is separated from the boundary and thus it contains only a finite number of nodes.

Keywords: two membranes problem; regular tree; mean value operators; game theory; obstacle problem

1. Introduction

One of the systems that attracted the attention of the partial differential equations community is the two membranes problem that models the behaviour of two elastic membranes that are clamped at the boundary of a prescribed domain, and they are assumed to be ordered (one membrane is assumed to be above the other) and they are subject to different external forces. The main assumption here is that the two membranes do not penetrate each other (they are assumed to be ordered in the whole domain). This situation can be modelled by two obstacle problems; the lower membrane acts as an obstacle from below for the free elastic equation that describes the location of the upper membrane, while, conversely, the upper membrane is an obstacle from above for the equation for the lower membrane. The mathematical formulation is as follows: given two differential operators $F(x, u, \nabla u, D^2u)$ and

$G(x, v, \nabla v, D^2 v)$ find a pair of functions (u, v) defined inside a prescribed domain $\Omega \subset \mathbb{R}^N$ such that

$$\begin{cases} \min\Big\{F(x,u(x),\nabla u(x),D^2u(x)),(u-v)(x)\Big\}=0, & x\in\Omega,\\ \max\Big\{G(x,v(x),\nabla v(x),D^2v(x)),(v-u)(x)\Big\}=0, & x\in\Omega,\\ u(x)=f(x), & x\in\partial\Omega,\\ v(x)=g(x), & x\in\partial\Omega. \end{cases}$$

The two membranes problem for the Laplacian with a right-hand side, that is, for $F(D^2u) = -\Delta u + h_1$ and $G(D^2v) = -\Delta v - h_2$, was first considered in [23] using variational arguments. When the equations that model the two membranes have a variational structure, this problem can be tackled using calculus of variations (one aims to minimize the sum of the two energies subject to the constraint that the functions that describe the position of the membranes are always ordered inside the domain), see [23]. However, when the involved equations are not variational the analysis relies on monotonicity arguments (using the maximum principle). Once existence of a solution (in an appropriate sense) is obtained, a lot of interesting questions arise, like uniqueness, regularity of the involved functions, a description of the contact set, the regularity of the contact set, etc. See [5, 6, 20], the dissertation [24], and references therein. We also mention that a more general version of the two membranes problem involving more than two membranes was considered by several authors (see for example [2, 7, 8]).

Our main goal here is to introduce and analyse the obstacle problem and the two membranes problem when the ambient space is an infinite graph with a regular structure (a regular tree) and the involved operators are given by mean value formulas.

A regular tree. Let us first describe the ambient space. Regular trees can be viewed as discrete models of the unit ball of $\mathbb{R}^N$. A tree is, informally, an infinite graph (that we will denote by $\mathbb{T}$) in which each node but one (the root of the tree denoted by $\emptyset$) has exactly $m+1$ connected nodes, m successors and one predecessor (the root has only m successors). An element x in $\mathbb{T}$ is called a vertex and it is represented as a k-tuple for some $k \in \mathbb{N}$ of natural numbers between 0 and m, that is, $x = (0, a_1, ..., a_k)$ where $a_i \in \{0, ..., m-1\}$ for all $1 \leq j \leq k$. Here k is the level of x, that we denote by $|x|$, according to the distance (in nodes) to the root. The root has zero level, its successors have level one, etc. Each vertex x has m successors, which we will denote by $S^1(x)$ and are described by

$$S^1(x) = \Big\{(0, a_1, ..., a_k, i) \colon i \in \{0, 1, \dots, m-1\}\Big\}$$

where $x = (0, a_1, ..., a_k)$. Notice that we used a digit $i \in \{0, 1, \dots, m-1\}$ to enumerate the successors of x. In general, we will denote

$$S^l(x) = \Big\{(0, a_1, ..., a_k, i_1, \dots, i_l) \colon i_j \in \{0, 1, \dots, m-1\} \quad \text{for} \quad 1 \leq j \leq l\Big\}$$

the set of successors of $x = (0, a_1, ..., a_k)$ with $|x| + l$ level. By convention, $S^0(x) = \{x\}$. If x is not the root then x has a unique immediate predecessor, which we will denote $\hat{x}$. We use the notation x^j to denote the predecessor of x with $|x^j| = j$, that is, for a node x with $|x| = k$, we have $x = (0, a_1, \dots, a_j, \dots, a_k)$, then $x^j = (0, a_1, \dots, a_j)$. Let $\mathbb{T}^x \subset \mathbb{T}$ be the subtree with root x (the subset of the tree composed of all the consecutive successors of x). A branch of $\mathbb{T}$ (that we denote by z) is an infinite sequence of vertices starting at the root, where each of the vertices in the sequence is followed by one of its immediate successors. The collection of all branches forms the boundary of $\mathbb{T}$, denoted by $\partial\mathbb{T}$. Observe that the mapping $\psi : \partial\mathbb{T} \to [0, 1]$ defined as

$$\psi(z) = \sum_{k=0}^{+\infty} \frac{a_k}{m^k}$$

is surjective, where $z = (0, a_1, \dots, a_k, \dots) \in \partial\mathbb{T}$ and $a_k \in \{0, 1, \dots, m-1\}$ for all $k \in \mathbb{N}$. When $x = (0, a_1, \dots, a_k)$ is a vertex, we set $\psi(x) = \psi((0, a_1, \dots, a_k, 0, \dots))$.

Mean value formulas and operators. Given a parameter $0 \leq \beta \leq 1$, we define the averaging operator L acting on functions $u : \mathbb{T} \mapsto \mathbb{R}$ as follows:

$$L(u)(x) = u(x) - \beta u(\hat{x}) - (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big), \qquad x \neq \emptyset \tag{1.1}$$

and

$$L(u)(\emptyset) = u(\emptyset) - \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big). \tag{1.2}$$

The operator L acting on u at a vertex $x \in \mathbb{T}$ is given by the difference between the value of u at that node and the mean value of u at the vertices that are connected with x (with a weight given by β for the predecessor and $(1-\beta)/m$ for the successors). Note that here the distinction in the definition of Lu based on whether $x = \emptyset$ or not relies on the fact that the root has no predecessor.

Now, given a function $h : \mathbb{T} \mapsto \mathbb{R}$, a solution to

$$L(u)(x) = h(x)$$

is a function $u : \mathbb{T} \mapsto \mathbb{R}$ that verifies

$$u(x) = \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x), \qquad x \neq \emptyset,$$

and the equation at the root of the tree that only involves the successors and is given by

$$u(\emptyset) = \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big) + h(\emptyset).$$

Notice that the equation $L(u)(x) = h(x)$ is an analogous in the tree of the partial differential equation $-\Delta u(x) = h(x)$ in the unit ball of $\mathbb{R}^N$ (recall the mean value formula for harmonic functions).

Next, in order to impose boundary conditions, we want to make precise what we understand for $u|_{\partial\mathbb{T}} = f$.

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DEFINITION 1. *Given $f : [0,1] \to \mathbb{R}$, we say that u verifies the boundary condition $u|_{\partial\mathbb{T}} = f$ if*

$$\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)),$$

that is, if $z = (0, a_1, \dots, a_k, \dots) \in \partial\mathbb{T}$, then

$$\lim_{n\to\infty} u((0, a_1, ..., a_n)) = f\Big(\sum_{k=1}^{+\infty} \frac{a_k}{m^k}\Big).$$

Here and in what follows we will take this limit uniformly, that is, given $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that $|u(x) - f(\psi(z))| < \varepsilon$ for every $|x| \geq K$ and every $z \in \partial\mathbb{T}^x$.

Note that, if $u|_{\partial\mathbb{T}} = f$, due to the fact that the limit is uniform, we have that

$$\lim_{N\to\infty} \frac{1}{m^N} \sum_{y\in S^N(x)} u(y) = \fint_{\partial\mathbb{T}^x} f(s)\mathrm{d}s. \tag{1}$$

Now, let us state the precise definition of being a subsolution/supersolution to $Lu = h$ in $\mathbb{T}$, with a boundary condition $u|_{\partial\mathbb{T}} = f$.

DEFINITION 2. *Given an operator L defined as before, a function $h : \mathbb{T} \to \mathbb{R}$, and a continuous function $f : [0,1] \to \mathbb{R}$, we say that $u : \mathbb{T} \to \mathbb{R}$ is a subsolution (resp. supersolution) to $Lu = h$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ if it verifies*

$$\begin{cases} L(u)(x) \leq h(x) \quad (resp. \geq), & x \in \mathbb{T}, \\ \limsup\limits_{x\to z\in\partial\mathbb{T}} u(x) \leq f(\psi(z)) \quad (resp. \liminf \geq). \end{cases}$$

We say that u is a solution if it is both a subsolution and a supersolution.

With this definition at hand, we can introduce the obstacle problem (see also §2).

DEFINITION 3. *Given an operator L, a function $h : \mathbb{T} \to \mathbb{R}$, a continuous boundary datum $f : [0,1] \to \mathbb{R}$, and a function $\varphi : \mathbb{T} \to \mathbb{R}$ (the obstacle), we say that $u : \mathbb{T} \to \mathbb{R}$ is the solution to the obstacle problem for $L - h$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} \geq f$ and obstacle φ from below (φ is assumed to satisfy $\limsup_{x\to z\in\partial\mathbb{T}} \varphi(x) < f(\psi(z))$) if it verifies*

$$u(x) = \inf \left\{ w(x) : \begin{array}{c} L(w)(x) \geq h(x), \quad x \in \mathbb{T}, \\ w(x) \geq \varphi(x), \quad x \in \mathbb{T}, \\ and \ \liminf\limits_{x\to z\in\partial\mathbb{T}} w(x) \geq f(\psi(z)) \end{array} \right\}.$$

Under adequate conditions on h (see below), this function u(x) is the unique function that satisfies

$$\begin{cases} 0 = \max\Big\{ -L(u)(x) + h(x), \varphi(x) - u(x)\Big\}, \quad x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

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Analogously, given an operator L, a function $h : \mathbb{T} \to \mathbb{R}$, a continuous boundary datum $g : [0,1] \to \mathbb{R}$, and a function $\varphi : \mathbb{T} \to \mathbb{R}$ (the obstacle), we say that $v : \mathbb{T} \to \mathbb{R}$ is the solution to the obstacle problem for $L - h$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle φ from above (here u is assumed to satisfy $\liminf_{x\to z\in\partial\mathbb{T}} \varphi(x) > g(\psi(z))$) if it verifies

$$v(x) = \sup \left\{ w(x) : \begin{array}{l} L(w)(x) \le h(x), \quad x \in \mathbb{T}, \\ w(x) \le \varphi(x), \quad x \in \mathbb{T}, \\ \textit{and } \limsup\limits_{x\to z\in\partial\mathbb{T}} w(x) \le g(\psi(z)) \end{array} \right\}.$$

Under adequate conditions on h, this can be written as

$$\begin{cases} 0 = \min\Big\{ -L(v)(x) + h(x), \varphi(x) - v(x) \Big\}, \quad x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases}$$

With the definition of the obstacle problem, we can describe the two membranes problem in $\mathbb{T}$.

Definition 4. *Let L_1 and L_2 be two averaging operators defined as in (1.1) and (1.2) (with different β_1, β_2), $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ two functions and $f, g : [0,1] \to \mathbb{R}$ two continuous boundary conditions. A pair $(u, v) : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is a solution to the two membranes problem if it solves the system*

$$\begin{cases} 0 = \max\Big\{ -L_1(u)(x) + h_1(x), v(x) - u(x) \Big\}, \ x \in \mathbb{T}, \\ 0 = \min\Big\{ -L_2(v)(x) + h_2(x), u(x) - v(x) \Big\}, \ x \in \mathbb{T}, \end{cases} \tag{1.3}$$

with

$$\begin{cases} \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)), \\ \lim\limits_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases} \tag{1.4}$$

Based on the previous definition 3, a pair (u, v) is a solution to the two membranes problem (1.3) with initial datum (1.4) if and only if u is the solution to the obstacle problem for $L_1 - h_1$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ and obstacle v from below and v is the solution to the obstacle problem for $L_2 - h_2$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle u from above.

Recall that u is the solution to the obstacle problem for $L_1 - h_1$ in $\mathbb{T}$ with $u|_{\partial\mathbb{T}} = f$ and obstacle v from below if it is the infimum of supersolutions for $L_1 - h_1$ with boundary datum f that are above the obstacle v, and analogously, v is the solution to the obstacle problem for $L_2 - h_2$ in $\mathbb{T}$ with $v|_{\partial\mathbb{T}} = g$ and obstacle v from above if it is the supremum of subsolutions for $L_2 - h_2$ with boundary datum g that are above the obstacle u.

Let us point out that (1.3) has a probabilistic interpretation that we will describe in §4.

Now, for short, we introduce a notation, for a node x, let

$$\mathcal{S}_h(x) := \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{k-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j}.$$

With this notation at hand, we are ready to state our main result for the two membranes problem.

THEOREM 1.1 *Given two averaging operators, L_1 and L_2 defined as in* (1.1) *and* (1.2) *(that involve two different parameters β_1, β_2), two different functions $h_1, h_2 : \mathbb{T} \to \mathbb{R}$, and two continuous boundary conditions $f, g : [0,1] \to \mathbb{R}$ with $f > g$ in $[0,1]$, such that*

$$\begin{array}{l} 0 \le \beta_1 < \frac{1}{2}, \qquad 0 \le \beta_2 < \frac{1}{2}, \\ \displaystyle\lim_{x\in\mathbb{T},|x|=k\to\infty}\sum_{j=1}^{k}\left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_{h_1}(x^j) = 0, \\ \displaystyle\lim_{x\in\mathbb{T},|x|=k\to\infty}\sum_{j=1}^{k}\left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_{h_2}(x^j) = 0, \end{array}$$

then, there exists a pair $(u, v) : \mathbb{T} \mapsto \mathbb{R}$ that is a solution to the two membranes problem. That is, (u, v) solves the system (1.3)*.*

Moreover, under these conditions, the coincidence set $\{x \in \mathbb{T} : u(x) = v(x)\}$ is finite.

To prove this result, we first deal with a single equation and find necessary and sufficient conditions for the existence of a solution. We have the following result that we believe that has its own interest.

THEOREM 1.2 *Given an averaging operator, L, a function $h : \mathbb{T} \to \mathbb{R}$, and a continuous boundary condition $f : [0,1] \to \mathbb{R}$, there exists a unique bounded solution to*

$$\begin{cases} L(u)(x) = h(x), & x \in \mathbb{T}, \\ \lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

if and only if

$$\begin{array}{l} 0 \le \beta < \frac{1}{2}, \\ \displaystyle\lim_{x\in\mathbb{T},|x|=k\to\infty}\sum_{j=1}^{k}\left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_h(x^j) = 0. \end{array} \tag{1.5}$$

Under the conditions in the statement of the previous theorem, we can also construct sub and supersolutions that are the key to obtain solvability of the obstacle problem (from above or below).

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Notice that when $\beta < \frac{1}{2}$ if h is such that

$$\lim_{x\in\mathbb{T},|x|=k\to\infty} S_h(x) = 0$$

then the second condition in (1.5) holds. In fact, we have

$$\begin{aligned}\sum_{j=1}^{k-1}\left(\tfrac{\beta}{1-\beta}\right)^{k-j}S_h(x^j) &= \sum_{j=1}^{k_0}\left(\tfrac{\beta}{1-\beta}\right)^{k-j}S_h(x^j) + \sum_{j=k_0}^{k-1}\left(\tfrac{\beta}{1-\beta}\right)^{k-j}S_h(x^j)\\ &\leq C\left(\tfrac{\beta}{1-\beta}\right)^{k}\sum_{j=1}^{k_0}\left(\tfrac{\beta}{1-\beta}\right)^{-j} + \max_{j\geq k_0}|S_h(x^j)|\sum_{j=k_0}^{k-1}\left(\tfrac{\beta}{1-\beta}\right)^{k-j}\end{aligned}$$

that is small if we first choose k_0 large (in order to make $\max_{j\geq k_0}|S_h(x^j)|$ small) and then send k to infinity.

To look for examples of functions h that satisfy our condition we take a function that depends only on the level k, that is, $h(y) = \widehat{H}(k)$. Then, for $0 < \beta < \frac{1}{2}$, the condition $S_h(x) \to 0$ reads as

$$\lim_{x\in\mathbb{T},|x|\to\infty} \frac{1}{\beta}\sum_{k=1}^{\infty} H(k)\sum_{j=0}^{k-1}\left(\frac{\beta m}{1-\beta}\right)^{k-j} = 0.$$

Therefore, for H constant, $H = cte$, this condition is not verified. When $\frac{\beta m}{1-\beta} < 1$ any $H(k)$ with $\lim_{k\to\infty} H(k) = 0$ satisfies the condition, but any $H(k) > 0$ with $\liminf_{k\to\infty} H(k) > 0$ does not. On the other hand, when $\frac{\beta m}{1-\beta} \geq 1$ we need that $H(k)$ goes to zero very fast in order to verify the condition.

With the result for a single equation at hand, to prove our main result concerning the two membranes problem, we use the strategy of iterate the obstacle problem from above or below. Starting with v_0 a subsolution to $L_2 - h_2$ with boundary datum g, we let u_1 the solution to the obstacle problem for $L_1 - h_1$ with boundary datum f and obstacle v_0. Then we take v_1 as the solution to the obstacle problem for the second operator $L_2 - h_2$ with boundary datum g and obstacle u_1 and so on. In this way, we obtain two sequences, u_n, v_n, that we prove that are monotone and converge to a solution to the two membranes problem. To show that the boundary values are attained we need to use super and subsolutions for a single equation. For a similar iteration procedure for second order elliptic partial differential equations in a bounded domain in the Euclidean space, we refer to [13].

Concerning previous results in the literature for mean value formulas on trees, we refer to [1, 3, 9–12, 14, 15, 17, 21, 22] and references therein. In [9], it was studied the solvability for the single equation with $h \equiv 0$ and proved that there is a bounded solution for any continuous boundary datum f if and only if $0 \leq \beta < \frac{1}{2}$. Our results include the case $h \neq 0$ with the condition (1.5). For systems of mean value formulas on trees, we quote [17] where two coupled equations (but not of obstacle type) were considered (a probabilistic interpretation of the equations is also provided there).

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The article is organized as follows: in §2, we analyse conditions for existence of a solution to a single equation. In §3, we deal with the two membranes problem. Finally, in §4, we present a probabilistic interpretation of the two membranes problem using game theory.

2. Conditions for existence of bounded solutions to a single equation

In this section, we want to find conditions on β and the function $h : \mathbb{T} \to \mathbb{R}$ such that for every continuous function $f : [0,1] \to \mathbb{R}$ there exists a bounded solution to the Dirichlet problem

$$\begin{cases} L(u)(x) = h(x), & x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases} \tag{2.6}$$

Proof of theorem 1.2. First, suppose that such a bounded solution exists, then we have a function $u : \mathbb{T} \mapsto \mathbb{R}$ such that

$$u(x) = \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u(y)\Big) + h(x),$$

for $x \in \mathbb{T}\setminus\{\emptyset\}$ and $|x| = k$, if we write $u(x) = \beta u(x) + (1-\beta)u(x)$ we get

$$\beta\Big(u(x) - u(\hat{x})\Big) = (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} (u(y) - u(x))\Big) + h(x).$$

We define $w(x)$ for $x \in \mathbb{T}\setminus\{\emptyset\}$ as the increment of u between x and its predecessor, i.e.

$$w(x) := u(x) - u(\hat{x}).$$

Then, the previous equation written in terms of w reads as

$$\begin{aligned}
\beta w(x) &= (1-\beta)\Big(\tfrac{1}{m}\sum_{y\in S^1(x)} w(y)\Big) + h(x) \\
w(x) &= \Big(\tfrac{1-\beta}{\beta}\Big)^i \tfrac{1}{m^i}\sum_{y\in S^i(x)} w(y) + \tfrac{1}{\beta}\sum_{j=0}^{i-1}\Big(\tfrac{1-\beta}{\beta}\Big)^j \sum_{y\in S^j(x)} \tfrac{h(y)}{m^j} \\
w(x)\Big(\tfrac{\beta}{1-\beta}\Big)^i &= \tfrac{1}{m^i}\sum_{y\in S^i(x)} w(y) + \tfrac{1}{\beta}\sum_{j=0}^{i-1}\Big(\tfrac{\beta}{1-\beta}\Big)^{i-j} \sum_{y\in S^j(x)} \tfrac{h(y)}{m^j}.
\end{aligned}$$

Adding from $i = 1$ to infinity, we get

$$w(x)\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^i = \sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i} + \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{i-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j}. \tag{2.7}$$

If $h \equiv 0$, we have existence of a solution such that

$$w(x)\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^i = \sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i}$$

and since w is bounded (this follows from the fact that u is assumed to be bounded) we have that the right hand side is finite. Hence, we obtain the convergence of the series

$$\sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{i}$$

that is equivalent to the condition

$$\frac{\beta}{1-\beta}<1 \iff 0<\beta<\frac{1}{2}.$$

Now, we use the notations

$$\mathcal{S}_h(x) := \frac{1}{\beta}\sum_{i=1}^{\infty}\sum_{j=0}^{i-1}\Big(\frac{\beta}{1-\beta}\Big)^{i-j}\sum_{y\in S^j(x)}\frac{h(y)}{m^j},$$
$$c_\beta := \sum_{i=0}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{i} = \frac{1-\beta}{1-2\beta}, \quad \text{for} \quad \beta\in\Big(0,\frac{1}{2}\Big).$$

With this notation, it is straightforward to check that

$$c_\beta - 1 = \sum_{i=1}^{\infty}\Big(\frac{\beta}{1-\beta}\Big)^{i} > 0.$$

On the other hand, we claim that

$$\sum_{i=1}^{\infty}\sum_{y\in S^i(x)}\frac{w(y)}{m^i} = \fint_{\partial\mathbb{T}^x} f(s)\mathrm{d}s - u(x). \tag{2.8}$$

Assuming this claim, (2.7) can be rewritten as

$$w(x)\Big(c_\beta - 1\Big) = \fint_{\partial\mathbb{T}^x} f(s)\mathrm{d}s - u(x) + \mathcal{S}_h(x).$$

From this, we obtain that the solution $u:\mathbb{T}\mapsto\mathbb{R}$ verifies

$$u(x) = \frac{c_\beta - 1}{c_\beta}u(\hat{x}) + \frac{1}{c_\beta}\fint_{\partial\mathbb{T}^x} f(s)\mathrm{d}s + \frac{1}{c_\beta}\mathcal{S}_h(x), \quad x\neq\emptyset. \tag{2.9}$$

Now, we have

$$u(\emptyset) = \frac{1}{m}\sum_{y\in S^1(\emptyset)} u(y) + h(\emptyset),$$

and, using (2.9) for $x\in S^1(\emptyset)$ we get

$$u(\emptyset) = \frac{1}{m}\sum_{y\in S^1(\emptyset)}\Big(\frac{c_\beta - 1}{c_\beta}u(\emptyset) + \frac{1}{c_\beta}\fint_{\partial\mathbb{T}^y} f(s)\mathrm{d}s + \frac{1}{c_\beta}\mathcal{S}_h(y)\Big) + h(\emptyset),$$

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that is

$$u(\emptyset) = \frac{1}{m} \sum_{y \in S^1(\emptyset)} \Big(\fint_{\partial \mathbb{T}^y} f(s)\mathrm{d}s + \mathcal{S}_h(y) \Big) + c_\beta h(\emptyset).$$

Note that

$$\int_0^1 f(s)ds = \frac{1}{m} \sum_{y \in S^1(\emptyset)} \fint_{\partial \mathbb{T}^y} f(s)\mathrm{d}s$$

and then we obtain

$$u(\emptyset) = \int_0^1 f(s)\mathrm{d}s + \frac{1}{m} \sum_{y \in S^1(\emptyset)} \mathcal{S}_h(y) + c_\beta h(\emptyset).$$

Therefore, we have that u satisfies the recurrence

$$\begin{cases} u(x) = \frac{c_\beta - 1}{c_\beta} u(\hat{x}) + \frac{1}{c_\beta} \fint_{\partial \mathbb{T}^x} f(s)\mathrm{d}s + \frac{1}{c_\beta} \mathcal{S}_h(x), & x \neq \emptyset, \\ u(\emptyset) = \int_0^1 f(s)\mathrm{d}s + \frac{1}{m} \sum\limits_{y \in S^1(\emptyset)} \mathcal{S}_h(y) + c_\beta h(\emptyset). \end{cases} \tag{2.10}$$

If we iterate this recurrence up to the root, we get the following formula for u; given $x \in \mathbb{T}$ with $|x| = k$ for $k \geq 1$,

$$\begin{aligned} u(x) = &\Big(\tfrac{\beta}{1-\beta}\Big)^k \int_0^1 f(s)ds + \tfrac{1}{c_\beta} \sum_{j=1}^k \Big(\tfrac{\beta}{1-\beta}\Big)^{k-j} \fint_{\partial \mathbb{T}^{x^j}} f(s)\mathrm{d}s \\ &+ \Big(\tfrac{\beta}{1-\beta}\Big)^k \Big[\tfrac{1}{m} \sum_{y \in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \tfrac{1}{c_\beta} \sum_{j=1}^k \Big(\tfrac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j) \end{aligned} \tag{2.11}$$

where x^j denotes the predecessor of x with $|x^j| = j$, that is, in terms of the notation given in the introduction, for a node x with $|x| = k$, we have $x = (a_1, \dots, a_j, \dots, a_k)$ with $0 \leq a_j \leq m-1$, then $x^j = (a_1, \dots, a_j)$.

This function u given by (2.11) is well-defined, finite, and solves the equation $L(u)(x) = h(x)$ for $x \in \mathbb{T}$.

Now we just take $f \equiv 0$ and we obtain that u, given by

$$u(x) = \Big(\tfrac{\beta}{1-\beta}\Big)^k \Big[\tfrac{1}{m} \sum_{y \in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \tfrac{1}{c_\beta} \sum_{j=1}^k \Big(\tfrac{\beta}{1-\beta}\Big)^{k-j} S_h(x^j)$$

must satisfy

$$\lim_{x \in \mathbb{T}, |x| \to \infty} u(x) = 0$$

uniformly. Since $\beta < \frac{1}{2}$, it is clear that

$$\lim_{k \to \infty} \Big(\frac{\beta}{1-\beta}\Big)^k \Big[\frac{1}{m} \sum_{y \in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] = 0.$$

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Hence, we must have

$$\lim_{k\to\infty}\sum_{j=1}^{k}\Big(\frac{\beta}{1-\beta}\Big)^{k-j}S_h(x^j)=0$$

as we wanted to show.

Proof of the claim (2.8). We will prove by induction in N that

$$\lim_{N\to\infty}\sum_{k=1}^{N}\sum_{y\in S^k(x)}\frac{w(y)}{m^k}=\fint_{\partial\mathbb{T}^x}f(s)\mathrm{d}s-u(x).$$

For $N=2$, we have that

$$\begin{aligned}\sum_{k=1}^{2}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} &= \sum_{y\in S^1(x)}\frac{w(y)}{m^1}+\sum_{y\in S^2(x)}\frac{w(y)}{m^2}\\ &= \frac{1}{m}\Big(\sum_{y\in S^1(x)}\Big(u(y)-u(x)\Big)\Big)+\frac{1}{m^2}\\ &\quad\times\Big(\sum_{y\in S^1(x)}\sum_{z\in S^1(y)}\Big(u(z)-u(y)\Big)\Big)\\ &= \frac{1}{m}\Big(\sum_{y\in S^1(x)}u(y)\Big)-u(x)+\frac{1}{m^2}\Big(\sum_{y\in S^2(x)}u(y)\Big)-\frac{1}{m}\\ &\quad\times\Big(\sum_{y\in S^1(x)}u(y)\Big)\\ &= \frac{1}{m^2}\Big(\sum_{y\in S^2(x)}u(y)\Big)-u(x).\end{aligned}$$

Now, our inductive hypothesis is

$$\sum_{k=1}^{N}\sum_{y\in S^k(x)}\frac{w(y)}{m^k}=\frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big)-u(x),$$

and we have to prove that for $N+1$,

$$\begin{aligned}\sum_{k=1}^{N+1}\sum_{y\in S^k(x)}\frac{w(y)}{m^k} &= \sum_{y\in S^{N+1}(x)}\frac{w(y)}{m^{N+1}}+\frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big)-u(x)\\ &= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^N(x)}\sum_{z\in S^1(y)}\Big(u(z)-u(y)\Big)\Big)+\frac{1}{m^N}\\ &\quad\times\Big(\sum_{y\in S^N(x)}u(y)\Big)-u(x)\\ &= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^{N+1}(x)}u(y)\Big)-\frac{1}{m^N}\Big(\sum_{y\in S^N(x)}u(y)\Big)+\frac{1}{m^N}\\ &\quad\times\Big(\sum_{y\in S^N(x)}u(y)\Big)-u(x)\\ &= \frac{1}{m^{N+1}}\Big(\sum_{y\in S^{N+1}(x)}u(y)\Big)-u(x).\end{aligned}$$

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Since u have supposed that $u|_{\partial\pi} = f$, we have that

$$\lim_{N\to\infty} \frac{1}{m^N} \sum_{y\in S^N(x)} u(y) = \fint_{\partial\mathbb{T}^x} f(s)\mathrm{d}s$$

due to the fact in the definition of $u|_{\partial\pi} = f$ the convergence is uniform. This ends the proof of the claim.

Conversely, assuming that the conditions on β and h, (1.5) hold, from our previous computations, (2.11) gives us a way to construct solutions to our problem. In fact, we can define the function $u : \mathbb{T} \mapsto \mathbb{R}$ given by

$$\begin{aligned} u(x) &= \left(\tfrac{\beta}{1-\beta}\right)^k \int_0^1 f(s)ds + \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta}\right)^{k-j} \fint_{\partial\mathbb{T}^{x^j}} f(s)\mathrm{d}s \\ &\quad + \left(\tfrac{\beta}{1-\beta}\right)^k \Big[\tfrac{1}{m} \sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_h(x^j) \end{aligned} \tag{2.12}$$

where, as before, x^j denotes the predecessor of x with $|x^j| = j$.

One can check that u, given by the explicit expression (2.12), satisfies

$$Lu(x) = h(x), \qquad x \in \mathbb{T}.$$

Now, let us check that, when we have the solvability conditions

$$\begin{aligned} &0 \le \beta < \tfrac{1}{2}, \\ &\lim_{x\in\mathbb{T}, |x|=k\to\infty} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_h(x^j) = 0, \end{aligned}$$

it holds that

$$\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z))$$

uniformly.

From (2.12), we get

$$\begin{aligned} u(x) - f(\psi(z)) &= \left(\tfrac{\beta}{1-\beta}\right)^k \int_0^1 f(s)ds + \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta}\right)^{k-j} \fint_{\partial\mathbb{T}^{x^j}} f(s)\mathrm{d}s - f(\psi(z)) \\ &\quad + \left(\tfrac{\beta}{1-\beta}\right)^k \Big[\tfrac{1}{m} \sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] + \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_h(x^j). \end{aligned}$$

Since $\beta < \frac{1}{2}$, it holds that

$$\lim_{k\to\infty} \left(\frac{\beta}{1-\beta}\right)^k \int_0^1 f(s)ds = 0$$

and also

$$\lim_{k\to\infty} \left(\frac{\beta}{1-\beta}\right)^k \Big[\frac{1}{m} \sum_{y\in S^1(\emptyset)} S_h(y) + c_\beta h(\emptyset)\Big] = 0.$$

Now, given $\gamma > 0$, since f is continuous, we have that there exists k_0 such that

$$\left| \fint_{\partial \mathbb{T}x^j} f(s)\mathrm{d}s - f(\psi(z)) \right| < \gamma$$

for every $j > k_0$. Now, we use that

$$c_\beta = \sum_{l=0}^{\infty} \left(\frac{\beta}{1-\beta} \right)^l$$

and that f is bounded to obtain

$$\begin{aligned}
&\left| \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta} \right)^{k-j} \fint_{\partial \mathbb{T}x^j} f(s)\mathrm{d}s - f(\psi(z)) \right| \\
&\quad = \left| \tfrac{1}{c_\beta} \sum_{j=1}^{k} \left(\tfrac{\beta}{1-\beta} \right)^{k-j} \fint_{\partial \mathbb{T}x^j} f(s)\mathrm{d}s - \tfrac{1}{c_\beta} \sum_{l=0}^{\infty} \left(\tfrac{\beta}{1-\beta} \right)^{l} f(\psi(z)) \right| \\
&\quad \le C \sum_{j=1}^{k_0-1} \left(\tfrac{\beta}{1-\beta} \right)^{k-j} \\
&\qquad + \left| \tfrac{1}{c_\beta} \sum_{l=0}^{k-k_0} \left(\tfrac{\beta}{1-\beta} \right)^{l} \fint_{\partial \mathbb{T}x^{k-l}} f(s)\mathrm{d}s - \tfrac{1}{c_\beta} \sum_{l=0}^{\infty} \left(\tfrac{\beta}{1-\beta} \right)^{l} f(\psi(z)) \right| \\
&\quad \le C \sum_{j=1}^{k_0-1} \left(\tfrac{\beta}{1-\beta} \right)^{k-j} + \gamma \tfrac{1}{c_\beta} \sum_{l=0}^{k-k_0} \left(\tfrac{\beta}{1-\beta} \right)^{l} + C \tfrac{1}{c_\beta} \sum_{l=k-k_0+1}^{\infty} \left(\tfrac{\beta}{1-\beta} \right)^{l} \le \tfrac{\varepsilon}{2},
\end{aligned}$$

for γ small enough and k large enough. Hence, we have that

$$\lim_{k\to\infty} \left| \frac{1}{c_\beta} \sum_{j=1}^{k} \left(\frac{\beta}{1-\beta} \right)^{k-j} \fint_{\partial \mathbb{T}x^j} f(s)\mathrm{d}s - f(\psi(z)) \right| = 0.$$

Finally, the condition

$$\lim_{x\in\mathbb{T}, |x|=k\to\infty} \sum_{j=1}^{k} \left(\frac{\beta}{1-\beta} \right)^{k-j} S_h(x^j) = 0$$

is just what we need to ensure that

$$\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)).$$

This shows that there exists a solution.

Uniqueness follows for our previous arguments. In fact, if u is a solution, the increments $w(x) = u(x) - u(\hat{x})$ solve (2.7) from where we obtain (by our previous computations) that u verifies (2.11). Hence, any solution to our problem is given by (2.11) and therefore we conclude uniqueness of solutions. In fact, (2.11) is a representation formula for the solution to (2.6) in terms of the functions h and f. Uniqueness also follows from the comparison principle (see lemma 2.1). □

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Remark 1. Under our hypothesis on β and h, for $f:[0,1]\to\mathbb{R}$ continuous the function $w_1:\mathbb{T}\to\mathbb{R}$ defined as

$$
\begin{aligned}
w_1(x) &= \left(\tfrac{\beta}{1-\beta}\right)^k\Big[\tfrac{1}{m}\sum_{y\in S^1(\emptyset)} S_h(y)+c_\beta h(\emptyset)\Big]\\
&\quad+\tfrac{1}{c_\beta}\sum_{j=1}^{k}\left(\tfrac{\beta}{1-\beta}\right)^{k-j} S_h(x^j), \qquad x\neq\emptyset, |x|=k,\\
w_1(\emptyset) &= \tfrac{1}{m}\sum_{y\in S^1(\emptyset)} \mathcal{S}_h(y)+c_\beta h(\emptyset),
\end{aligned}
$$

is the solution to

$$
\begin{cases}
L(w_1)(x)=h(x), & x\in\mathbb{T},\\
\lim\limits_{x\to z\in\partial\mathbb{T}} w_1(x)=0.
\end{cases}
$$

Moreover, the function $w_2:\mathbb{T}\to\mathbb{R}$ defined as

$$
\begin{aligned}
w_2(x) &= \left(\tfrac{\beta}{1-\beta}\right)^k\int_0^1 f(s)ds+\tfrac{1}{c_\beta}\sum_{j=1}^{k}\left(\tfrac{\beta}{1-\beta}\right)^{k-j} ⨍_{\partial\mathbb{T}^{x^j}} f(s)\mathrm{d}s, \qquad x\neq\emptyset, |x|=k,\\
w_2(\emptyset) &= \int_0^1 f(s)\mathrm{d}s,
\end{aligned}
$$

is the solution to

$$
\begin{cases}
L(w_2)(x)=0, & x\in\mathbb{T},\\
\lim\limits_{x\to z\in\partial\mathbb{T}} w_2(x)=f(\psi(z)).
\end{cases}
$$

Therefore, we can write

$$
u(x)=w_1(x)+w_2(x)
$$

being w_1 the solution to (2.6) for a given h with $f=0$ and w_2 the solution for a given f with $h=0$.

Remark 2. Notice that analogous arguments allow us to show that, under the same conditions, we can build supersolutions as large as we want enlarging h (and also large subsolutions) to the equation that satisfy the boundary condition

$$
\lim_{x\to z\in\partial\mathbb{T}} u(x)=f(\psi(z)).
$$

To this end, consider L defined as in (1.1) and (1.2) and $h_1,h_2:\mathbb{T}\to\mathbb{R}$ two functions in the tree such that $h_1\leq h_2$ and f a boundary continuous function. Moreover, suppose that β, h_1, and h_2 satisfy the solvability condition. Let u_i be the unique solution for the Dirichlet problem of the single equation associated with the operator $L-h_i$ and boundary function f, that is, u_i is given by (2.10) for $i=1,2$. Then, u_1 a supersolution the Dirichlet problem with $L-h_2$ and u_2 is a subsolution for the Dirichlet problem with $L-h_1$ and f. Now, notice that we can make u_1 as large as we want in a finite number of nodes just taking $h_1(\emptyset)$ large enough.

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Let us prove the uniqueness for solutions to (2.6). To this end, we prove that the operator L verifies the comparison principle.

LEMMA 2.1. (Comparison Principle) *Given $u, v : \mathbb{T} \to \mathbb{R}$ such that*

$$\begin{cases} L(u)(x) \geq h(x), & x \in \mathbb{T}, \\ \liminf_{x \to z \in \partial\mathbb{T}} u(x) \geq f(\psi(z)) \end{cases}$$

and

$$\begin{cases} L(v)(x) \leq h(x), & x \in \mathbb{T}, \\ \limsup_{x \to z \in \partial\mathbb{T}} v(x) \leq g(\psi(z)). \end{cases}$$

Then, if $f \geq g$ in $[0,1]$, we get

$$u(x) \geq v(x), \qquad x \in \mathbb{T}.$$

Proof. Let us argue by contradiction, i.e. assume that

$$\sup_{x \in \mathbb{T}} (v-u)(x) = \theta > 0.$$

Let us suppose that there exists $k \geq 1$, and $|x_0| = k$ such that $(v-u)(x_0) \geq \frac{\theta}{2}$ and $(v-u)(y) < \frac{\theta}{2}$ for all $|y| < k$. At that point, we have

$$L(v)(x_0) - L(u)(x_0) \leq 0.$$

Solving this inequality, we arrive to

$$0 < \frac{\theta}{2} \leq (v-u)(x_0) \leq \beta(v-u)(\hat{x_0}) + \frac{1-\beta}{m} \sum_{y \in S(x_0)} (v-u)(y)$$

$$\leq \beta(v-u)(x_0) + \frac{1-\beta}{m} \sum_{y \in S(x_0)} (v-u)(y),$$

then

$$(v-u)(x_0) \leq \frac{1}{m} \sum_{y \in S(x_0)} (v-u)(y).$$

This implies that there exists $x_1 \in S(x_0)$ such that $(v-u)(x_1) \geq (v-u)(x_0)$. If we repeat this argument, we will obtain a sequence $(x_n)_{n \geq 0}$ such that $x_{n+1} \in S(x_n)$

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and $(v-u)(x_{n+1}) \geq (v-u)(x_n)$. Thus, we get

$$\begin{aligned} 0 < \tfrac{\theta}{2} &\leq \liminf_{x_k \to z \in \partial\mathbb{T}} (v-u)(x_k) \\ &\leq \limsup_{x_k \to z \in \partial\mathbb{T}} v(x_k) - \liminf_{x_k \to z \in \partial\mathbb{T}} u(x_k) \leq g(\psi(z)) - f(\psi(z)) \leq 0, \end{aligned}$$

which is a contradiction. Now, let us consider the case

$$(v-u)(\emptyset) \geq \frac{\theta}{2}.$$

In this case, we have

$$0 < \frac{\theta}{2} \leq (v-u)(\emptyset) \leq \frac{1}{m} \sum_{y \in S(\emptyset)} (v-u)(y),$$

and this implies that there exists $x_1 \in S(\emptyset)$ such that $(v-u)(x_1) \geq (v-u)(\emptyset)$. If from this point we argue as before we obtain a contradiction. This ends the proof. □

3. The two membranes problem

3.1. The obstacle problem on trees

We defined a solution for the obstacle problem from below and from above in the introduction, see definition 3. Now, we claim that the problem can be regarded from two perspectives.

Definition 5. *Given an operator L defined as before and a function $h : \mathbb{T} \to \mathbb{R}$, such that the conditions for solvability, (1.5), hold, $\varphi : \mathbb{T} \to \mathbb{R}$ a bounded function, and $f : [0,1] \to \mathbb{R}$ a continuous function such that*

$$\limsup_{x \to z \in \partial\mathbb{T}} \varphi(x) < f(\psi(z)),$$

we say that u is a solution to the obstacle problem from below, and we note $u = \underline{O}(L, h, \varphi, f)$ if u verifies

$$u(x) = \inf \left\{ w(x) : \begin{array}{c} L(w)(x) \geq h(x), \quad x \in \mathbb{T}, \\ w(x) \geq \varphi(x), \quad x \in \mathbb{T}, \\ \text{and } \liminf\limits_{x \to z \in \partial\mathbb{T}} w(x) \geq f(\psi(z)) \end{array} \right\}, \tag{3.13}$$

that is equivalent to

$$\begin{cases} u(x) \geq \varphi(x), & x \in \mathbb{T}, \\ L(u)(x) \geq h(x), & x \in \mathbb{T}, \\ L(u)(x) = h(x), & x \in \{u > \varphi\}, \\ \lim\limits_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases} \tag{3.14}$$

Notice that (3.14) *can be written as*

$$\begin{cases} 0 = \max\Big\{ -L(u)(x) + h(x), \varphi(x) - u(x) \Big\}, & x \in \mathbb{T}, \\ \displaystyle\lim_{x \to z \in \partial\mathbb{T}} u(x) = f(\psi(z)). \end{cases}$$

Analogously, for $g : [0,1] \to \mathbb{R}$ *and* $\varphi : \mathbb{T} \to \mathbb{R}$ *such that*

$$\liminf_{x \to z \in \partial\mathbb{T}} \varphi(x) > g(\psi(z))$$

we define v *a solution from above and denote* $v = \overline{\mathrm{O}}(L, h, \varphi, g)$ *if*

$$v(x) = \sup \left\{ w(x) : \begin{array}{c} L(w)(x) \le h(x), \quad x \in \mathbb{T}, \\ w(x) \le \varphi(x), \quad x \in \mathbb{T}, \\ \textit{and } \displaystyle\limsup_{x \to z \in \partial\mathbb{T}} w(x) \le g(\psi(z)) \end{array} \right\}, \tag{3.15}$$

that is equivalent to

$$\begin{cases} v(x) \le \varphi(x), & x \in \mathbb{T}, \\ L(v)(x) \le h(x), & x \in \mathbb{T}, \\ L(v)(x) = h(x), & x \in \{v < \varphi\}, \\ \displaystyle\lim_{x \to z \in \partial\mathbb{T}} v(x) = g(\psi(z)), \end{cases} \tag{3.16}$$

that can be written as

$$\begin{cases} 0 = \min\Big\{ -L(v)(x) + h(x), \varphi(x) - v(x) \Big\}, & x \in \mathbb{T}, \\ \displaystyle\lim_{x \to z \in \partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases}$$

Let us prove that both definitions (the one as inf/sup of super/subsolutions and the one solving inequalities) of being a solution to the obstacle problem are equivalent.

PROPOSITION 3.1. *Consider an operator* L *defined as in (*1.1*) and (*1.2*), and a function* $h : \mathbb{T} \to \mathbb{R}$ *such that the conditions for solvability, (*1.5*), hold,* $\varphi : \mathbb{T} \to \mathbb{R}$ *a bounded function, and* $f : [0,1] \to \mathbb{R}$ *a continuous function such that*

$$\limsup_{x \to z \in \partial\mathbb{T}} \varphi(x) < f(\psi(z)).$$

Then, the function u *defined by (*3.13*) is well-defined and is solution to (*3.14*). Conversely, a solution* u *to (*3.14*) is also the minimizer in (*3.13*).*

Analogously, for a given $g : [0,1] \to \mathbb{R}$*, a continuous function such that*

$$\liminf_{x \to z \in \partial\mathbb{T}} \varphi(x) > g(\psi(z)),$$

we have that the function v *defined by (*3.15*) is well-defined and is solution to (*3.16*). Conversely, a solution* v *to (*3.16*) is the maximizer in (*3.15*).*

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Proof. Let us consider the set

$$\underline{\Lambda}_{f,h,\varphi} = \Big\{ w : L(w) \geq h \ \ w \geq \varphi \ \ \liminf_{x \to z \in \partial \mathbb{T}} w(x) \geq f(\psi(z)) \Big\}. \tag{3.17}$$

This set is non-empty and is bounded from below. In fact, if we consider $M = \max\{\|f\|_\infty, \|\varphi\|_\infty\}$, and w_0 the unique bounded solution to

$$\begin{cases} L(w_0)(x) = h(x), & x \in \mathbb{T}, \\ \lim\limits_{x \to z \in \partial \mathbb{T}} w_0 = f(\psi(z)). \end{cases}$$

Note that $\underline{\Lambda}_{f,h,\varphi}$ is not empty due to the fact that the function $w_0 + M \in \underline{\Lambda}_{f,h,\varphi}$. Moreover, if $w \in \underline{\Lambda}_{f,h,\varphi}$, $w(x) \geq \varphi(x) \geq -M$, then, is bounded from below. Let us define

$$u(x) = \inf_{w \in \underline{\Lambda}_{f,h,\varphi}} w(x). \tag{3.18}$$

Let us show that this function u verifies (3.14). In fact, using that $w \geq \varphi$ for all $w \in \underline{\Lambda}_{f,h,\varphi}$, taking infimum we get $u \geq \varphi$. In addition, given $w \in \underline{\Lambda}_{f,h,\varphi}$, we have

$$L(w)(x) \geq h(x) \quad \Rightarrow \quad w(x) \geq \beta w(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} w(y)\Big) + h(x), \quad x \neq \emptyset.$$

As a consequence, if we take infimum in the right hand of the above inequality, we obtain

$$w(x) \geq \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x), \quad x \neq \emptyset.$$

Furthermore, taking infimum in the left hand

$$u(x) \geq \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u(y)\Big) + h(x) \quad \Rightarrow \quad L(u)(x) \geq h(x), \quad x \neq \emptyset.$$

Analogously, we can do the same computation on the root $\emptyset$ and obtain

$$u(\emptyset) \geq \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} u(y)\Big) + h(\emptyset) \quad \Rightarrow \quad L(u)(\emptyset) \geq h(x).$$

Finally, we have

$$\liminf_{x \to z \in \partial \mathbb{T}} w(x) \geq f(\psi(z))$$

for all $w \in \underline{\Lambda}_{f,h,\varphi}$, taking infimum we get

$$\liminf_{x \to z \in \partial \mathbb{T}} u(x) \geq f(\psi(z)).$$

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We just proved that $u \in \underline{\Lambda}_{f,h,\varphi}$. Let us prove now that

$$L(u)(x) = 0$$

if $x \in \{u > \varphi\}$. Suppose that this is not true, given $x_0 \in \{u > \varphi\}$ such that $L(u)(x_0) > h(x_0)$, i.e.

$$u(x_0) > \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u(y)\Big) + h(x).$$

Let us define

$$\delta_1 = u(x_0) - \beta u(\hat{x}) + (1-\beta)\Big(\frac{1}{m}\sum_{y\in S^k(x)} u(y)\Big) + h(x),$$

$$\delta_2 = u(x_0) - \varphi(x_0)$$

and

$$\delta_0 = \min\{\delta_1, \delta_2\}.$$

If we consider

$$u_0(x) = \begin{cases} u(x), & x \neq x_0, \\ u(x) - \frac{\delta_0}{2}, & x = x_0. \end{cases}$$

This function verifies $u_0 \geq \varphi$,

$$\liminf_{x\to z\in\partial\mathbb{T}} u_0(x) \geq f(\psi(z))$$

and $L(u_0)(x) \geq h(x)$. In fact, $L(u_0)(x_0) \geq h(x)$ and then $L(u_0)(\hat{x_0}) \geq h(\hat{x_0})$, and for $y_0 \in S(x_0)$, $L(u_0)(y_0) \geq h(y_0)$. Thus, $u_0 \in \underline{\Lambda}_{f,h,\varphi}$ and $u_0 < u$ which is a contradiction. We have proved that

$$L(u)(x) = h(x)$$

in the set $\{u > \varphi\}$.

Let us verify that

$$\limsup_{x\to z\in\partial\mathbb{T}} u(x) \leq f(\psi(z)).$$

Suppose that this is not true, i.e.

$$\limsup_{x\to z\in\partial\mathbb{T}} u(x) > f(\psi(z)).$$

Using that

$$\limsup_{x\to z\in\partial\mathbb{T}} \varphi(x) < f(\psi(z)),$$

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there exists $k \in \mathbb{N}$ such that $|x| \geq k$ and $\varphi(x) < f(\psi(z))$. Then, there exists x_0 such that $|x_0| \geq k$ and $u(x_0) > f(\psi(z))$. We can argue as before to make a function $u_0 \in \underline{\Lambda}_{f,h,\varphi}$ and $u_0 < u$ which is a contradiction.

Conversely, assume that u solves (3.14) and let us prove that u is the minimizer in (3.13). Since u solves (3.14), we have that it satisfies $L(u)(x) \geq h(x)$ for $x \in \mathbb{T}$, $\lim_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z))$ and $w(x) \geq \varphi(x)$ and then u is a competitor in the minimization problem (3.13). Therefore,

$$u(x) \geq \inf_{w\in\underline{\Lambda}_{f,h,\varphi}} w(x).$$

To prove the reverse inequality, call z the minimizer, that is,

$$z(x) = \inf_{w\in\underline{\Lambda}_{f,h,\varphi}} w(x).$$

Since we have that $u \geq z$ we get an inclusion for the sets where these functions touch the obstacle,

$$\{x : u(x) = \varphi(x)\} \subset \{x : z(x) = \varphi(x)\}.$$

Outside the set $\{x : u(x) = \varphi(x)\}$ we have that u solves $L(u)(x) = h(x)$ and z is a supersolution to this equation, while the boundary conditions give

$$\liminf_{x\to z\in\partial\mathbb{T}} z(x) \geq f(\psi(z)) = \lim_{x\to z\in\partial\mathbb{T}} u(x).$$

From the proof of the comparison principle, we get that

$$u(x) \leq z(x),$$

and we conclude that $u \equiv z$, the minimizer of (3.13).

Analogously, we consider

$$\overline{\Lambda}_{g,h,\varphi} = \Big\{ w : L(w) \leq h \;\; w \leq \varphi \;\; \limsup_{x\to z\in\partial\mathbb{T}} w(x) \leq g(\psi(z)) \Big\}. \tag{3.19}$$

This set is non-empty and bounded from above. Then, we define

$$v(x) = \sup_{w\in\overline{\Lambda}_{g,h,\varphi}} w(x). \tag{3.20}$$

This function v verifies (3.16), i.e. $v = \overline{\mathrm{O}}(L, h, \varphi, g)$, and conversely, a function that verifies (3.16) is given by (3.20). This ends the proof. □

Corollary 1. *Given an operator L defined as in (1.1) and (1.2), and a function $h : \mathbb{T} \to \mathbb{R}$ such that the conditions for solvability, (1.5), hold, $\varphi : \mathbb{T} \to \mathbb{R}$ a bounded*

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function, and $f : [0, 1] \to \mathbb{R}$ *a continuous function. If*

$$\limsup_{x \to z \in \partial \mathbb{T}} \varphi(x) < f(\psi(z)),$$

there exists a unique solution for the obstacle problem from below,

$$\underline{\mathrm{O}}(L, h, \varphi, f) = \inf_{w \in \underline{\Lambda}_{f,h,\varphi}} w,$$

where $\underline{\Lambda}_{f,h,\varphi}$ *is defined in* (3.17).

Analogously, if

$$\liminf_{x \to z \in \partial \mathbb{T}} \varphi(x) > f(\psi(z)),$$

there exists a unique solution for the obstacle problem from above,

$$\overline{\mathrm{O}}(L, h, \varphi, g) = \sup_{w \in \overline{\Lambda}_{f,h,\varphi}} w,$$

where $\overline{\Lambda}_{g,h,\varphi}$ *is defined in* (3.19).

Proof. The existence of $\underline{\mathrm{O}}(L, h, \varphi, f)$ and of $\overline{\mathrm{O}}(L, h, \varphi, g)$ relies on the fact that the sets $\underline{\Lambda}_{f,h,\varphi}$ and $\overline{\Lambda}_{g,h,\varphi}$ are not empty, and there exists a $M > 0$ such that $w \geq -M$ for all $w \in \underline{\Lambda}_{f,h,\varphi}$ and $v \leq M$ for all $v \in \overline{\Lambda}_{f,h,\varphi}$, as we showed in the proof of the equivalence of the definitions of being a solution to the obstacle problem. The uniqueness is a direct consequence of taking the infimum or the supremum on these sets. □

3.2. The two membranes problem: existence via iterations of the obstacle problem

Let us show that the two membranes problem in the tree has a solution.

Proof of theorem 1.1*.* Let us consider L_1 and L_2 two operators given by the mean value formula and $h_1, h_2 : \mathbb{T} \to \mathbb{R}$ that verify the solvability condition with β_1, h_1 and β_2, h_2, respectively, and $f, g : [0, 1] \to \mathbb{R}$ two continuous functions such that $f > g$. Let us start the method with v_0 a bounded subsolution of the operator with boundary condition g, that is,

$$\begin{array}{ll} L_2(v_0)(x) \leq h_2(x), & x \in \mathbb{T}, \\ \limsup_{x \to z \in \partial \mathbb{T}} v_0(x) \leq g(\psi(z)). & \end{array}$$

With this function v_0, we let u_1 be the solution to the obstacle problem from below for the operator L_1, right hand side h_1, boundary datum f, and obstacle v_0, that is,

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$$u_1 = \underline{\mathrm{O}}(L_1, h_1, v_0, f).$$

Notice that the set

$$\underline{\Lambda}_{f,h_1 v_0} = \Big\{ w : L_1(w) \geq h_1 \;\; w \geq v_0 \;\; \liminf_{x \to z \in \partial \mathbb{T}} w(x) \geq f(\psi(z)) \Big\}$$

is not empty, since by the results in §2, we can construct a supersolution to our problem with $L_1 - h_1$ and boundary condition f as large as we want (see remark 2). Therefore, the function u_1 can be obtained solving the minimization problem $\inf\{w(x) :\; w \in \underline{\Lambda}_{f,h_1 v_0}\}$.

With this function u_1, we take

$$v_1 = \overline{\mathrm{O}}(L_2, h_2, u_1, g).$$

Here, notice that the corresponding set

$$\overline{\Lambda}_{g,h_2,u_1} = \Big\{ w : L_2(w) \leq h_2 \;\; w \leq u_1 \;\; \limsup_{x \to z \in \partial \mathbb{T}} w(x) \leq g(\psi(z)) \Big\}$$

is not empty (here we recall that we can construct large subsolutions to $L_2 - h_2$) and hence v_1 can be obtained from the maximization problem $\sup\{w(x) :\;\; w \in \overline{\Lambda}_{g,h_2,u_1}\}$.

Now, we iterate this procedure and define

$$u_n = \underline{\mathrm{O}}(L_1, h_1, v_{n-1}, f) \quad \text{and} \qquad v_n = \overline{\mathrm{O}}(L_2, h_2, u_n, g).$$

In this way, we obtain two sequences $\{u_n\}_{n\geq 1}, \{v_n\}_{n \geq 1}$. Our goal is to show that these sequences are monotone and that they converge to a pair of functions (u, v) that is a solution to the two membranes problem.

CLAIM # 1: The sequences are increasing, i.e. $u_n \geq u_{n-1}, v_n \geq v_{n-1}$.

Let us start with v_{n-1}. By definition of being a solution to the obstacle problem, v_{n-1} satisfies $u_{n-1} \geq v_{n-1}$, $L_2(v_{n-1}) \leq h_2$ and $\lim_{x\to z \in \partial\mathbb{T}} v_{n-1}(x) = g(\psi(z))$. Hence, $v_{n-1} \in \overline{\Lambda}_{g,h_2,u_{n-1}}$. Then, using again the definition of being a solution to the obstacle problem, we get

$$v_n = \sup_{w \in \overline{\Lambda}_{g,h_2,u_{n-1}}} w \geq v_{n-1}.$$

On the other hand, $u_n \geq v_n \geq v_{n-1}$, $L_1(u_n) \geq h_1$, and $\lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z))$, then, using one more time the definition of being a solution to the obstacle problem, we get

$$u_{n-1} = \inf_{w \in \underline{\Lambda}_{f,h_1,v_{n-1}}} w \leq u_n.$$

This ends the proof of the claim.

CLAIM # 2: The sequences are bounded.

Let us start with $\{u_n\}_{n\geq 1}$. Let w be the solution to

$$\begin{cases} L_2(w)(x) = h_2(x), & x \in \mathbb{T}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} w(x) = g(\psi(z)). \end{cases}$$

By the comparison principle $v_n \leq w$. Let us consider

$$\rho = \underline{\mathrm{O}}(L_1, h_1, w, f).$$

Note that ρ is bounded since w is bounded and the operators satisfy the solvability conditions. Since $v_n \leq w$, we have that $\rho \in \underline{\Lambda}_{g,h_2,v_n}$ for all $n \in \mathbb{N}$. Using (3.18), we get

$$u_n \leq \rho$$

for all $n \in \mathbb{N}$.

Now we observe that $\{v_n\}_{n\geq 1}$ is bounded, since it holds that

$$v_n \leq u_{n-1} \leq \rho$$

for all $n \in \mathbb{N}$. This ends the proof the second claim.

Thanks to CLAIM # 1 and CLAIM # 2, we can take limits as $n \to \infty$ to obtain

$$u_n(x) \to u_\infty(x) \quad \text{and} \quad v_n(x) \to v_\infty(x),$$

for all $x \in \mathbb{T}$.

CLAIM # 3: The limit pair (u_∞, v_∞) solves the two membranes problem.

Let us prove that

$$u_\infty = \underline{\mathrm{O}}(L_1, h_1, v_\infty, f) \quad \text{and} \quad v_\infty = \overline{\mathrm{O}}(L_2, h_2, u_\infty, g).$$

First, we observe that we have

$$\begin{cases} u_n(x) = \max\Big\{\beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} u_n(y)\Big) + h_1(x), v_{n-1}(x)\Big\}, \ x \neq \emptyset, \\ u_n(\emptyset) = \max\Big\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} u_n(y)\Big) + h_1(\emptyset), v_{n-1}(x)\Big\}, \\ \lim\limits_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)). \end{cases}$$

Therefore, from the monotonicity of the sequence u_n, we get

$$u_\infty(x) \geq u_n(x) \geq \beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum_{y\in S^1(x)} u_n(y)\Big) + h_1(x),$$

$$u_\infty(\emptyset) \geq u_n(\emptyset) \geq \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} u_n(y)\Big) + h_1(\emptyset),$$

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and

$$u_\infty(x) \geq u_n(x) \geq v_{n-1}(x).$$

Letting $n \to \infty$ in the right hand sides, we get

$$\begin{aligned} &u_\infty(x) \geq \beta_1 u_\infty(\hat{x}) + (1-\beta_1)\Big(\tfrac{1}{m} \sum_{y \in S^1(x)} u_\infty(y)\Big) + h_1(x), \\ &u_\infty(\emptyset) \geq \Big(\tfrac{1}{m} \sum_{y \in S^1(\emptyset)} u_\infty(y)\Big) + h_1(\emptyset), \end{aligned} \tag{3.21}$$

and

$$u_\infty(x) \geq v_\infty(x). \tag{3.22}$$

Now, for v_n, we have

$$v_n(x) \leq \beta_2 v_n(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m} \sum_{y \in S^1(x)} v_n(y)\Big) + h_2(x),$$

$$v_n(\emptyset) \leq \Big(\frac{1}{m} \sum_{y \in S^1(\emptyset)} v_n(y)\Big) + h_2(\emptyset),$$

and using again the monotonicity of the sequence we get (after passing to the limit as $n \to \infty$),

$$\begin{aligned} &v_\infty(x) \leq \beta_2 v_\infty(\hat{x}) + (1-\beta_2)\Big(\tfrac{1}{m} \sum_{y \in S^1(x)} v_\infty(y)\Big) + h_2(x), \quad x \neq \emptyset, \\ &v_\infty(\emptyset) \leq \Big(\tfrac{1}{m} \sum_{y \in S^1(\emptyset)} v_\infty(y)\Big) + h_2(\emptyset). \end{aligned} \tag{3.23}$$

Now, for a point x in the set $\{x : u_\infty(x) > v_\infty(x)\}$ with $x \neq \emptyset$, we have that $u_n(x) > v_n(x)$ for n large and hence we obtain, for $x \in \{x : u_\infty(x) > v_\infty(x)\}$,

$$u_n(x) = \beta_1 u_n(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m} \sum_{y \in S^k(x)} u_n(y)\Big) + h_1(x),$$

and

$$v_n(x) = \beta_2 v_n(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m} \sum_{y \in S^k(x)} v_n(y)\Big) + h_2(x).$$

Passing to the limit as $n \to \infty$, we conclude that

$$u_\infty(x) = \beta_1 u_\infty(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m} \sum_{y \in S^1(x)} u_\infty(y)\Big) + h_1(x), \tag{3.24}$$

and

$$v_\infty(x) = \beta_2 v_\infty(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum_{y\in S^1(x)} v_\infty(y)\Big) + h_2(x). \tag{3.25}$$

An analogous computation can be done when $u_\infty(\emptyset) > v_\infty(\emptyset)$ to obtain

$$u_\infty(\emptyset) = \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} u_\infty(y)\Big) + h_1(\emptyset), \tag{3.26}$$

and

$$v_\infty(\emptyset) = \Big(\frac{1}{m}\sum_{y\in S^1(\emptyset)} v_\infty(y)\Big) + h_2(\emptyset). \tag{3.27}$$

Now, concerning the boundary condition of u_∞, we have

$$\lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)).$$

Hence, from the monotonicity of the sequence, we obtain

$$\liminf_{x\to z\in\partial\mathbb{T}} u_\infty(x) \geq \lim_{x\to z\in\partial\mathbb{T}} u_n(x) = f(\psi(z)).$$

Now, since from §2, we know that we can construct a large supersolution for L_1 with boundary datum f that we call $\overline{u}$ and we have a comparison principle, we obtain

$$u_n(x) \leq \overline{u}(x)$$

for every $x \in \mathbb{T}$ and every n and hence we get

$$u_\infty(x) \leq \overline{u}(x).$$

Thanks to this inequality we obtain

$$\limsup_{x\to z\in\partial\mathbb{T}} u_\infty(x) \leq \lim_{x\to z\in\partial\mathbb{T}} \overline{u}(x) = f(\psi(z))$$

and hence we conclude that

$$\lim_{x\to z\in\partial\mathbb{T}} u_\infty(x) = f(\psi(z)). \tag{3.28}$$

A similar argument shows that

$$\lim_{x\to z\in\partial\mathbb{T}} v_\infty(x) = g(\psi(z)). \tag{3.29}$$

To conclude, we just observe that (3.21), (3.22), (3.24), (3.26), and (3.28) show that

$$u_\infty = \underline{\mathrm{O}}(L_1, h_1, v_\infty, f).$$

In the same way, (3.23), (3.22), (3.25), (3.27), and (3.29) imply that

$$v_\infty = \overline{\mathrm{O}}(L_2, h_2, u_\infty, g).$$

Finally, we observe that, since we have a subsolution (called $\underline{u}$) for L_1 with datum f and a supersolution (called $\overline{v}$) for L_2 with datum g and we assumed that $f > g$, we have

$$\lim_{x\to z\in\partial\mathbb{T}} \overline{v}(x) = g(\psi(z)) < f(\psi(z)) = \lim_{x\to z\in\partial\mathbb{T}} \underline{u}(x).$$

Therefore, we have that

$$v_\infty(x) \le \overline{v}(x) < \underline{u}(x) \le u_\infty(x)$$

for every x with $|x|$ large enough, say $|x| > C$. This proves that the contact set $\{x : u_\infty(x) = v_\infty(x)\}$ does not contain nodes in $\{x : |x| > C\}$ and therefore the contact set is finite. □

4. A probabilistic interpretation of the two membranes problem in the tree

Recall that in §1, we mentioned that the system (1.3), that is given by

$$\begin{cases} u(x) = \max\left\{\beta_1 u(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} u(y)\Big) + h_1(x), v(x)\right\}, \ x \neq \emptyset, \\ u(\emptyset) = \max\left\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} u(y)\Big) + h_1(\emptyset), v(\emptyset)\right\}, \\ v(x) = \min\left\{\beta_2 v(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} v(y)\Big) + h_2(x), u(x)\right\}, \ x \neq \emptyset, \\ v(\emptyset) = \min\left\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} v(y)\Big) + h_2(\emptyset), u(\emptyset)\right\}, \end{cases} \tag{4.30}$$

has a probabilistic interpretation. In this final section, we include the details. We refer to the books [4] and [16] and papers [18] and [19] for more information concerning games and mean value properties.

The game is a two-player zero-sum game played in two boards (each board is a copy of the m-regular tree) with the following rules: the game starts with a token at some node in one of the two trees (x_0, i) with $x_0 \in \mathbb{T}$ and $i = 1, 2$ (we add an index to denote at which board is the position of the game). In the first board, the token is moved to the predecessor of x or to one of the m successors using probabilities β_1 and $(1-\beta_1)/m$ while in the second board the probabilities are changed to β_2 and $(1-\beta_2)/m$. At the root of the tree, the token moves to one of the successors with probability $1/m$. In the first board, we add a running payoff $h_1(x)$ and in the second board we add $h_2(x)$. In addition to these rules for the movements of the token, the players have a choice to play at the same board or to change boards. If x_0 is in the first board then Player I (who aims to maximize the expected payoff) decides to remain in the same board and play one round of the game moving to the

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predecessor of x or to one of the m successors using probabilities β_1 and $(1-\beta_1)/m$ and collecting a running payoff $h_1(x)$ or to jump to the other board. On the other hand, when x_0 is in the second board then it is Player II (who aims to minimize the expected payoff) who decides to remain in the same board and play one round of the game with probabilities β_2 and $1-\beta_2$ and collecting a running payoff $h_2(x)$ or to jump to the first board.

We take a finite level M (large) and we add the rule that the game ends when the position arrives to a node at level M, x_τ. We also have two final payoffs f and g. In the first board, Player I pays to Player II the amount encoded by $f(\psi(x_\tau))$ while in the second board the final payoff is given by $g(\psi(x_\tau))$ plus the amount encoded in the running payoff. That is, the total payoff of one occurrence of this game is given by

$$\begin{aligned} \text{total payoff} \quad := \quad & f(x_\tau)\chi_{\{j=1\}}(j_\tau) + g(x_\tau)\chi_{\{j=2\}}(j_\tau) \\ & -\sum_{k=0}^{\tau-1}\Big(-h_1(x_k)\chi_{\{j=1\}}(j_{k+1}) - h_2(x_k)\chi_{\{j=2\}}(j_{k+1})\Big). \end{aligned}$$

Notice that the total payoff is the sum of the final payoff (given by $f(x_\tau)$ or by $g(x_\tau)$ according to the board at which the position leaves the domain) and the running payoff that is given by $h_1(x_k)$ and $h_2(x_k)$ corresponding to the board in which we play at step $k+1$.

Notice that the successive positions of the token are determined by the jumping probabilities given by β_i (to jump to the predecessor), $(1-\beta_i)/m$ (to jump to one of the successors) $i = 1, 2$ according to the board at which the game is played and the strategies of both players (the choice that one of them makes at each turn regarding the possible change of board). We refer to [4] for more details and precise definitions of strategies. We will denote by S_I a strategy for the first player and S_{II} a strategy for the second player.

Then the value function for this game is defined as

$$w_M(x,i) = \inf_{S_I}\sup_{S_{II}} \mathbb{E}^{(x,i)}(\text{final payoff}) = \sup_{S_{II}}\inf_{S_I} \mathbb{E}^{(x,i)}(\text{final payoff}).$$

Here the inf and sup are taken among all possible strategies of the players. In this definition of the value of the game, we penalize games that never end (both players may choose to change boards at the same node for ever producing a game that never ends). The value of the game $w_M(x,i)$ encodes the amount that the players expect to get/pay playing their best with final payoffs f and g at level M.

We have that the pair of functions (u_M, v_M) given by $u_M(x) = w_M(x,1)$ and $v_M(x) = w_M(x,2)$ is a solution to the system (4.30) in the finite subgraph of the tree composed by nodes of level less than M.

Notice that the first equation encodes all the possibilities for the next position of the game in the first board and includes a maximum since the first player has the choice to play or change to the second board. Similarly, the second equation takes into account all the possibilities for the game in the second board and includes a minimum since in this case it is the second player who decides to play or to change boards.

Now our goal is to take the limit as $M \to \infty$ in these value functions for this game and obtain that the limit is the unique solution to our system that verifies the boundary conditions

$$\begin{cases} \lim\limits_{x\to z\in\partial\mathbb{T}} u(x) = f(\psi(z)), \\ \lim\limits_{x\to z\in\partial\mathbb{T}} v(x) = g(\psi(z)). \end{cases} \tag{4.31}$$

THEOREM 4.1 *Fix two continuous functions $f, g : [0,1] \to \mathbb{R}$. Let (u_L, v_L) be the values of the game in the finite subgraph of the tree with nodes of level less than M, that is, (u_M, v_M) is the solution to (4.30) with conditions $u_M(x) = f(\psi(x))$ and $v_L(x) = g(\psi(x))$ at nodes of level L. Then (u_M, v_M) converge, along subsequences, as $M \to \infty$ to (u, v) solutions to the two membranes problem, that is, a solution to (4.30) with (4.31) in the whole tree.*

Proof. From the estimates that we have proved in the previous sections for a solution (u_M, v_M), we know that these functions are uniformly bounded in M. Therefore, we can extract a subsequence $M_j \to \infty$ such that

$$u_{M_j}(x) \to u(x) \qquad \text{and} \qquad v_{M_j}(x) \to v(x),$$

for every $x \in \mathbb{T}$. Passing to the limit in the equations

$$\begin{cases} u_M(x) = \max\Big\{\beta_1 u_M(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} u_M(y)\Big) + h_1(x), v_M(x)\Big\}, \\ u_M(\emptyset) = \max\Big\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} u_M(y)\Big) + h_1(\emptyset), v_M(\emptyset)\Big\}, \\ v_M(x) = \min\Big\{\beta_2 v_M(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} v_M(y)\Big) + h_2(x), u_M(x)\Big\}, \\ v_M(\emptyset) = \min\Big\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} v_M(y)\Big) + h_2(\emptyset), u_M(\emptyset)\Big\}, \end{cases}$$

we get that the limit (u, v) solves

$$\begin{cases} u(x) = \max\Big\{\beta_1 u(\hat{x}) + (1-\beta_1)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} u(y)\Big) + h_1(x), v(x)\Big\}, \\ u(\emptyset) = \max\Big\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} u(y)\Big) + h_1(\emptyset), v(\emptyset)\Big\}, \\ v(x) = \min\Big\{\beta_2 v(\hat{x}) + (1-\beta_2)\Big(\frac{1}{m}\sum\limits_{y\in S^1(x)} v(y)\Big) + h_2(x), u(x)\Big\}, \\ v(\emptyset) = \min\Big\{\Big(\frac{1}{m}\sum\limits_{y\in S^1(\emptyset)} v(y)\Big) + h_2(\emptyset), u(\emptyset)\Big\}, \end{cases}$$

in the whole tree.

On the other hand, since f and g are continuous, given $\eta > 0$ there exists M large enough such that

$$|u_M(x) - \max_{I_x} f| = |f(\psi(x)) - \max_{I_x} f| < \eta,$$

and

$$|v_M(x) - \max_{I_x} g| = |g(\psi(x)) - \max_{I_x} g| < \eta$$

for every x at level M with M large enough. Therefore, we get

$$u_M(x) \le \max_{I_x} f + \eta, \qquad \text{and} \qquad v_M(x) \le \max_{I_x} g + \eta$$

for every x at level M large. Using that (u_M, v_M) converge as $M \to \infty$ to (u, v) we conclude that

$$\begin{cases} \limsup_{x \to z \in \partial \mathbb{T}} u(x) \le f(\psi(z)) + \eta, \\ \limsup_{x \to z \in \partial \mathbb{T}} v(x) \le g(\psi(z)) + \eta. \end{cases}$$

A similar argument using that

$$u_M(x) \ge \min_{I_x} f - \eta, \qquad \text{and} \qquad v_M(x) \ge \min_{I_x} g - \eta$$

for M large gives that

$$\begin{cases} \liminf_{x \to z \in \partial \mathbb{T}} u(x) \ge f(\psi(z)) - \eta, \\ \liminf_{x \to z \in \partial \mathbb{T}} v(x) \ge g(\psi(z)) - \eta. \end{cases}$$

Therefore, since η is arbitrary, we conclude that (u, v) satisfies

$$\begin{cases} \lim_{x \to z \in \partial \mathbb{T}} u(x) = f(\psi(z)), \\ \lim_{x \to z \in \partial \mathbb{T}} v(x) = g(\psi(z)), \end{cases}$$

and the proof is completed. □

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