NEKHOROSHEV STABILITY IN QUASI–INTEGRABLE DEGENERATE HAMILTONIAN SYSTEMS

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Many classical problems of Mechanics can be studied regarding them as perturbations of integrable systems; this is the case of the fast rotations of the rigid body in an arbitrary potential, the restricted three body problem with small values of the mass–ratio, and others. However, the application of the classical results of Hamiltonian Perturbation Theory to these systems encounters difficulties due to the presence of the so-called 'degeneracy'. More precisely, the Hamiltonian of a quasi–integrable degenerate system looks like

$$H(I,\varphi,p,q) = h(I) + \varepsilon f(I,\varphi,p,q)\,, \tag{1}$$

where $(I,\varphi) \in U \times \mathbf{T}^n$, $U \subseteq \mathbf{R}^n$, are action–angle type coordinates, while the degeneracy of the system manifests itself with the presence of the 'degenerate' variables $(p,q) \in \mathcal{B} \subseteq \mathbf{R}^{2m}$. The KAM theorem has been applied under quite general assumptions to degenerate Hamiltonians (Arnold, 1963), while the Nekhoroshev theorem (Nekhoroshev, 1977) provides, if h is convex, the following bounds: there exist positive ε_0, a_0, t_0 such that if $\varepsilon < \varepsilon_0$ then $|I(t)-I(0)| \le a_0\varepsilon^{\frac{1}{2n}}$ if $|t| \le \min\{T_e, t_0\exp(\varepsilon_0/\varepsilon)^{\frac{1}{2n}}\}$ where T_e is the escape time of the solution from the domain of (1). An escape is possible because the motion of the degenerate variables can be bounded in principle only by $(\dot p,\dot q) = \mathcal{O}(\varepsilon)$, and so over the time $\exp(\varepsilon_0/\varepsilon)^{\frac{1}{2n}}$ they can experience large variations. Therefore, there is the problem of individuating which assumptions on the perturbation and on the initial data allow to control the motion of the degenerate variables over long times. The main assumptions we consider are: (I) the so–called secular Hamiltonian of H, that is

$$\mathcal{K}(I,p,q) = h(I) + \frac{\varepsilon}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} f(I,\varphi,p,q)d\varphi_1\ldots d\varphi_n$$

is integrable, or quasi–integrable. Then, one can choose the $(p,q) \in V \times \mathbf{T}^m$ (with $V \subseteq \mathbf{R}^m$) to be action–angle coordinates and there is a second small parameter η such that $\mathcal{K} = h(I)+\varepsilon(K_0(I,p)+\eta K_1(I,p,q))$. The problem poses even when the secular Hamiltonian is integrable. (II) the secular Hamiltonian provides convexity with respect to all actions I,p (though with a convexity constant of order ε). (III) the perturbation is analytic in a complex neighbourhood of the real domain. We can state now the result

J. Henrard and S. Ferraz-Mello (eds.), Impact of Modern Dynamics in Astronomy, 443–444.

https://doi.org/10.1017/S0252921100073085 Published online by Cambridge University Press

Theorem *Let H be as in (1) satisfying* I, II, III. *There exist positive ε_*, η_*, r_*, R_0, a, T_* such that if $\varepsilon \leq \varepsilon_*$ and $\eta \leq \eta_*$ then any motion with initial datum such that $|\nu \cdot \omega(I(0))| > \varepsilon^a$ for any $\nu \in \mathbf{Z}^n$ with*

$$|\nu| \leq r_* \ln \frac{1}{\varepsilon} ,$$

satisfies

$$|I(t) - I(0)| \leq \varepsilon^{\frac{5}{6}} R_0 \zeta^{\frac{1}{2n(n+m)}} \ , \ |p(t) - p(0)| \leq R_0 \zeta^{\frac{1}{2n(n+m)}} ,$$

where $\zeta = \max \left\{ \varepsilon^{\frac{n(n+m)}{n(n+m)+3n-2}}, \eta \right\}$, *for any time t with*

$$|t| \leq T_* \sqrt{\zeta} \min \left\{ \exp \left[\left(\frac{\varepsilon_*}{\varepsilon} \right)^{\frac{1}{2n(n+m)+6n-4}} \right], \exp \left[\left(\frac{\eta_*}{\eta} \right)^{\frac{1}{2n(n+m)}} \right] \right\} .$$

An analogous stability result was proved for the first time in (Guzzo and Morbidelli, 1997 – see also Morbidelli and Guzzo, 1997) in connection with stability problems in the Asteroid Main Belt. The above theorem is proved with all details in (Guzzo, 1998).

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