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A FURSTENBERG TRANSFORMATION OF THE 2-TORUS WITHOUT QUASI-DISCRETE SPECTRUM

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ABSTRACT. R. Ji asked whether or not a Furstenberg transformation of the 2-torus of the form $(x, y) \longmapsto (e^{2\pi i\theta}x, f(x)y)$, where θ is irrational and $f : T \longrightarrow T$ is continuous with non-zero degree k, is topologically conjugate to the Anzai transformation $(x, y) \longmapsto (e^{2\pi i\theta}x, x^k y)$ or its inverse. In this paper this question is settled in the negative. Further, some sufficient conditions are given under which the crossed product C^*-algebra associated with a Furstenberg transformation of the 2-torus has a unique tracial state.

Introduction. In his Ph.D. thesis Ji asked whether or not a transformation of the 2-torus $\mathbf{T}^2$ of the form $\varphi(x,\ y) = (\lambda x,\ f(x)y)$, where $\lambda = e^{2\pi i\theta}$, θ irrational, and $f : \mathbf{T} \longrightarrow \mathbf{T}$ is continuous with non-zero degree k, is topologically conjugate to the Anzai tranformation $\varphi_\theta(x,\ y) = (\lambda x,\ x^k y)$ or to its inverse ([5], p. 76). He calls such a transformation φ a Furstenberg transformation. In the present paper we shall use a construction of Furstenberg [3] to settle this question in the negative.

It is known that every minimal homeomorphism of the unit circle $\mathbf{T}$ is uniquely ergodic, meaning that it has a unique invariant Borel probability measure (see, for example, [3], Theorem 1.3). This is so because such a transformation is essentially a rotation of $\mathbf{T}$ by angle $2\pi\theta$, where θ is an irrational number. However, Furstenberg showed that this is no longer the case for the 2-torus. He constructed an irrational real number θ and a continuous real-valued function r on $\mathbf{T}$ such that the transformation $(x,\ y) \longrightarrow (e^{2\pi i\theta}x,\ e^{2\pi i r(x)}y)$ is minimal but not uniquely ergodic (see [3], p. 585; or [6], p. 85).

In our case we shall use the same θ and r and show that the (minimal) transformation $\varphi(x,\ y) = (e^{2\pi i\theta}x,\ e^{2\pi i r(x)}xy)$ does not have topological quasi-discrete spectrum, whereas clearly φ_θ (and its inverse) has topological quasi-discrete spectrum. Therefore, φ cannot be topologically conjugate to φ_θ or its inverse. Although our φ differs from Furstenberg's merely by the factor "x" in the second variable, it turns out that in contrast it is uniquely ergodic. Thus the associated crossed product C^*-algebra $A(\varphi) = C(\mathbf{T}^2) \times_\varphi \mathbf{Z}$ has a unique tracial state.

From earlier work [8] the C^*-algebras $A(\varphi)$ and $A(\varphi_\theta)$ are both simple, have unique tracial states, isomorphic K-groups, and the same tracial range $\mathbf{Z} + \theta\mathbf{Z}$. We are as yet unable to distinguish their C^*-isomorphism classes. In fact, this raises a more general

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and interesting question: If φ_1 and φ_2 are Furstenberg transformations of $\mathbf{T}^2$ and if $A(\varphi_1) \cong A(\varphi_2)$ and if φ_1 has topological quasi-discrete spectrum, does it necessarily follow that φ_2 has topological quasi-discrete spectrum? In other words, does the C^*-algebraic structure "carry" information about the topological quasi-discrete spectrum? If so, how?

1. **Preliminaries**. Let $f : \mathbf{T}^2 \longrightarrow \mathbf{T}$ be a continuous function. Then by the Homotopy Lifting Theorem we can write f as $f(x,\ y) = x^m y^n e^{2\pi i F(x,y)}$ for some integers $m,\ n$ and a continuous function $F : \mathbf{T}^2 \longrightarrow \mathbf{R}$. We call the 1×2 integral matrix $D(f) = [m\ n]$ the bidegree of f.

Let $\varphi : \mathbf{T}^2 \longrightarrow \mathbf{T}^2$ be a homeomorphism and write $\varphi = (\varphi_1, \varphi_2)$, where $\varphi_i : \mathbf{T}^2 \longrightarrow \mathbf{T}$ are continuous. Then the degree matrix of φ is the 2×2 matrix

$$D(\varphi) = \begin{pmatrix} D(\varphi_1) \\ D(\varphi_2) \end{pmatrix}.$$

It is easy to see that if φ and ψ are two homeomorphisms on $\mathbf{T}^2$, then $D(\varphi \circ \psi) = D(\varphi)D(\psi)$. Thus clearly $D(\varphi) \in GL(2,\ \mathbf{Z})$. A homeomorphism φ of a topological space X is said to be ergodic with respect to a Borel probability measure μ on X if whenever E is a Borel measurable set such that $\varphi(E) \subseteq E$, then $\mu(E) = 0$ or 1. One says that φ is uniquely ergodic if it has a unique invariant Borel probability measure. One says that φ is minimal if whenever E is a closed φ-invariant subset of X, then E is empty or all of X. Equivalently, for each $x \in X$ the orbit $\{x,\ \varphi(x),\ \varphi(\varphi(x)),\ \dots\}$ is dense in X. Hence it follows that if $f : X \longrightarrow C$ is continuous and X is connected, and if $f \circ \varphi = f$, then f is constant. Two homeomorphism φ_1 and φ_2 are said to be topologically conjugate if there exists a homeomorphisms h such that $h \circ \varphi_1 = \varphi_2 \circ h$.

Let φ be a homeomorphism of a topological space X. Consider the sets

$$G_0(\varphi) = \{\lambda \in C : \lambda \text{ is an eigenvalue of } \varphi\} \subseteq \mathbf{T},$$

$$G_1(\varphi) = \{f \in C(X) : f \circ \varphi = \lambda f \text{ for some } \lambda \in G_0(\varphi), \text{ and } |f| = 1\},$$

$$\vdots$$

$$G_j(\varphi) = \{g \in C(X) : g \circ \varphi = fg \text{ for some } f \in G_{j-1}(\varphi), \text{ and } |g| = 1\}, \text{ for } j \geqq 1.$$

Their union $\cup_{j \geqq 0} G_j(\varphi)$, is known as the set of quasi-eigenfunctions of φ. The homeomorphism φ is said to have topologically quasi-discrete spectrum if the C^*-algebra generated by all its quasi-eigenfunctions is $C(X)$. It is easy to see that the property of possessing topological quasi-discrete spectrum is invariant under topological conjugation.

2. **A Furstenberg Transformation Without Quasi-Discrete Spectrum**. This section will be devoted mainly to proving the following theorem.

THEOREM 2.1. *There exists a minimal homeomorphism φ of $\mathbf{T}^2$ of the form*

$$\varphi(x,\ y) = (e^{2\pi i\theta}x,\ e^{2\pi i r(x)}xy),$$

for suitable irrational number θ, and $r : \mathbf{T} \longrightarrow \mathbf{R}$ continuous, such that φ does not have topologically quasi-discrete spectrum. Furthermore, φ is uniquely ergodic so that the associated crossed product $A(\varphi)$ has a unique tracial state.

Hence, it follows from this theorem that φ is not topologically conjugate to the Anzai transformation $\varphi_\theta(x,\ y) = (e^{2\pi i\theta}x,\ xy)$ or to its inverse.

To prove this theorem we shall need three lemmas. The proof of the following lemma may be found in [4] (p. 135).

LEMMA 2.2. *Let φ be a minimal homeomorphism of a compact metric space X. Let $f \in C(X)$. Then the following conditions are equivalent.*

(1) *$f = g \circ \varphi - g$, for some $g \in C(X)$,*

(2) *$\{\sum_{k=0}^{n} f \circ \varphi^{(k)}\}_{n \geqq 1}$ is a uniformly bounded sequence of functions on X.*

LEMMA 2.3. *([3], p. 585; [2], p. 18) There exists $\lambda = e^{2\pi i\theta}$, where θ is irrational, and a continuous function $r : \mathbf{T} \longrightarrow \mathbf{R}$ such that the equation*

$$F(\lambda x) - F(x) = r(x),\ (x \in \mathbf{T})$$

has a real $L^2(\mathbf{T})$-solution F which is not equal to a continuous function almost everywhere (with respect to Lebesgue measure). Consequently, this equation has no $C(\mathbf{T})$-solution.

PROOF. (The following construction is due to Furstenberg). Let $\nu_1 = 1$ and recursively define $\nu_{k+1} = 2^{\nu_k} + \nu_k + 1$. Then $\theta = \sum_{k=1}^{\infty} 2^{-\nu_k}$ is an irrational number. Let $n_k = 2^{\nu_k}$ for $k \geqq 1$, so that one easily checks the inequality

$$0 < n_k\theta - [n_k\theta] \leqq 2^{-n_k},\ (k \geqq 1)$$

where $[t]$ denotes the greatest integer less than or equal to t. Letting $n_{-k} = -n_k$, $(k \geqq 1)$, we set

$$r(t) = \sum_{k \neq 0} \frac{1}{|k|}(e^{2\pi i n_k\theta} - 1)e^{2\pi i n_k t},\ t \in \mathbf{R},$$

where the series converges uniformly since

$$|e^{2\pi i n_k\theta} - 1| = |e^{2\pi i(n_k\theta - [n_k\theta])} - 1| \leqq |e^{2\pi i 2^{-n_k}} - 1| \leqq 2\pi 2^{-n_k},\ k \geqq 1,$$

so that r is a continuous function.

Now let

$$F(t) = \sum_{k \neq 0} \frac{1}{|k|} e^{2\pi i n_k t},\ t \in \mathbf{R},$$

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so that $F \in L^2(\mathbf{T})$. It is then easy to check that

$$F(t+\theta) - F(t) = r(t), \ \text{(a.e. } t \in \mathbf{R}).$$

Now if F is equal almost everwhere to a continuous function g, then by Fejer's theorem the arithmetic (Cesaro) means of the partial sums of the Fourier series converge uniformly to g. But it is easy to check that they fail to converge at $t = 0$, since $\sum_{k \neq 0} 1/|k| = \infty$. Hence the result.

To prove the last part of the lemma, assume that $f \in C(\mathbf{T})$ and $f(\lambda x) - f(x) = r(x)$, for almost every $x \in \mathbf{T}$. Then upon subtracting we have $(F - f)(\lambda x) = (F - f)(x)$, (a.e.). However, since $x \longmapsto \lambda x$ is ergodic, as θ is irrational, it follows that $F - f$ is constant (a.e.), and so F is equal almost everywhere to a continuous function, a contradiction to what we just proved. □

LEMMA 2.4. *Let $\lambda = e^{2\pi i\theta}$ and r be as in the preceding lemma, and write $h(x) = \lambda x$. Then for any real number α the sequence of functions*

$$\sum_{k=0}^{n} (r + \alpha) \circ h^{(k)} = (n+1)\alpha + \sum_{k=0}^{n} r \circ h^{(k)},$$

for $n \geqq 1$, is not uniformly bounded. (Here, $h^{(k)} = h \circ h \circ \ldots \circ h$, k times).

PROOF. Fix $\alpha \in \mathbf{R}$. Assume that the sequence of functions in the statement of the lemma is uniformly bounded. Since h is minimal, θ being irrational, Lemma 2.2 gives us a continuous function g on $\mathbf{T}$ such that $g \circ h - g = r + \alpha$. Now Lemma 2.3 has that $r = F \circ h - F$, where $F \in L^2(\mathbf{T})$ and F is not equal to a continuous function (a.e.). Thus, $f \circ h - f = \alpha$, where $f = g - F \in L^2(\mathbf{T})$. By induction, we obtain $f \circ h^{(n)} - f = n\alpha$, (a.e.) for all $n \geqq 1$. Now $\|f \circ h^{(n)}\|_2 = \|f\|_2$, ($L^2$-norms) by the Lebesgue invariance of $x \longmapsto \lambda^n x$. Hence

$$n|\alpha| \leqq \|f \circ h^{(n)}\|_2 + \|f\|_2 = 2\|f\|_2 < \infty,$$

for all $n \geqq 1$. Therefore, $\alpha = 0$ and substituting this back into the above formula we obtain $g \circ h - g = r$, where $g \in C(\mathbf{T})$, which contradicts the second part of Lemma 2.3. □

PROOF OF THEOREM 2.1. With $\lambda = e^{2\pi i\theta}$ and r as in Lemma 2.3, consider the transformation of $\mathbf{T}^2$ defined by

$$\varphi(x, y) = (e^{2\pi i\theta}x, e^{2\pi i r(x)}xy).$$

It is not hard to see that $\{\lambda^k : k \in \mathbf{Z}\}$ are all the eigenvalues of φ. By the minimality criterion (cf. [6] (p. 84), or [8] (1.1.4, for details)), φ is minimal if and only if for any non-zero integer n the equation

$$M(\lambda x) = (e^{2\pi i r(x)}x)^n M(x), \ (x \in \mathbf{T}),$$

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has no continuous solution $M : \mathbf{T} \longrightarrow \mathbf{T}$. Since $n \neq 0$, the degrees of both sides are not equal, so equality cannot hold.

It is easy to see that for a minimal homeomorphism φ of a connected metric space X, if $f \circ \varphi = \lambda f$ and $g \circ \varphi = \lambda g$ where $\lambda \in \mathbf{T}$ and $g \neq 0$, then f is a scalar multiple of g. Thus it follows that the only eigenfunctions of φ (of modulus 1) are

$$G_1(\varphi) = \{au^k : k \in \mathbf{Z},\ |a| = 1\},$$

where $u(x,\ y) = x$.

Since the C^*-algebra generated by u is not all of $C(\mathbf{T}^2)$, to show that φ does not have topological quasi-discrete spectrum it will suffice to check that there is no $g \in C(\mathbf{T}^2)$ with $|g| = 1$ satisfying $g \circ \varphi = au^k g$, where $|a| = 1$ and k is a non-zero interger. (If $k = 0$, then g is just an eigenfunction). This shows that $G_2(\varphi) = G_1(\varphi)$ and so $\cup_{j \geqq 0} G_j(\varphi) = G_1(\varphi)$ which does not generate $C(\mathbf{T}^2)$ as a C^*-algebra.

So assume that for some $k \neq 0$ there is a solution $g \in C(\mathbf{T}^2)$ such that $g \circ \varphi = au^k g$ and $|g| = 1$. Writing $g(x,\ y) = x^m y^n e^{2\pi i S(x,y)}$, for some $S : \mathbf{T}^2 \longrightarrow \mathbf{R}$ continuous and some integers $m,\ n$, the equation $g \circ \varphi = au^k g$ implies, upon looking at the x-degrees of both sides, that $n = k$ so that the equation reduces to

$$e^{2\pi i\{S(\varphi(x,y)) - S(x,y) + kr(x)\}} = a\lambda^{-m}.$$

Since the right hand side is constant, we have

$$S(\varphi(x,\ y)) - S(x,\ y) + kr(x) = c,$$

a real constant. By induction this becomes

$$\frac{S(\varphi^{(p)}(x,\ y)) - S(x,\ y)}{-k} = r(x) + r(\lambda x) + \ldots + r(\lambda^{p-1}x) + p(-c/k),$$

for all $p \geqq 1$. But the left side is a uniformly bounded sequence of functions, so the right side contradicts Lemma 2.4. This proves that φ does not have topologically quasi-discrete spectrum.

Now it remains to prove that φ is uniquely ergodic. Since φ has the form

$$\varphi(x,\ y) = (e^{2\pi i\theta}x,\ e^{2\pi i r(x)}xy),$$

we may apply a result of Furstenberg (cf. [6], p. 17, Theorem 3) so that it suffices to show that φ is ergodic with respect to Lebesgue product measure $m \times m$ on $\mathbf{T}^2$. This means that if E is a Borel subset of $\mathbf{T}^2$ which is φ-invariant, then E has Lebesgue measure 0 or 1. To show, in turn, that φ is ergodic it suffices to show that the equation $G(\lambda x) = (e^{2\pi i r(x)}x)^n G(x)$, (a.e. on $\mathbf{T}$), for any $n \neq 0$, has no measurable solution $G : \mathbf{T} \longrightarrow \mathbf{T}$ (cf. [6], ergodicity criterion on pp. 84f). So let us assume that such a G exists, so that $G \in L^2(\mathbf{T})$. By Lemma 2.3 we have $r(x) = F(\lambda x) - F(x)$, where F is measurable. Thus the above equation becomes

$$G(\lambda x)e^{-2\pi i n F(\lambda x)} = x^n G(x) e^{-2\pi i n F(x)},$$

or

$$(*) \quad f(\lambda x) = x^n f(x),$$

where $f(x) = G(x)e^{-2\pi i n F(x)}$ is measureable with $|f| = 1$ (a.e.). So now it remains to check that the equation $(*)$ has no such solution. Assume it has a solution $f \in L^2(\mathbf{T})$ so that it can be represented by its Fourier series (which is L^2-convergent), say

$$f(x) = \sum_{k=-\infty}^{\infty} a_k x^k.$$

Substituting this into $(*)$ we obtain

$$\sum_k a_k \lambda^k x^k = \sum_k a_k x^{k+n},$$

so that $a_k \lambda^k = a_{k-n}$ or $|a_k| = |a_{k-n}|$, for $k \in \mathbf{Z}$. But since $\sum |a_k|^2 < \infty$ and $n \neq 0$, we necessarily have $a_k = 0$ for all k. Thus $f = 0$ (a.e.), a contradiction to $|f| = 1$. □

In a similar manner one can show that for every non-zero interger n the (irrational) Furstenberg transformation

$$(x, y) \longmapsto (e^{2\pi i \theta} x, e^{2\pi i r(x)} x^n y)$$

does not have topologically quasi-discrete spectrum and hence cannot be topologically conjugate to the Anzai transformation $(x, y) \longmapsto (e^{2\pi i \theta} x, x^n y)$, nor to its inverse, where θ is as in Theorem 2.1. The above suggests that (in the notation of Theorem 2.1) the C^*-algebra $A(\varphi)$ is not isomorphic to $A(\varphi_\theta)$. However, we do not know how to prove this, for none of the invariants we know so far distinguish these algebras.

The proof of the unique ergodicity in the previous theorem can readily be generalized in the following manner. Let us say that a continuous function $S : \mathbf{T} \longrightarrow \mathbf{R}$ can be "split" with respect to $\mu \in \mathbf{T}$ if it can be written as $S(x) = F(\mu x) - F(x) + c$, (a.e.) for some measurable real-valued function F on $\mathbf{T}$ and some real constant c.

PROPOSITION 2.5. *Suppose that*

$$\psi(x, y) = (\mu x, e^{2\pi i S(x)} x^m y),$$

where μ is irrational, m is a non-zero integer, and S is continuous and can be split with respect to μ. Then the associated crossed product $A(\psi)$ has a unique tracial state.

By the minimality criterion, $A(\psi)$ is a simple C^*-algebra if μ is irrational and $m \neq 0$ (we do not need the split condition for the simplicity). This follows on applying a theorem of Power (cf. [7]).

EXAMPLE. A simple lemma due to L. Baggett (cf. [1], Lemma 2.1) shows that (in the notation of the above proposition) if S is absolutely continuous with derivative S' in L^2, and if θ is "badly approximable" (i.e., $\exists\delta > 0$ such that $n|e^{2\pi in\theta} - 1| \geqq \delta$ for all n, in contrast to Furstenberg's θ), then S can be split with respect to $e^{2\pi i\theta}$ and hence by the above proposition $A(\psi)$ has a unique tracial state.

QUESTION 2.6. If we drop the assumption that S can be split with respect to μ in the above proposition, can we still conclude that ψ is uniquely ergodic?

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