CONVERGENCE OF ITERATES AND DIFFERENTIABILITY

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1. Introduction

Given a function T mapping a Hausdorff locally convex topological vector space Φ into Φ and a point ϕ_0 of Φ, convergence of the elementary filter associated with the sequence of iterates determined by T and ϕ_0 is investigated. Sufficient conditions that the limit $\bar{\phi}$, if it exists, be a fixed point of T are given and in the case Φ is the space of real valued functions of a real variable differentiability of the limit function $\bar{\phi}$ is investigated.

It should be noted that it is not assumed that T is continuous and/or linear.

2. Notations, Conventions, and Preliminaires

For each $x \in E$, a nonempty set, let (Ξ_x, ξ_x) be a topological vector space and denote by (Φ, ξ) the topological vector space $\Pi_{x \in E} \Xi_x$ equipped with the product topology. Then (Φ, ξ) is a Hausdorff locally convex topological vector space (l.c.t.v.s.) if and only if (Ξ_x, ξ_x) is a Hausdorff l.c.t.v.s. for each $x \in E$.

The dual of Φ equipped with a topology ρ will be denoted by Φ'_ρ and it is assumed throughout that ρ is compatible with the duality between Φ and Φ'.

The filter base of ρ-open neighborhoods of a point ϕ' of Φ' will be denoted by $N_\rho(\phi')$ and in particular $N_\rho(\theta')$ denotes the filter base of ρ-open neighborhoods of the identity element of Φ'. The associated filter of neighborhoods will be denoted by $\mathscr{G}N_\rho(\phi')$.

If T is any function mapping Φ into Φ and ϕ_0 is any element of Φ, the following sequence of iterates can be formed

$$\phi_0, T\phi_0, T^2\phi_0, \cdots, T^n\phi_0, \cdots,$$

and will be denoted by $\{\phi_0\}_T$. $T^n\phi_0$ is defined as follows: $T^0\phi_0 = \phi_0$, $T^n\phi_0 = T(T^{n-1}\phi_0)$ for $n = 1, 2, \cdots$. Finally, $\mathscr{F}\{\phi_0\}_T$ will denote the elementary filter associated with the sequence $\{\phi_0\}_T$.

For convenience, the definition of convergence on a filter is given in terms of uniform spaces.

DEFINTION 2.1. [1; 287] Let Y be a uniform space, $\mathscr{F}$ a filter of subsets of Y, and $\mathscr{G}$ a filter of subsets of X^Y, where X denotes either Φ or the real numbers $R^\#$, regarded as uniform spaces. *$\mathscr{G}$ converges to f_0* on $\mathscr{F}$, denoted $\mathscr{G} \to f_0$ on $\mathscr{F}$, if for every member of U of the uniformity on X there is a D in $\mathscr{G}$ such that for each f in D there is an F_f in $\mathscr{F}$ with the property that $(f(s), f_0(s))$ is in U for all s in F_f.

Since X^Y is also a uniform space, the roles of Y and X^Y may be interchanged and the convergence of $\mathscr{F}$ to s_0 on $\mathscr{G}$ is defined in an exactly analogous manner.

In particular, for a sequence $\{\phi_n\} \subseteq \Phi$ and the filter base $N_\rho(\phi')$ in Φ', $N_\rho(\phi') \to \phi'_0$ on $\{\phi_n\}$ means: *for every $\varepsilon > 0$ there is a D in $N_\rho(\phi')$ such that for each ψ' in D there is a number $N > 0$ with the property that $|\langle\phi_n, \psi'\rangle - \langle\phi_0, \psi\rangle| < \varepsilon$ for all $n \geqq N$.*

Similarly, $\{\phi_n\} \to \phi_0$ on $N_\rho(\phi')$ means: *for every $\varepsilon > 0$ there is a number $N > 0$ such that for each $n \geqq N$ there is a $D \in N_\rho(\phi')$ with the property that $|\langle\phi_n, \psi'\rangle - \langle\phi_0, \psi'\rangle| < \varepsilon$ for all $\psi' \in D$.*

Two topologies are of particular interest: $\rho = \sigma(\Phi', \Phi)$ and if Φ is semi-reflexive, $\rho = \beta(\Phi', \Phi)$, the so called norm-topology.

3. Principal Results

Given a function T, a point ϕ_0 of Φ, and a topology ρ for Φ', the results obtained below center on the relationship between the two filters $\mathscr{F}\{\phi_0\}_T$ and $\mathscr{G}N_\rho(\theta')$.

LEMMA 3.1. *Let $\mathscr{F}$ be any filter of subsets of Φ and let $\mathscr{G} = \mathscr{G}N_\rho(\theta')$. If $\mathscr{F} \to \bar{\phi}$ on $\mathscr{G}$ then $\bar{\phi}$ is ρ-continuous.*

PROOF. See [2] Theorem 2.

The fundamental relationship between the two filters $\mathscr{G}N_\rho(\theta')$ and $\mathscr{F}\{\phi_0\}_T$ is given by the theorem which follows.

THEOREM 3.2. *If Ξ_x is Hausdorff for each $x \in E$, and*

i) $\mathscr{F}\{\phi_0\}_T$ is $\sigma(\Phi, \Phi')$-Cauchy,

ii) ρ is any topology compatible with the duality between Φ and Φ',

iii) $\mathscr{G}N_\rho(\theta') \to \theta'$ on $\mathscr{F}\{\phi_0\}_T$,

then there is an element $\bar{\phi}$ of Φ''_ρ for which $\mathscr{F}\{\phi_0\}_T \to \bar{\phi}$ on $\mathscr{G}N_\rho(\theta')$.

PROOF. Since $\mathscr{F}\{\phi_0\}_T$ is $\sigma(\Phi, \Phi')$-Cauchy, $\mathscr{F}\{\phi_0\}_T$ determines an element $\bar{\phi}$ of Φ'^* for which $\mathscr{F}\{\phi_0\}_T \to \bar{\phi}$ weakly.

Let F be any member of $\mathscr{F}\{\phi_0\}_T$.

Since Φ' is Hausdorff, $\mathscr{G}N_\rho(\theta') \to \theta'$ and so

$$\langle\psi, \mathscr{G}N_\rho(\theta')\rangle \to \langle\psi, \theta'\rangle$$

for every $\psi' \in F$, indeed, convergence holds for all $\psi \in \Phi'^* \supseteq \Phi$.

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If $D \in \mathscr{G}N_\rho(\theta')$ and since $\mathscr{F}\{\phi_0\}_T$ is $\sigma(\Phi,\Phi')$-Cauchy, then $\langle \mathscr{F}\{\phi_0\}_T, \psi' \rangle \to \langle \bar{\phi}, \psi' \rangle$ for every $\psi' \in D$.

By the duality theorem of Brace [1; 292], $\mathscr{F}\{\phi_0\}_T \to \bar{\phi}$ on $\mathscr{G}N_\rho(\theta')$. By the lemma, $\bar{\phi} \in \Phi''_\rho$.

The third hypothesis of the above theorem may be replaced by the following equivalent expression which is given in terms of the iterates of T:

iii′) *For every* $\varepsilon > 0$ *there is a* ρ*-neighborhood* D *of* $\theta' \in \Phi'$ *such that for each* $\phi' \in D$ *there exists an* $N = N(\phi') > 0$ *for which* $|\langle T^k\phi_0, \phi' \rangle| < \varepsilon$ *whenever* $k \geqq N$.

The following example shows that the weak limit of $\mathscr{F}\{\phi_0\}_T$ need not be a fixed point of T.

EXAMPLE 3.3. Let $\Phi = \Phi' = L_2[0,2\pi]$. Define the continuous nonlinear function T on a subset of Φ as follows:

$$T\phi(x) = \phi(x)\cos x + [1-\phi^2(x)]^{1/2}\sin x$$

whenever $\sup_{x\in[0,2\pi]}|\phi(x)| \leqq 1$.

The function T has the unique fixed point $\bar{\phi}(x) = \cos(x/2)$ and on the other hand $\mathscr{F}\{0\}_T$ converges weakly to zero.

Note too that $\mathscr{F}\{\phi_0\}_T$ need not be $\sigma(\Phi,\Phi')$-Cauchy for every choice of ϕ_0; for example, this is the case for $\mathscr{F}\{1\}_T$.

With additional hypotheses, sufficient conditions for the weak limit $\bar{\phi}$ of $\mathscr{F}\{\phi_0\}_T$ to be a fixed point of T are given below.

Let C denote the space of continuous mappings of Φ into Φ.

THEOREM 3.4. *If* Ξ_x *is Hausdorff for each* $x \in E$ *and*

i) *there exists* $\phi_0 \in \Phi$ *for which* $\mathscr{F}\{\phi_0\}_T$ *is* $\sigma(\Phi,\Phi')$*-Cauchy (and so determines an element* $\bar{\phi}$ *of* Φ'^**),*

ii) *there exists a filter* $\mathscr{T}$ *of subsets of* C *for which:*

a) $\mathscr{T} \to T$

b) $\mathscr{T}(\psi) \to T\psi$ *for all* ψ *in at least one* F *in* $\mathscr{F}\{\phi_0\}_T$*, and*

c) $K(\mathscr{F}) \to K\bar{\phi}$ *for all* K *in at least one* $D \in \mathscr{T}$*, where* $\mathscr{F} = \mathscr{F}\{\phi_0\}_T$*,*

iii) $\mathscr{F}\{\phi_0\}_T \to \bar{\phi}$,

then

i) $\mathscr{T} \to T$ *on* $\mathscr{F}$ *if and only if* $\mathscr{F} \to \bar{\phi}$ *on* $\mathscr{T}$*, and*

ii) $T\bar{\phi} = \bar{\phi}$.

PROOF. (i) follows by direct application of the duality theorem of Brace [1; 292].

The duality theorem also insures that

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$$\lim_{F \in \mathscr{F}} \lim_{D \in \mathscr{T}} D(F) = \lim_{D \in \mathscr{T}} \lim_{F \in \mathscr{F}} D(F).$$

However, the left hand side is equal to $\bar{\phi}$ and the right hand side is equal to $T\bar{\phi}$.

Turning now to the special case $E = \Xi_x = R^{\#}$, it is of interest to give conditions insuring that the weak limit $\bar{\phi}$ of $\mathscr{F}\{\phi_0\}_T$ be r-times continuously differentiable at a point x_0 of E (respectively, on all of E), briefly $\bar{\phi} \in C^r(x_0)$ (respectively, $C^r(E)$).

To this end, let $\{f_n\}$ be a sequence of elements of ϕ satisfying

i) $f_n \in C^r(x_0)$ *for each* $n = 1,2,\cdots$,

ii) $\{f_n\}$ *converges pointwise to* $f(x)$ *on some neighborhood of* x_0.

Define $q_n^{[s]}(x)$ and $q^{[s]}(x)$ as follows:

$$q_n^{[s]}(x) = \begin{cases} [f_n^{(s-1)}(x) - f_n^{(s-1)}(x_0)]/(x - x_0) & x \neq x_0 \\ f_n^{(s)}(x_0) & x = x_0, \end{cases}$$

and

$$q^{[s]}(x) = \begin{cases} [f^{(s-1)}(x) - f^{(s-1)}(x_0)]/(x - x_0) & x \neq x_0 \\ \lim_{n \to \infty} f_n^{(s)}(x_0) \text{ when it exists} & x = x_0 \end{cases}$$

where $1 \leqq s \leqq r$ and $n = 1,2,\cdots$.

Lemma 3.5. *Given a sequence* $\{f_n(x)\}$ *satisfying (i) and (ii) above, the following are equivalent:*

i) $f^{(s)}(x_0)$ *exists and* $\lim_{n \to \infty} f_n^{(s)}(x_0) = f^{(s)}(x_0)$,

ii) *the sequence* $\{f_n^{(s)}(x_0)\}$ *of constant functions converges to* $q^{[s]}(x)$ *on* $\mathscr{G}N(x_0)$, *the filter generated by the neighborhoods of* x_0,

iii) $\{q_n^{[s]}(x)\} \to q^{[s]}(x)$ *on* $\mathscr{G}N(x_0)$ *for every* s, $1 \leqq s \leqq r$.

Proof. The case $s = 1$ is Theorem 1.2 in [3].

For s such that $1 < s \leqq r$, apply Theorem 1.2 [3] to the sequence $\{f^{(s-1)}(x_0)\}$.

Theorem 3.6. *If for some* $x_0 \in E$

i) $T(C^r(x_0)) \subseteq C^r(x_0)$,

ii) $\mathscr{F}\{\phi_0\}_T$ *converges pointwise to* $\bar{\phi}$ *on some neighborhood of* x_0,

iii) $\phi_0 \in C^r(x_0)$,

iv) $\{(T^n\phi_0)^{(r)}(x_0) \mid n = 1,2,\cdots\}$ *converges on* $\mathscr{G}N(x_0)$, *then* $\bar{\phi} \in C^r(x_0)$.

Proof. If $\phi_0 \in C^r(x_0)$, the above hynotheses insure that the sequence $\{T^n\phi_0\}$ satisfies the hypotheses of the lemma.

Corollary 3.7. *If the above hypotheses are satisfied at each* $x \in E$, *then* $\bar{\phi} \in C^r(E)$.

It should be remarked that while the above hypotheses are sufficient to insure $\bar{\phi} \in C^r(E)$ and the existence of the pointwise limit $\bar{\phi}$ the conditions are also necessary by virtue of the three equivalent statements of the lemma.

4. Applications

If Φ is semireflexive, $\beta(\Phi', \Phi) = \tau(\Phi', \Phi)$.

THEOREM 4.1. *If the hypotheses of Theorem* 3.2 *are satisfied for the choice* $\rho = \beta(\Phi', \Phi)$, *then* $\bar{\phi} \in \Phi''_\beta$.

PROOF. The conditions of Theorem 3.2 imply that $\bar{\phi}$ is ρ-continuous [3; 2]

LEMMA 4.2. *If* Ξ_x *is Hausdroff for each* $x \in E$, *and*

i) Φ *is semireflexive,*

ii) $\mathscr{F}\{\phi_0\}_T$ *is a bounded* $\sigma(\Phi, \Phi')$*-Cauchy filter, then there exists an element* $\bar{\phi}$ *of* Φ'^* *such that*

i) $\mathscr{F}\{\phi_0\}_T \to \bar{\phi}$ *on* $\mathscr{G}N_\sigma(\theta')$, *and*

ii) $\mathscr{G}N_\sigma(\theta') \to \theta'$ *on* $\mathscr{F}\{\phi_0\}_T$.

PROOF. The assumption that Φ is semireflexive is equivalent to the statement: every bounded $\sigma(\Phi, \Phi')$-Cauchy filter in Φ converges to a point of Φ'^* on $\mathscr{G}N_\sigma(\theta')$, by virtue of Theorem 5 [2; 240]. In particular therefore, the second hypothesis insures that this is the case for $\mathscr{F}\{\phi_0\}_T$ and hence the first conclusion

Finally since Φ' is Hausdorff, $\mathscr{G}N_\sigma(\theta') \to \theta'$ and thus in particular $\mathscr{G}N_\sigma(\theta') \to \theta'$ on $\mathscr{F}\{\phi_0\}_T$.

THEOREM 4.3. *If* $\rho = \sigma(\Phi', \Phi)$, *the hypotheses of Theorem* 3.2 *and Lemma* 4.2 *are satisfied, then* $\bar{\phi} \in \Phi$.

References

[1] J. W. Brace, 'Convergence on Filters and Simple Equicontinuity', *Ill. J. Math.*, IX, 1965, 286–296.

[2] R. M. Nielsen, 'Completion in the Bidual', *J. Australian Math. Soc.*, VIII (2) 1968, 238–241.

[3] R. M. Nielsen, 'Pointwise Convergence and Differentiability', *J. Math. Anal. and Appl.*, XXV 1969, 337–340.

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