Concurrency of lines joining vertices of a triangle to opposite vertices of triangles on its sides.

By A. G. Burgess, M.A., F.R.S.E.

(*Read 13th February 1914. Received 10th April 1914*).

(1). Let ABC be the given triangle; A′BC, B′CA, C′BA triangles described externally on its sides, and let the angles of these triangles be $A'BC = \mu_1$, $A'CB = \nu_1$, $B'AC = \lambda_2$, $B'CA = \nu_2$, $C'AB = \lambda_3$, $C'BA = \mu_3$ (Fig. 1).

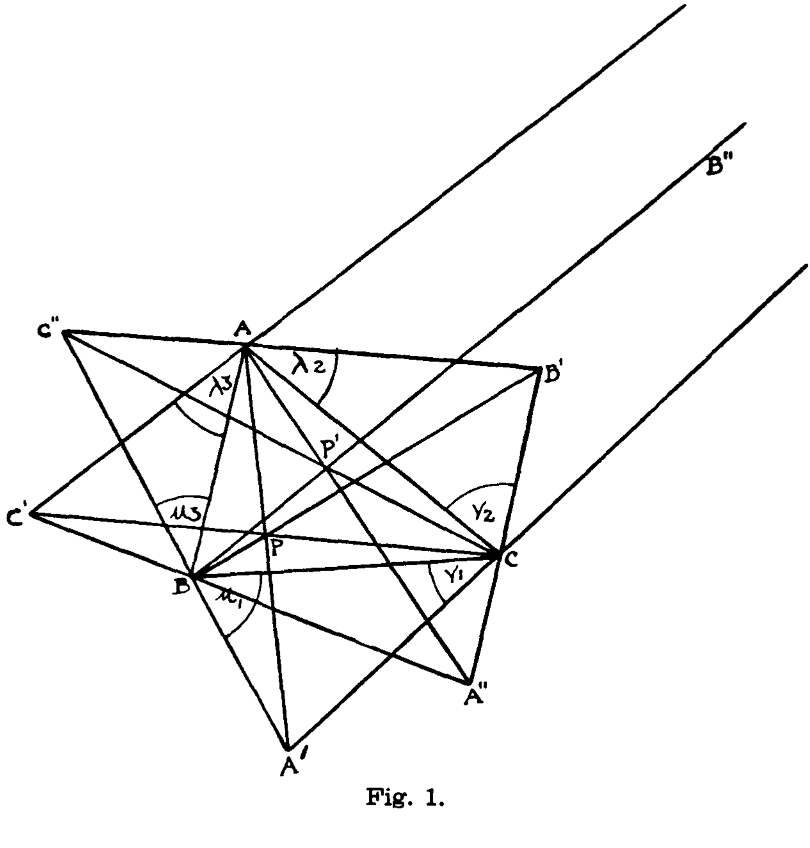

Fig. 1.

https://doi.org/10.1017/S0013091500035045 Published online by Cambridge University Press

Using normal coordinates, then for AA′, $\dfrac{\beta}{\gamma}=\dfrac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}$,

∴ the lines AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}\cdot\frac{\sin\nu_2/\sin(C+\nu_2)}{\sin\lambda_2/\sin(A+\lambda_2)}\cdot\frac{\sin\lambda_3/\sin(A+\lambda_3)}{\sin\mu_3/\sin(B+\mu_3)}=1.$$

e.g. if $\mu_1=\lambda_2=90-C$, $\nu_1=\lambda_3=90-B$, $\nu_2=\mu_3=90-A$,

$$\frac{\beta}{\gamma}=\frac{\sin(90-C)/\sin(B+90-C)}{\sin(90-B)/\sin(C+90-B)}=\frac{\sec B}{\sec C},$$

and the lines are concurrent at the orthocentre $\sec A:\sec B:\sec C$. (The points A′, B′, C′ are the points in which the altitudes intersect the circumcircle of $\triangle ABC$).

(2). If the angles at the vertices be interchanged, *e.g.* μ_1 and μ_3, then AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_3/\sin(B+\mu_3)}{\sin\nu_2/\sin(C+\nu_2)}\cdot\frac{\sin\nu_1/\sin(C+\nu_1)}{\sin\lambda_3/\sin(A+\lambda_3)}\cdot\frac{\sin\lambda_2/\sin(A+\lambda_2)}{\sin\mu_1/\sin(B+\mu_1)}=1,$$

which is the same condition as in (1). Hence if the lines are concurrent, they are concurrent when the angles are interchanged, *e.g.* interchanging the angles of example in (1), so that $A'BC=90-A$, $A'CB=90-A$, etc., then

$$\frac{\beta}{\gamma}=\frac{\sin(90-A)/\sin(B+90-A)}{\sin(90-A)/\sin(C+90-A)}=\frac{\cos(C-A)}{\cos(A-B)},$$

and the point of concurrence is $\cos(B-C):\cos(C-A):\cos(A-B)$, A′, B′, C′ being the images of the circumcentre in the sides.

(3). If C′BA′, etc., are straight lines, so that $\mu_1+\mu_3=180-B$, etc., and the lines be concurrent, then if the angles be interchanged as in (2), the two points of concurrence are isogonals.

e.g. if $\mu_1=\nu_1=A$, $\nu_2=\lambda_2=B$, $\lambda_3=\mu_3=C$, the point of concurrence is the Lemoine point $\sin A:\sin B:\sin C$, the points A′, B′, C′ being the points in which perpendiculars from the circumcentre O on the sides of the triangle meet the circles round OBC, OCA, OAB.

Interchanging the angles, so that $\mu_1=\lambda_2=C$, $\nu_1=\lambda_3=B$, $\nu_2=\mu_3=A$, the point of concurrence is the centroid

$$\operatorname{cosec}A:\operatorname{cosec}B:\operatorname{cosec}C,$$

the isogonal to the Lemoine point. The lines AA′, BB′, CC′ are bisected by the sides of the triangle.

(4). If $\mu_1 = \nu_1 = A \pm \theta$, $\nu_2 = \lambda_2 = B \pm \theta$, $\mu_3 = \lambda_3 = C \pm \theta$,

then $$\frac{\beta}{\gamma} = \frac{\sin(A \pm \theta)/\sin(B + A \pm \theta)}{\sin(A \pm \theta)/\sin(C + A \pm \theta)} = \frac{\sin(B \mp \theta)}{\sin(C \mp \theta)},$$

and the lines are concurrent at the point

$$\sin(A \mp \theta) : \sin(B \mp \theta) : \sin(C \mp \theta).$$

This point lies on the line joining the Lemoine point $\sin A : \sin B : \sin C$ to the circumcentre $\cos A : \cos B : \cos C$, for $\Sigma \sin(A \mp \theta)\sin(B - C) = 0$.

e.g. If $\theta = 0$, the point is the Lemoine point, and if $\theta = 60°$, the points are $\sin(A \mp 60) : \sin(B \mp 60) : \sin(C \mp 60)$, the two Isodynamic points.

(5). If all six angles $= \theta$, the point of concurrence is $\operatorname{cosec}(A + \theta) : \operatorname{cosec}(B + \theta) : \operatorname{cosec}(C + \theta)$, the isogonal to the point derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

e.g. If $\theta = 60°$, the point is the Inner Isogonic point, the isogonal of one of the Isodynamic points.

(6). If $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, the lines AA′, BB′, CC′ are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A + \lambda)} : \frac{\sin\mu}{\sin(B + \mu)} : \frac{\sin\nu}{\sin(C + \nu)}.$$

e.g. If $\lambda = 90 - A$, $\mu = 90 - B$, $\nu = 90 - C$, the point of concurrence is the circumcentre $\cos A : \cos B : \cos C$, A′, B′, C′ lying on the circumcircle of $\triangle ABC$.

If $\lambda = 90 - C$, $\mu = 90 - A$, $\nu = 90 - B$, the point is

$$\frac{\cos C}{\cos(C - A)} : \frac{\cos A}{\cos(A - B)} : \frac{\cos B}{\cos(B - C)}.$$

(7). If P be a point within $\triangle ABC$, and if AP, BP, CP be produced to meet the circumcircles of $\triangle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB in A′, B′, C′, $\mu_1 = \mu_3$, $\nu_1 = \nu_2$, $\lambda_2 = \lambda_3$, and if these angles be λ, μ, ν, $\lambda + \mu + \nu = 180°$, and the $\angle$s BPC, CPA, APB are $180 - \lambda$, $180 - \mu$, $180 - \nu$, for $A'BC = A'PC = C'PA = C'BA$, etc., and $BA'C = CPB = CAB = \lambda$, etc., and $BPC = 180 - \lambda$, etc.

e.g. If $\lambda = 90 - \dfrac{A}{2}$, $\mu = 90 - \dfrac{B}{2}$, $\nu = 90 - \dfrac{C}{2}$, the point P is the incentre $1 : 1 : 1$, and the $\angle$s at P are $90 + \dfrac{A}{2}$, etc., A′, B′, C′ being the three excentres of $\triangle ABC$.

If $\lambda = A$, $\mu = B$, $\nu = C$, the point is the orthocentre.

(8). If the lines C′B, B′C be produced to meet in A″, A′C, C′A in B″, A′B, B′A in C″, and AA′ BB′, CC′ be concurrent, then AA″, BB″, CC″ are concurrent, for the condition for concurrence is

$$\frac{\sin(B+\mu_3)/\sin\mu_3}{\sin(C+\nu_2)/\sin\nu_2}\cdot\frac{\sin(C+\nu_1)/\sin\nu_1}{\sin(A+\lambda_3)/\sin\lambda_3}\cdot\frac{\sin(A+\lambda_2)/\sin\lambda_2}{\sin(B+\mu_2)/\sin\mu_1}=1,$$

the same condition as in (1); the point of concurrence is the isogonal to the point got by interchanging the angles μ_1 and μ_3, etc., as in (2).

e.g. If as in (1) $\mu_1 = \lambda_2 = 90 - C$, $\nu_1 = \lambda_3 = 90 - B$, $\nu_2 = \mu_3 = 90 - A$, the point of concurrence is $\sec(B - C) : \sec(C - A) : \sec(A - B)$, the isogonal to the point $\cos(B - C) : \cos(C - A) : \cos(A - B)$ derived in (2) by interchanging the angles.

(9). If A′B′C′ be regarded as the original triangle with triangles C′BA′, A′CB′, B′AC′ described on the sides, then if the lines AA′, BB′, CC′ are concurrent, B′B″, C′C″, A′A″ are concurrent, the points A″, B″, C″ being the points derived as in (8).

(10). If $\mu_1 = \mu_3 = \mu$, etc., the point of concurrence of AA″, BB″, CC″ is

$$\frac{\sin(A+\lambda)}{\sin\lambda}:\frac{\sin(B+\mu)}{\sin\mu}:\frac{\sin(C+\nu)}{\sin\nu},$$

and the lines AA″, BB″, CC″ are the isogonals of AA′, BB′, CC′.

e.g. If $\lambda = C$, $\mu = A$, $\nu = B$, AA′, BB′, CC′ meet in
$\dfrac{\sin C}{\sin B} : \dfrac{\sin A}{\sin C} : \dfrac{\sin B}{\sin A}$ and AA″, BB″, CC″ in $\dfrac{\sin B}{\sin C} : \dfrac{\sin C}{\sin A} : \dfrac{\sin A}{\sin B}$, the two Brocard points.

If $\lambda = \mu = \nu = \theta$, AA′, BB′, CC′ meet in $\operatorname{cosec}(A + \theta) : \operatorname{cosec}(B + \theta) : \operatorname{cosec}(C + \theta)$ and AA″, BB″, CC″ in $\sin(A + \theta) : \sin(B + \theta) : \sin(C + \theta)$, the same point as was derived in (4) by making $\mu_1 = \nu_1 = A - \theta$, etc.

(11). If $\lambda + \mu + \nu = 180°$, and AA″, BB″, CC″ meet in P′, BP′CA″, etc., are concyclic, for $BP'C = A + \lambda$, and $BA''C = 180 - A - \lambda$.

Again, $BAB' = A + \lambda$ and $BA''B' = BA''C = 180 - A - \lambda$, ∴ BAB′A″ and the other five sets of corresponding points are concyclic.

e.g. If $\lambda = \mu = \nu = 60°$, P′ is the isogonal of the Inner Isogonic point and $BP'C = A + 60$.

(12). If the triangles BA′C, CB′A, AC′B be described internally on the sides, AA′, BB′, CC′ are concurrent if

$$\frac{\sin\mu_1/\sin(B-\mu_1)}{\sin\nu_1/\sin(C-\nu_1)}\cdot\frac{\sin\nu_2/\sin(C-\nu_2)}{\sin\lambda_2/\sin(A-\lambda_2)}\cdot\frac{\sin\lambda_3/\sin(A-\lambda_3)}{\sin\mu_3/\sin(B-\mu_3)}=1.$$

As in the case of the external triangles, the lines are concurrent if μ_1 and μ_3, etc., be interchanged, and if $\mu_1=\mu_3=\mu$, $\nu_1=\nu_2=\nu$, $\lambda_2=\lambda_3=\lambda$, the lines are always concurrent, the point being

$$\frac{\sin\lambda}{\sin(A-\lambda)}:\frac{\sin\mu}{\sin(B-\mu)}:\frac{\sin\nu}{\sin(C-\nu)}.$$

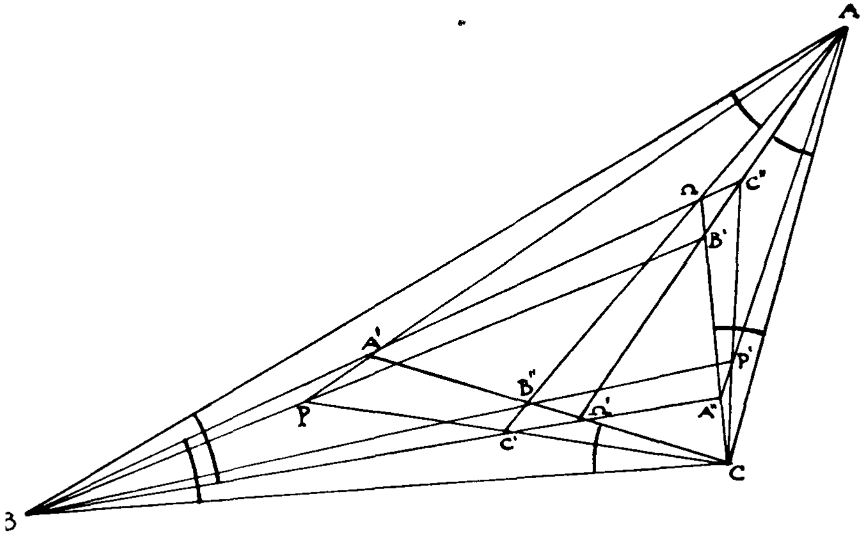

Fig. 2.

e.g. If $\mu_1=\mu_3=90-A$, $\nu_1=\nu_2=90-B$, $\lambda_2=\lambda_3=90-C$, the point of concurrence is $\dfrac{\cos C}{\cos B}:\dfrac{\cos A}{\cos C}:\dfrac{\cos B}{\cos A}$.

If $\lambda+\mu+\nu=180$, then as in the case of the external triangles, A′BPC, etc., are concyclic points, the angles between the lines AP, BP, CP being λ, μ, ν.

e.g. If $\lambda=\mu=\nu=60°$, the point is

$$\operatorname{cosec}(A-60):\operatorname{cosec}(B-60):\operatorname{cosec}(C-60),$$

the Outer Isogonic point.

(13). If CB′, BC′ be produced to meet in A″, etc., then if AA′, BB′, CC′ be concurrent, so are AA″, BB″, CC″, and the point of concurrence is the isogonal of the point got by interchanging the angles μ_1 and μ_3, etc., just as in the case of the external mentioned in (8). A′A″, B′B″, C′C″ are also concurrent as in (9).

e.g. If $\mu_1=\mu_3=\frac{B}{3}$, $\nu_1=\nu_2=\frac{C}{3}$, $\lambda_2=\lambda_3=\frac{A}{3}$, AA′, BB′, CC′ meet in the point $\sec\frac{A}{3}:\sec\frac{B}{3}:\sec\frac{C}{3}$ and AA″, BB″, CC″ in the point $\cos\frac{A}{3}:\cos\frac{B}{3}:\cos\frac{C}{3}$. The first point with reference to triangle A′B′C′ can be shown to be

$$\frac{\sin(60-A/3)}{\sin(60+A/3)}:\frac{\sin(60-B/3)}{\sin(60+B/3)}:\frac{\sin(60-C/3)}{\sin(60+C/3)}.$$

e.g. If all six angles equal the Brocard angle ω (Fig. 2), AA′, BB′, CC′ meet in the point $\operatorname{cosec}(A-\omega):\operatorname{cosec}(B-\omega):\operatorname{cosec}(C-\omega)$, and AA″, BB″, CC″ in the isogonal point $\sin(A-\omega):\sin(B-\omega):\sin C-\omega)$, a point lying on the line joining the Lemoine point to the circumcentre. The points A′, B′, C′ and A″, B″, C″ form the Brocard triangles.

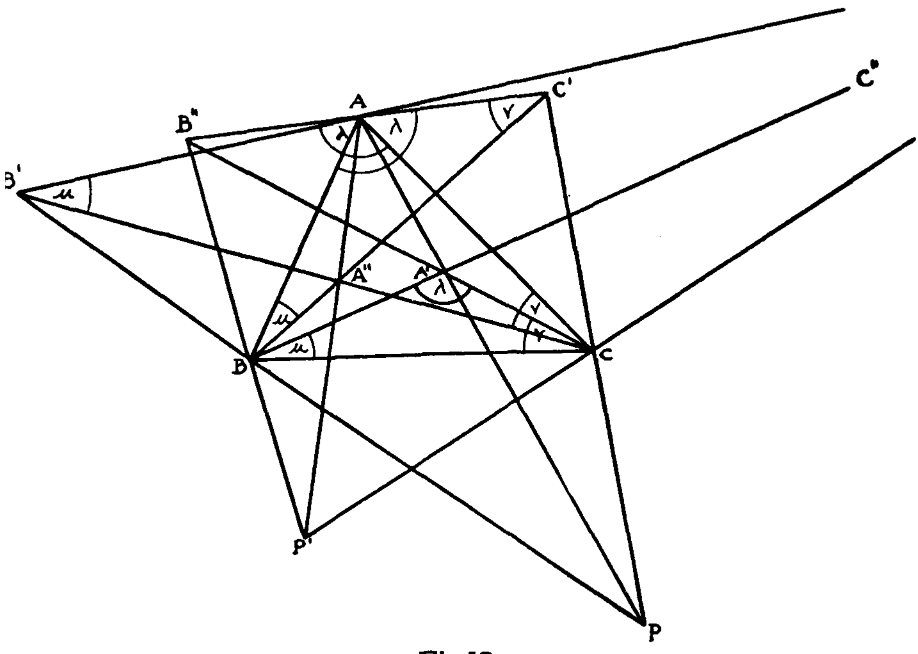

Fig. 13.

(14). If (Fig. 3) $\mu_1 = \mu_3 = \mu$, $\nu_1 = \nu_2 = \nu$, $\lambda_2 = \lambda_3 = \lambda$, and if $\lambda + \mu + \nu = 180°$, and AA′, BB′, CC′ meet in P, and AA″, BB″, CC″ in P′, then as in case of the external triangles mentioned in (10), the following sets of four points are concyclic, BP′CA″, CP′AB″, AP′BC″, CAC′A″, CBC′B″, ABA′B″, ACA′C″, BCB′C″, BAB′A″.

e.g. If $\lambda = \mu = \nu = 60$, P′ is $\sin(A - 60) : \sin(B - 60) : \sin(C - 60)$, one of the Isodynamic points; see (4).