ON MEASURABILITY FOR VECTOR-VALUED FUNCTIONS

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1. Introduction. The problem of developing an abstract integration theory has been approached from many angles **(6)**. The most general of several definitions based on the norm topology is that of Birkhoff **(1)**, which includes the well-known and widely used Bochner integral **(3)**.

The original Birkhoff formulation was based on the notion of unconditional convergence of an infinite series of elements in a Banach space and the closed convex extensions of certain approximating sums. Later simplifications by Birkhoff **(2)**, Kunisawa **(8)**, and others, showed that it was possible to bypass the convex extension and closure, and also to avoid the use of unconditional convergence. In connection with two of these simplifications **(8; 7)** certain classes of "measurable" functions were defined which included the functions measurable in the sense of Bochner as subclasses. Kunisawa, in particular, defines integrability in terms of "*-measurable" functions and shows that every Birkhoff-integrable function is *-measurable.

A classical characterization of Lebesgue-measurable functions is that they are "almost" continuous, in the sense of the well-known Lusin theorem **(10**, p. 72**)**. The Bourbaki **(4**, p. 180**)**, definition of measurability for a function f, defined on a locally compact set E with values in an arbitrary topological space, is based on the Lusin property in that f is called *measurable* if it is continuous on each of a collection of compact sets with total measure approximating that of E. It turns out that when the range space is a Banach space this definition is equivalent to Bochner measurability **(4**, Theorem 3, p. 189**)**. There are, however, fairly simple vector-valued functions which are not measurable according to the Bourbaki definition, or in the Bochner sense of being the limit almost everywhere of a sequence of step functions, or according to any definition that implies the Lusin property. A classical example **(5**, p. 166**)** involves the space M of bounded real functions $f(t)$ on $0 \leqslant t \leqslant 1$ with

$$||f(t)|| = \sup_{0 \leqslant t \leqslant 1} |f(t)|.$$

Let $x(s) = f_s(t)$, where $f_s(t) = 0$ on $0 \leqslant t \leqslant s$, and $f_s(t) = 1$ on $s < t \leqslant 1$. Thus $x(s)$ is defined on $0 \leqslant s \leqslant 1$ and is everywhere discontinuous there. Nevertheless, this function is measurable in the Kunisawa sense and is in fact Riemann (Graves) integrable.

In this note we show that if one considers functions defined on a separable, complete, metric space Ω, with a measure defined on a class of subsets of Ω

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which include the Borel sets, then the *-measurable, or Birkhoff-integrable, functions studied in **(8)** satisfy a generalized form of the Lusin condition in that they are "almost" Riemann-integrable on $E \subset \Omega$, $m(E) < \infty$, in terms of a natural extension of the Riemann (Graves) integral to closed compact sets. A definition of measurability based on this idea of a weakened Lusin condition has been previously discussed **(11)** for functions defined on the real line with values in a Banach space.

2. Notation. Ω denotes a separable, complete, metric space, with metric ρ. m^* is a metric outer measure constructed from a sequential covering class $\mathfrak{C}$, consisting of open sets, which covers Ω, and is such that $\Omega = \bigcup_{i=1}^{\infty} A_n$, where $A_n \in \mathfrak{C}$ and $m^*(A_n) < \infty$. $m(E)$ is the measure function determined by m^* and defined for the class $\mathfrak{M}$ of sets E in Ω which are measurable with respect to m^*. In particular, all Borel sets are included in $\mathfrak{M}$ **(9**, p. 101**)**. Also, m is a *regular* measure **(9**, p. 111**)** in the sense that for any measurable set E and any given $\epsilon > 0$ there exists an open set $G \supset E$ such that $m(G - E) < \epsilon$ and a closed set $P \subset E$ such that $m(E - P) < \epsilon$.

F, F', C will denote compact subsets of Ω, and P will denote a closed subset of Ω, not necessarily compact. In addition, X will denote an arbitrary linear normed complete space, or Banach space, $x(s)$ and $y(s)$ functions defined on a subset of points s in Ω and valued in X, and $f(s)$ a real-valued function defined on a subset of Ω.

For any subset $E \subset \Omega$ we define the diameter of E as follows:

$$d(E) = \sup\{\rho(s, s') \mid s \in E, s' \in E\}.$$

3. An extension of the Riemann (Graves) integral definition. Let F be a compact set in Ω, and let S_i, $i = 1, \ldots, n$, be a set of closed spheres, with positive finite diameters, which cover F. Let S denote the ordered collection $S_1, S_2, \ldots, S_n$.

DEFINITION 3.1. *A subdivision of F, generated by a covering S, is the finite collection of subsets of F constructed as follows:*

$$F_1 = S_1 \cap F, \quad F_2 = S_2 \cap (F - F_1), \quad \ldots, \quad F_n = S_n \cap (F - F_{n-1}).$$

We denote a subdivision by Δ, and the maximum $d(S_i)$, $S_i \in S$, by $N(\Delta)$, which we call the *norm* of Δ.

DEFINITION 3.2. *Let $x(s)$ be defined and bounded on a compact set F. Let Δ be a subdivision of F. If X contains an element L such that for every $\eta > 0$ there exists $\delta > 0$ with*

$$\left\| \sum_{i=1}^{n} x(\xi_i)m(F_i) - L \right\| < \eta$$

for every subdivision with $N(\Delta) < \delta$ and every choice of ξ_i in F_i $(i = 1, \ldots, n)$,

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then L is the Riemann (Graves) integral, or RG-*integral of $x(s)$ over F and we write*

$$(\mathrm{RG})\int_F x(s)\, ds = L.$$

We choose a compact set F as the domain of $x(s)$ over which we define our integral because this ensures that F will be closed and totally bounded, i.e., for any $\epsilon > 0$, there exists a *finite* covering of F by open spheres of radius ϵ.

It is not difficult to see that when F is a closed interval of the real line the RG-integral is equivalent to the original Graves formulation **(5)**.

The following necessary and sufficient condition for RG-integrability parallels that of Graves and is easily proved by standard arguments.

THEOREM 3.1. *Let $x(s)$ be defined and bounded on a compact set F. A necessary and sufficient condition for the existence of the* RG-*integral of $x(s)$ over F is that, given $\eta > 0$, there exist $\delta > 0$ such that for any two subdivisions Δ_1, Δ_2 of F with $N(\Delta_1) < \delta$, $N(\Delta_2) < \delta$,*

$$||\textstyle\sum_{\Delta_1} x(\xi_{1i})\, m(F_{1i}) - \sum_{\Delta_2} x(\xi_{2i})\, m(F_{2i})|| < \eta,$$

where ξ_{1i}, ξ_{2i} may be any points on F_{1i}, F_{2i}, respectively.

The elementary properties of the RG-integral listed in the next theorem are obvious extensions of the corresponding properties of the Riemann integral and follow directly from the definition and Theorem 3.1.

THEOREM 3.2. (i) *If x, y, and f are* RG-*integrable over F and $||x(s)|| \leqslant f(s)$ for s on F, then*

$$(\mathrm{RG})\int_F (x + y)\, ds = (\mathrm{RG})\int_F x\, ds + (\mathrm{RG})\int_F y\, ds$$

and

$$||(\mathrm{RG})\int_F x\, ds\, || \leqslant (\mathrm{RG})\int_F f\, ds.$$

(ii) *If $F \cap F' = 0$ and if $x(s)$ is* RG-*integrable over F and F', then it is integrable over $F \cup F'$ and*

$$(\mathrm{RG})\int_{F\cup F'} x\, ds = (\mathrm{RG})\int_F x\, ds + (\mathrm{RG})\int_{F'} x\, ds.$$

(iii) *If $x_n(s)$ $(n = 1, 2, \ldots)$ is* RG-*integrable over F for each n and if $\{x_n(s)\}$ converges uniformly to $x(s)$ in F, then $x(s)$ is* RG-*integrable over F and*

$$(\mathrm{RG})\int_F x_n(s)\, ds \to (\mathrm{RG})\int_F x(s)\, ds.$$

A further property of the RG-integral is contained in the following theorem.

THEOREM 3.3. *Let F be a compact set contained in Ω, and C be any closed, hence compact, subset of F. If $x(s)$ is* RG-*integrable over F, then it is* RG-*integrable over C.*

Proof. We shall make use of the following result, which has been proved in a variety of ways by Birkhoff **(1)**, Jeffery **(7)**, and others.

LEMMA. *Let $e_1, e_2, \ldots, e_n$ be any n disjoint measurable sets on a measurable subset $E \subset \Omega$, $m(E) < \infty$, ξ_i an arbitrary point of e_i, and $T = \sum x(\xi_i)\, m(e_i)$, where $x(s)$ is a bounded function on E with values in X. Let*

$$e_{i1}, e_{i2}, \ldots, e_{ik}.$$

be a partition of e_i into disjoint measurable sets, and ξ_{ij}, ξ'_{ij} any points on e_{ij}. Then

$$\left\| \sum_{i=1}^{n} \sum_{j=1}^{k_i} x(\xi_{ij})\, m(e_{ij}) - T \right\| \leqslant \sup \left\| \sum_{i=1}^{n} \{x(\xi_i) - x(\xi'_i)\} m(e_i) \right\|$$

for ξ_i, ξ_i' any two points on e_i.

Let $\eta > 0$ be given. Then there exists $\delta > 0$ such that for any two subdivisions Δ, Δ' of F into sets F_i, F_j', with $N(\Delta) < \delta$, $N(\Delta') < \delta$, we have

$$\|\textstyle\sum_{\Delta} x(\xi_i)\, m(F_i) - \sum_{\Delta'} x(\xi_j')\, m(F_j')\| < \tfrac{1}{3}\eta.$$

We note in particular that

$$\|\textstyle\sum_{\Delta} \{x(\xi_i) - x(\xi_i')\}\, m(F_i)\| < \tfrac{1}{3}\eta.$$

for all ξ_i, ξ_i' in F_i.

Now let Δ_1, Δ_2 be any two subdivisions of C into subsets C_{1i}, C_{2j} $(i = 1, \ldots, n; j = 1, \ldots, m)$ generated by coverings S_1, S_2 of C, consisting of closed spheres, with $N(\Delta_1) < \delta$, $N(\Delta_2) < \delta$.

Finally, let Δ_3, Δ_4 be two subdivisions of F into subsets F_{3i}, F_{4j} $(i = 1, \ldots, p; j = 1, \ldots, q)$ generated by coverings S_3 and S_4 of F, where S_3 consists of the ordered set of spheres in S_1 followed by the spheres in the covering used to construct the subdivision Δ. Similarly, S_4 consists of the ordered set of spheres in S_2 followed by the spheres used in constructing Δ'. Clearly, $N(\Delta_2) < \delta$, $N(\Delta_4) < \delta$.

We observe that $C_{1i} \subseteq F_{3i}$ $(i = 1, \ldots, n)$ and $C_{2j} \subseteq F_{4j}$ $(j = 1, \ldots, m)$.

Set $F_{3i} - C_{1i} = Q_{3i}$ $(i = 1, \ldots, n)$ and $F_{3i} = Q_{3i}$ for $i > n$. Also, set $F_{4j} - C_{2j} = Q_{4j}$ $(j = 1, \ldots, m)$ and $F_{4j} = Q_{4j}$ for $j > m$, and set

$$Q = \bigcup_{i=1}^{p} Q_{3i} = \bigcup_{j=1}^{q} Q_{4j}.$$

Let $\pi(Q)$ denote a partition of Q into a finite number of disjoint measurable sets Q_i by intersecting the sets Q_{3i}, Q_{4j} in all possible ways.

Now if ξ_i, ξ_{1i}, ξ_{2j}, ξ'_{3i}, ξ'_{4j} are arbitrarily chosen points of Q_i, C_{1i}, C_{2j}, F_{3i}, F_{4j}, respectively, we have

$$\begin{aligned}
&\|\textstyle\sum_{\Delta_1} x(\xi_{1i})\, m(C_{1i}) - \sum_{\Delta_2} x(\xi_{2j})\, m(C_{2j})\| \\
&\quad \leqslant \|(\textstyle\sum_{\Delta_1} x(\xi_{1i})\, m(C_{1i}) + \sum_{\pi(Q)} x(\xi_i)\, m(Q_i)) - \sum_{\Delta_3} x(\xi'_{3i})\, m(F_{3i})\| \\
&\qquad + \|\textstyle\sum_{\Delta_3} x(\xi'_{3i})\, m(\mathrm{F}_{3i}) - \sum_{\Delta_4} x(\xi'_{4j})\, m(F_{4j})\| \\
&\qquad + \|\textstyle\sum_{\Delta_4} x(\xi'_{4j})\, m(F_{4j}) - (\sum_{\Delta_2} x(\xi_{2j})\, m(C_{2j}) + \sum_{\pi(Q)} x(\xi_i)\, m(Q_i))\| \\
&\quad < \tfrac{1}{3}\eta + \tfrac{1}{3}\eta + \tfrac{1}{3}\eta = \eta.
\end{aligned}$$

Hence $x(s)$ is RG-integrable over C.

4. Measurability and Riemann-integrability. In his development of the Birkhoff integral, Kunisawa **(8)** considers a function $x(s)$ defined on the class $\mathfrak{B} = \{E\}$ of measurable subsets of a space Ω with $m(\Omega) < \infty$. A decomposition of a measurable set E into a *finite* number of mutually disjoint measurable sets is denoted by $\pi = \{E_i \mid i = 1, 2, \ldots, n\}$. For any two partitions π_1 and π_2, $\pi_1 \leqslant \pi_2$ means that every set of π_2 is contained in some set of π_1.

If $x(s)$ is a function defined on Ω, and π is any partition of Ω, then, by definition,

$$\pi(x, E) = \sum_{i=1}^{n} x(E \cap E_i)\ m(E \cap E_i),$$

where $x(E) = \{x(s) | s \in E\}$, i.e., $\pi(x, E)$ denotes the set of all sums of the form $\sum x(s_i)\, m(E \cap E_i)$, where $s_i \in E_i$.

The following definitions and lemma summarize, for convenience, the main features of *-measurable functions **(8**, pp. 525–526**)**.

DEFINITION 4.1. *$x(s)$ is called basic on Ω if there exists for every $\epsilon > 0$ a partition π_ϵ of Ω such that $d(\pi_\epsilon(x, \Omega)) < \epsilon$.*

LEMMA 4.1. *A necessary and sufficient condition for $x(s)$ to be basic on Ω is the existence of an X-valued set-function $I(x, E)$ defined on $\mathfrak{B}$ with the property that for every $\epsilon > 0$ there exists a partition π_ϵ such that $\pi_\epsilon < \pi$ implies*

$$||\pi(x, E) - I(x, E)|| < \epsilon$$

for any $E \in \mathfrak{B}$.

DEFINITION 4.2. *$I(x, E)$ is the (Birkhoff) integral of the basic function $x(s)$ over E.*

DEFINITION 4.3. *A sequence of functions $\{x_n(s) | n = 1, \ldots\}$ on Ω is approximately convergent to an X-valued function $x(s)$ if there exists for every $\epsilon > 0$ a sequence $\{E_n | n = 1, \ldots\}$ of measurable sets such that*

$$\{s | \, ||x_n(s) - x(s)|| \geqslant \epsilon\} \subseteq E_n, \qquad n = 1, 2, \ldots,$$

and $m(E_n) \to 0$.

DEFINITION 4.4. *A function $x(s)$ is *-measurable if there exists a sequence of basic functions converging approximately to $x(s)$.*

It turns out that $x(s)$ is Birkhoff-integrable if it is *-measurable, and if a sequence $\{x_n(s) \mid n = 1, 2, \ldots\}$ of basic functions converging approximately to $x(s)$ can be taken in such a way that

$$\lim_{n\to\infty} I(x_n, E)$$

exists strongly for each $E \in \mathfrak{B}$. Conversely, every Birkhoff-integrable function has this property.

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THEOREM 4.1. *If a measurable set E is contained in Ω, $m(E) < \infty$, then given $\epsilon > 0$ there exists a subset F in E such that F is compact and*

$$m(F) > m(E) - \epsilon.$$

Proof. Because the measure is regular there exists a closed subset P contained in E with $m(E - P) < \frac{1}{2}\epsilon$.

Since Ω is separable let $\{s_n\}$ be a sequence of points dense in P and write S_n^k for the closed sphere of radius $1/k$ with centre s_n. Set

$$F_t^k = \bigcup_{n=1}^{t} (S_n^k \cap P).$$

Now given any $\epsilon > 0$ there exists a positive integer n_1 such that

$$m(F_{n_1}^1) > m(P) - \tfrac{1}{4}\epsilon,$$

because if m^* is any regular outer measure and $\{A_n\}$ is an expanding sequence of sets, then

$$m^*\Big(\lim_n A_n\Big) = \lim_n m^*(A_n)$$

and our outer measure, constructed as described in § 2, is regular (**9**, p. 109). Similarly there exists a positive integer n_2 such that

$$m(F_{n_2}^2) > m(P) - \tfrac{1}{8}\epsilon.$$

Hence

$$m\Big(\bigcap_{i=1}^{2} F_{n_i}^i\Big) > m(P) - \tfrac{1}{2}\epsilon$$

because it is easy to verify that if $m(E - A) < \epsilon_1$, $m(E - B) < \epsilon_2$, then $m(E - (A \cap B)) < \epsilon_1 + \epsilon_2$.

In general, we define t_k $(k = 1, 2, \ldots)$ as the smallest positive integer such that

$$m\Big(\bigcap_{i=1}^{k} F_{t_i}^i\Big) > m(P) - \tfrac{1}{2}\epsilon.$$

Let

$$F = \bigcap_{i=1}^{\infty} F_{t_i}^i.$$

Then F is compact. First of all, F is closed, being the intersection of closed sets. Also, for each k, F is covered by a finite set of spheres of the form S_n^k. Hence given any infinite set K of points in F there clearly exists a sequence of nested closed sets of the form

$$F_1 = S_{n(1)}^1 \cap F, \qquad F_2 = S_{n(2)}^2 \cap F_1, \qquad F_k = S_{n(k)}^k \cap F_{k-1}, \ldots,$$

each containing an infinite number of points of K and with $d(F_k) \to 0$.

This leads at once to a Cauchy sequence of points $\{s_i\}$, $s_i \in F_i \cap K$, having a limit point s_0, which is the unique point contained in every F_i by

Cantor's theorem (**9**, p. 68), and hence $s_0 \in F$. Then s_0 is a limit point of K and so F has the Bolzano-Weierstrass property.

It is clear, from the way F is obtained, that $m(F) \geqslant m(P) - \frac{1}{2}\epsilon$. Hence F is a compact set, contained in E, with $m(F) > m(E) - \epsilon$.

DEFINITION 4.5 *$x(s)$ is almost Riemann-integrable over a measurable set $E \subset \Omega$ if, given $\epsilon > 0$ there exists a compact set $F \subset E$, $m(F) > m(E) - \epsilon$, and such that $x(s)$ is* RG-*integrable over F.*

THEOREM 4.2. *If $x(s)$ is basic on a set $E \subset \Omega$, $m(E) < \infty$, then it is almost Riemann-integrable over E.*

Proof. Let $\epsilon > 0$ be given, and let $\{\epsilon_n\}$ be a sequence of positive numbers with $\epsilon_n = \epsilon/2^n$.

For each ϵ_n take a partition

$$\pi_{\epsilon_n} = \{E_{ni}| \ i = 1, \ldots, j\}$$

satisfying the condition of Lemma 4.1.

For each E_{ni} of π_{ϵ_n} let F_{ni} be a compact set contained in E_{ni} with

$$m(E_{ni} - F_{ni}) < \epsilon_n/2^i$$

and let $F_n = \bigcup_{i=1}^{j} F_{ni}$. Then $m(E - F_n) < \epsilon_n$. Let $F = \bigcap_1^\infty F_n$. F is therefore closed and compact and $m(E - F) < \epsilon$. We shall show that $x(s)$ is RG-integrable over F.

Given any $\eta > 0$ choose a positive integer m such that $\epsilon_m < \eta$ and take $\pi = \pi_{\epsilon_m}$.

Let $F'_{mi} = F_{mi} \cap F$ and let d be the minimum distance apart for the closed sets F'_{mi}.

We observe that for the partition π' composed of the sets F'_{mi} and $E_{mi} - F'_{mi}$, $i = 1, \ldots, j(m)$, we have

$$||\pi'(x, E) - I(x, E)|| < \eta$$

for every $E \in B$. In particular we have

$$||\pi'(x, F) - I(x, F)|| < \eta.$$

Then if we take $\delta = d$ we see that any two subdivisions Δ_1, Δ_2, with $N(\Delta_1) < \delta$, $N(\Delta_2) < \delta$, are equivalent to two partitions π_1 and π_2 with $\pi_1 \geqslant \pi'$ and $\pi_2 > \pi'$ and we have

$$\begin{aligned} ||\textstyle\sum_{\Delta_1} x(\xi_{1i})\, m(F_{1i}) - \sum_{\Delta_2} x(\xi_{2i})\, m(F_{2i})|| &= ||\pi_1(x, F) - \pi_2(x, F)|| \\ &\leqslant ||\pi_1(x, F) - I(x, F)|| + ||\pi_2(x, F) - I(x, F)|| < 2\eta. \end{aligned}$$

Thus $x(s)$ is almost Riemann-integrable over E.

THEOREM 4.3. *Let $x(s)$ be defined on a measurable set $E \subset \Omega$, with $m(E) < \infty$. Then a necessary and sufficient condition for $x(s)$ to be *-measurable on E is that $x(s)$ is almost Riemann-integrable over E.*

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Proof. If $x(s)$ is *-measurable there is a sequence of basic functions $\{x_n(s)\}$ satisfying the condition of Definition 4.3. Let any $\epsilon > 0$ be given. Then there exists a sub-sequence

$$\{x_{n_k}(s)\}, \qquad k = 1, 2, \ldots,$$

of $\{x_n(s)\}$ and a sequence of measurable sets $\{E_k\}$, $k = 1, 2, \ldots$, such that $m(E_k) < \epsilon/2^{k+1}$ and $||x_{n_k}(s) - x(s)|| < \epsilon/2^k$ in $E - E_k$. On each measurable set $E - E_k$ there exists a compact set F_k such that $m(E - F_k) < \epsilon/2^k$ and on which $x_{n_k}(s)$ is RG-integrable by Theorem 4.2. Let $F = \bigcap_{k=1}^{\infty} F_k$. F is closed and $m(E - F) < \epsilon$. Also $x_{n_k}(s)$ is RG-integrable on F, for every n_k, by Theorem 3.3.

Let us choose, and fix, an n' from among the n_k such that

$$||x_{n'}(s) - x(s)|| < \eta/m(F)$$

for all s in F.

Then, given any $\eta < 0$, there exists $\delta > 0$ such that for any two subdivisions Δ_1, Δ_2 with $N(\Delta_1) < \delta$, $N(\Delta_2) < \delta$, we have

$$\begin{aligned} ||\textstyle\sum_{\Delta_1} x(\xi_{1i})\, m(F_{1i}) - \sum_{\Delta_2} x(\xi_{2i})\, m(F_{2i})|| \\ \leqslant ||\textstyle\sum\Delta_1 \{x(\xi_{1i}) - x_{n'}(\xi'_{1i})\}\, m(F_{1i})|| \\ + ||\textstyle\sum_{\Delta_1} x_{n'}(\xi'_{1i})\, m(F_{1i}) - \sum\Delta_2\, x_{n'}(\xi'_{2i})\, m(F_{2i})|| \\ + ||\textstyle\sum_{\Delta_2} \{x_{n'}(\xi'_{2i}) - x(\xi_{2i})\}\, m(F_{2i})|| \\ < \frac{\eta}{m(F)} \cdot (mF) + \eta + \frac{\eta}{m(F)} \cdot m(F) = 3\eta, \end{aligned}$$

for any points ξ_{1i}, ξ'_{1i} on F_{1i}, and ξ_{2i}, ξ'_{2i} on F_{2i}.

Thus we see that $x(s)$ is RG-integrable over E and hence the stated condition is necessary.

Next, let ϵ_n be a sequence of positive numbers with $\epsilon_n \to 0$. If $x(s)$ is almost Riemann-integrable on E, there exists a compact set $F_n \subset E$ with $m(E - F_n) < \epsilon_n$ on which $x(s)$ is RG-integrable.

For each n, set

$$x_n(s) = \begin{cases} x(s) & \text{on } F_n, \\ 0 & \text{on } E - F_n. \end{cases}$$

For each n, $x_n(s)$ is basic on E, because for any $\eta > 0$ there exists $\delta > 0$ such that with $N(\Delta) < \delta$ we have

$$||\textstyle\sum_{\Delta} x(\xi_i)\, m(F_i) - \sum_{\Delta} x(\xi_i')\, m(F_i)|| < \eta.$$

Then taking the subsets of F_n under the subdivision Δ, plus the set $E - F_n$, we have a partition π_n of E such that $d(\pi_n(x, E)) < \eta$. Moreover, the sequence $\{x_n(s)\}$ converges approximately to $x(s)$ on E because given any $\epsilon > 0$ we can set $E_n = E - F_n$ and the sequence $\{E_n\}$ is such that

$$\{s|\ ||x_n(s) - x(s)|| \geqslant \epsilon\} \subseteq E_n, \qquad n = 1, 2, \ldots,$$

and $m(E_n) \to 0$. This proves the sufficiency of the given condition.

Thus we see that the *-measurable and Birkhoff-integrable functions defined on a subset E, $m(E) < \infty$, of a separable complete metric space, satisfy a modified Lusin condition in the sense of being almost Riemann-integrable over E. It is easy to verify that this modified Lusin condition coincides with the original Lusin condition in the case of a real-valued function defined on a Lebesgue-measurable set, of finite measure, on the real line.

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