NEW COMPLEX ANALYTIC METHODS IN THE THEORY OF MINIMAL SURFACES: A SURVEY

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Abstract

In this paper we survey recent developments in the classical theory of minimal surfaces in Euclidean spaces which have been obtained as applications of both classical and modern complex analytic methods; in particular, Oka theory, period dominating holomorphic sprays, gluing methods for holomorphic maps, and the Riemann–Hilbert boundary value problem. Emphasis is on results pertaining to the global theory of minimal surfaces, in particular, the Calabi–Yau problem, constructions of properly immersed and embedded minimal surfaces in $\mathbb{R}^n$ and in minimally convex domains of $\mathbb{R}^n$, results on the complex Gauss map, isotopies of conformal minimal immersions, and the analysis of the homotopy type of the space of all conformal minimal immersions from a given open Riemann surface.

1. Introduction

An immersed surface $M \to \mathbb{R}^n$ in the n-dimensional Euclidean space ($n \geq 3$) is said to be a *minimal surface* if it locally minimizes its area, meaning that sufficiently small pieces of the surface have the smallest area among all surfaces with the same boundary. Such surfaces were first studied by Euler in 1744 who showed that the only area minimizing surfaces of rotation are planes and catenoids. The subject was taken up by Lagrange in 1760 who studied the area functional and came up with the differential equation of minimal graphs in $\mathbb{R}^3$. It was discovered by Meusnier in 1776 that a surface is minimal if and only if its mean curvature vector field $\mathbf{H} \colon M \to \mathbb{R}^n$ vanishes identically. Plateau pointed out in 1873 that minimal surfaces appear naturally as soap films, and Douglas [39] and Radó [101] independently proved in 1932 that every Jordan curve in $\mathbb{R}^n$ spans a minimal surface.

The influence of complex analysis in the study of minimal surfaces was apparent already in the third quarter of the 19th century when Enneper and Weierstrass provided an analytic formula for representing any minimal surface in $\mathbb{R}^n$. The so-called *Enneper–Weierstrass representation formula* (2.11) relies on the fact that for an isometric immersion $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ from a Riemannian surface M (a smooth surface endowed with a Riemannian metric), the metric Laplacian of the immersion equals two times its mean curvature vector:

$$\Delta X = (\Delta X_1, \ldots, \Delta X_n) = 2\mathbf{H}.$$

Since the vanishing of the Laplacian depends only on the conformal class of the metric, it follows that a conformal (angle preserving) immersion $X \colon M \to \mathbb{R}^n$ from a Riemann surface M is minimal if and only if it is harmonic, $\Delta X = 0$; equivalently, the $(1, 0)$-derivative $\partial X = (\partial X_1, \ldots, \partial X_n)$ is a holomorphic $\mathbb{C}^n$-valued 1-form. Furthermore, X is conformal if and only if ∂X satisfies the null equation

$$(\partial X_1)^2 + (\partial X_2)^2 + \cdots + (\partial X_n)^2 = 0. \tag{1.1}$$

(See Section 2.4.) This reduces the construction of oriented conformal minimal surfaces $M \to \mathbb{R}^n$ to the construction of holomorphic maps from the Riemann surface M to the subvariety $\mathfrak{A}^{n-1}$ of $\mathbb{C}^n$ defined by the equation $z_1^2 + z_2^2 + \cdots + z_n^2 = 0$, the *null quadric* (see (2.4)). This is the basis for the Enneper–Weierstrass formula; see Theorem 2.4 and (2.15). The analogous formula applies to *null holomorphic immersions* $Z = (Z_1, \ldots, Z_n) \colon M \to \mathbb{C}^n$, that is, holomorphic immersions satisfying

$$(dZ_1)^2 + (dZ_2)^2 + \cdots + (dZ_n)^2 = 0.$$

Every conformal minimal immersion $M \to \mathbb{R}^n$ is locally (on any simply connected subset of M) the real part of a null holomorphic immersion into $\mathbb{C}^n$; conversely, the real and the imaginary part of any null holomorphic immersion $M \to \mathbb{C}^n$ are conformal minimal immersions $M \to \mathbb{R}^n$. See Section 2.7 for more details.

In the mid-20th century Osserman [98] renewed the interest in the theory of minimal surfaces by showing in particular that the Enneper–Weierstrass representation formula is very useful for the construction of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature. This was the true starting point for the study of the global theory of minimal surfaces by complex analytic methods. However, as late as in the 1980s the prevailing thought was that hyperbolic Riemann surfaces (that is, those carrying nonconstant negative subharmonic functions) play only a marginal role in the global theory of minimal surfaces. This belief was partially refuted by the pioneering works of Jorge and Xavier [75] from 1980, Nadirashvili [92] from 1996, and Morales [91] from 2003 which combined the Enneper–Weierstrass formula with the classical Runge approximation theorem for holomorphic functions.

Nevertheless, the true power and versatility of this approach was revealed only in the last few years by bringing into the picture some of the more powerful complex analytic methods originating in Oka theory (which amounts to holomorphic

approximation techniques combined with a nonlinear version of the $\bar{\partial}$-problem), and by adapting the classical Riemann–Hilbert boundary value problem to the constructions of conformal minimal surfaces in $\mathbb{R}^n$ and holomorphic null curves in $\mathbb{C}^n$. One of the key advantages of these stronger complex analytic methods over the classical ones is that they allow a more precise control of the placement of the whole surface in the space. This enabled the authors, often in collaboration with Drinovec Drnovšek and López, to construct minimal surfaces with interesting global properties and with complete control of the conformal structure on the surface. In other words, not only it is now possible to find minimal surfaces with interesting global properties and with prescribed topological type, one can also control their conformal (holomorphic) type, a major advance in the theory.

The goal of this article is to present these recent developments in a way that is accessible not only to researchers, but also to graduate students in both the field of minimal surfaces and in complex analysis. What transpires from our narrative is that these two fields are much more closely intertwined than believed up to now, with major influences going in both directions. On the one hand, complex analytic methods are a powerful tool in the classical minimal surface theory; on the other hand, many questions about minimal surfaces lead to analogous questions about complex curves. Several lines of thought have been pursued separately by researchers in these two fields without having been aware of the analogies and synergies. Other problems have been considered only in one field and overlooked in the other one, even though they are perfectly natural and interesting in both fields.

Let us consider an example. It has been known since the early 1960s that every open Riemann surface embeds properly holomorphically into $\mathbb{C}^3$ (see [52, Theorem 2.4.1]; this is a special case of the Bishop–Narasimhan–Remmert embedding theorem for Stein manifolds). However, the question of whether every such surface embeds as a smooth closed complex curve in the complex Euclidean plane $\mathbb{C}^2$ remains one of the most difficult open problems of complex analysis, known as the *Forster conjecture* [47] or the *Bell–Narasimhan conjecture* [28]. On the minimal surfaces side, there is the equally natural problem of determining the smallest dimension $d \geq 3$ for which every open Riemann surface embeds as a proper conformal minimal surface in $\mathbb{R}^d$. Since every complex curve in $\mathbb{C}^n$ is also a minimal surface in $\mathbb{R}^{2n}$ by Wirtinger [108], we have that $d \leq 6$. The following recent result says in particular that $d \leq 5$.

THEOREM 1.1. *Let M be an open Riemann surface.*

(a) *There is a proper conformal minimal immersion $M \to \mathbb{R}^3$. Moreover, proper immersions are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^3$ in the compact-open topology.*

(b) *There exists a proper conformal minimal immersion $M \to \mathbb{R}^4$ with simple double points, and such immersions are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^4$.*

(c) *There is a proper conformal minimal embedding $M \hookrightarrow \mathbb{R}^5$. Moreover, proper conformal minimal embeddings are dense in the space of all conformal minimal immersions $M \to \mathbb{R}^n$ for any $n \geq 5$.*

Part (a) is due to Alarcón and López [23, 24], while parts (b) and (c) were proved in 2016 by the authors and López [19]. A more precise result in this direction is Theorem 4.8 which provides conformal minimal immersions and embeddings with a proper projection to a coordinate 2-plane, thereby giving an optimal negative answer to both Schoen–Yau's and Sullivan's conjectures (see Section 4.2).

It is known that only a few open Riemann surfaces embed as proper minimal surfaces in $\mathbb{R}^3$ (see, for example, [33, 34, 80, 89, 90]), so the smallest embedding dimension for open Riemann surfaces as minimal surfaces satisfies $d \geq 4$. This leaves us with the question of whether $d = 4$ or $d = 5$. An affirmative answer to the Forster–Bell–Narasimhan conjecture would imply $d = 4$.

In the last decade, powerful new methods for constructing proper holomorphic embeddings of open Riemann surfaces into $\mathbb{C}^2$ have been developed by Wold and Forstnerič, using the technique of exposing boundary points and pushing the boundary of the surface in $\mathbb{C}^2$ to infinity by holomorphic automorphisms; see the recent survey in [52, Ch. 9]. For example, every circular domain in $\mathbb{C}$ embeds properly holomorphically into $\mathbb{C}^2$ [57]. These results, along with the absence of any conceptual obstructions, speak in favor of the Forster–Bell–Narasimhan conjecture. Since minimal surfaces are much more abundant than complex curves, the following conjecture has an even better chance of being true.

CONJECTURE 1.2. *Every open Riemann surface admits a proper conformal minimal embedding into* $\mathbb{R}^4$.

On the other hand, every open Riemann surface which is currently known to properly embed as a conformal minimal surface in $\mathbb{R}^4$ is also known to properly embed as a complex curve in $\mathbb{C}^2$, the main reason being that no automorphisms of $\mathbb{R}^n$ other than the rigid motions map minimal surfaces to minimal surfaces.

Another example where the analogies between the fields of complex analysis and minimal surfaces become even more apparent is the problem of constructing complete bounded minimal surfaces in $\mathbb{R}^n$ (the Calabi–Yau problem) and complete bounded complex submanifolds in $\mathbb{C}^n$ (Yang's problem). We describe this topic briefly, referring to Section 5.3 for a more complete presentation.

Recall that an immersion $X \colon M \to \mathbb{R}^n$ from an open manifold M is said to be *complete* if the pullback $g = X^*(ds^2)$ of the Euclidean metric ds^2 on $\mathbb{R}^n$ by the immersion X is a complete Riemannian metric on M. Equivalently, given any divergent path $\gamma \colon [0, 1) \to M$, meaning that the point $\gamma(t)$ leaves every compact subset of M as t approaches 1, the path $t \mapsto X(\gamma(t)) \in \mathbb{R}^n$ has infinite Euclidean length.

The *Calabi–Yau problem for minimal hypersurfaces* asks whether there exist complete bounded minimal hypersurfaces in $\mathbb{R}^n$ for $n \geq 3$. Calabi conjectured in 1965 that such hypersurfaces do not exist (see [76, page 170]). The first counterexample to Calabi's conjecture was given by Nadirashvili [92] who in 1996 constructed a complete bounded immersed minimal disc in $\mathbb{R}^3$. A plethora of results followed, extending Nadirashvili's construction to more general surfaces; see Section 5.3. However, with the techniques available at that time it was impossible to control the conformal

structure or the boundary behavior of the examples. The following considerably more precise result was proved in 2015 by the authors together with Drinovec Drnovšek and López (see [7, Theorem 1.1]).

THEOREM 1.3. *For every compact bordered Riemann surface M and integer $n \geq 3$ there is a continuous map $X \colon M \to \mathbb{R}^n$ whose restriction to $\mathring{M} = M \backslash bM$ is a complete conformal minimal immersion $X \colon \mathring{M} \to \mathbb{R}^n$ and whose restriction to bM is a topological embedding. If $n \geq 5$ then $X \colon M \to \mathbb{R}^n$ can be chosen as a topological embedding.*

If X is as in the theorem, then $X(\mathring{M})$ is a complete minimal surface in $\mathbb{R}^n$ bounded by finitely many Jordan curves $X(bM)$, and we have a control of its conformal structure.

One of the tools that made this construction possible is the adaptation of the Riemann–Hilbert boundary value problem to null holomorphic curves and conformal minimal immersions (see Sections 5.1 and 5.2). This topic was started by the authors [15] in dimension $n = 3$. A more general result in any dimension $n \geq 3$ was obtained by the authors together with Drinovec Drnovšek and López [7]. We showed that one can increase the intrinsic boundary distance in the Riemann surface M by an arbitrarily large amount by changing the conformal minimal immersion $M \to \mathbb{R}^n$ as little as desired in the $\mathscr{C}^0$-norm (see Lemma 5.5). This is achieved by applying the Riemann–Hilbert method in a certain spiralling construction, somewhat resembling Nash's method [93] of constructing $\mathscr{C}^1$ isometric immersions of Riemannian manifolds into Euclidean spaces. The same technique applies to complex curves in $\mathbb{C}^n$ for any $n \geq 2$, to holomorphic null curves in $\mathbb{C}^n$ for any $n \geq 3$, and to holomorphic Legendrian curves in complex contact manifolds, thereby yielding an analogue of Theorem 1.3 in that setting (see [17, 21]).

On the other hand, Calabi's original conjecture holds for embedded minimal surfaces of finite topology in $\mathbb{R}^3$ since these are necessarily proper in $\mathbb{R}^3$ according to Colding and Minicozzi [31, Corollary 0.13]; their result was extended to surfaces of finite genus and countably many ends by Meeks *et al.* [88]. Nothing seems known about Calabi's conjecture in dimensions $n > 3$.

The analogous problem in complex analysis was raised in 1977 by Yang [109] who asked whether there exist complete bounded complex submanifolds in $\mathbb{C}^n$. The first such examples were found in 1979 by Jones [74] who showed that the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ admits a complete bounded holomorphic immersion into $\mathbb{C}^2$, embedding into $\mathbb{C}^3$, and proper embedding into the ball of $\mathbb{C}^4$. Interest in this subject was revived only recently, due mainly to the influence of the developments in minimal surface theory. In 2013 the authors showed in [12] that every bordered Riemann surface admits a complete proper holomorphic immersion into the ball of $\mathbb{C}^2$ and embedding into the ball of $\mathbb{C}^3$. A flurry of recent activity followed and we refer to [52, Section 4.18] for a survey. Contrary to the case of minimal hypersurfaces in $\mathbb{R}^n$ for $n > 3$, complete bounded complex hypersurfaces in $\mathbb{C}^n$ are known to exist in arbitrary dimension $n \geq 2$; see Alarcón and López [26] for $n = 2$ and Globevnik [61] for any n. Indeed, Globevnik showed in [61] that the ball $\mathbb{B}^n$ of $\mathbb{C}^n$ can be foliated

by complete closed complex hypersurfaces, and subsequently Alarcón [4] proved that every smooth complete complex hypersurface in $\mathbb{B}^n$ can be embedded as a leaf into a *nonsingular* holomorphic foliation of $\mathbb{B}^n$ all of whose leaves are complete.

Another classical topic of minimal surface theory is to understand which Riemann surfaces properly immerse or embed as conformal minimal surfaces into a given domain $\Omega \subset \mathbb{R}^n$. The case $\Omega = \mathbb{R}^n$ is covered by Theorem 1.1. When considering minimal surfaces in proper domains $\Omega \subsetneq \mathbb{R}^n$, especially bounded ones, one must restrict attention to surfaces of hyperbolic conformal type. Classically this problem was studied for convex domains (see [83]). The authors, jointly with Drinovec Drnovšek and López, proved in [7] that every bordered Riemann surface admits a complete proper conformal minimal immersion into any convex domain $\Omega \subset \mathbb{R}^n, n \geq 3$, which can be chosen as an embedding if $n \geq 5$ and an immersion with simple double points if $n = 4$. The Riemann–Hilbert boundary value problem for conformal minimal immersions (see Theorem 5.1) plays a major role in the proof. It provides an inductive construction by which all of the surface is kept in the domain Ω at every step while pushing its boundary closer and closer to $b\Omega$.

In the subsequent work [8] of the same authors this result was extended to the substantially bigger class of all *minimally convex* (also called *2-convex*) domains. A domain $\Omega \subset \mathbb{R}^n$ is minimally convex if it admits a smooth exhaustion function $\rho \colon \Omega \to \mathbb{R}_+$ such that the smallest two eigenvalues $\lambda_1(x), \lambda_2(x)$ of its Hessian $H_\rho(x) = (\partial^2\rho(x)/\partial x_j \partial x_k)$ at any point $x \in \Omega$ satisfy $\lambda_1(x) + \lambda_2(x) > 0$. If Ω is smoothly bounded and $\nu_1(x) \leq \nu_2(x) \leq \cdots \leq \nu_{n-1}(x)$ are the principal curvatures of the boundary $b\Omega$ at the point $x \in b\Omega$ from the inner side, then Ω is minimally convex if and only if $\nu_1(x) + \nu_2(x) \geq 0$ holds for every $x \in b\Omega$. In particular, if $n = 3$ and S is a properly embedded minimal surface in $\mathbb{R}^3$ (in which case $\nu_1(x) + \nu_2(x) = 0$ holds identically on S), then each connected component of $\mathbb{R}^3 \setminus S$ is a 2-convex domain. The following main result in this direction is [8, Theorem 1.1] (see Section 5.4).

THEOREM 1.4. *Assume that Ω is a minimally convex domain in $\mathbb{R}^n$ for some $n \geq 3$, M is a compact bordered Riemann surface, and $X \colon M \to \Omega$ is a conformal minimal immersion. Then X can be approximated uniformly on compacts in $\mathring{M} = M \setminus bM$ by proper (and complete if so desired) conformal minimal immersions $\mathring{M} \to \Omega$.*

Examples in [8] show that, in dimension $n = 3$, minimally convex domains form the biggest class of domains for which one can expect general approximation results for conformal minimal immersions by proper ones. This line of results is intimately related to the construction, due to Drinovec Drnovšek and Forstnerič [40], of proper holomorphic maps from bordered Riemann surfaces into any complex manifold Ω admitting an exhaustion function whose Levi form has at least two positive eigenvalues at every point (this holds in particular for any Stein manifold of dimension > 1). Analysis of the proof in [8] reveals a deeper reason behind this connection.

Another interesting and important object in the theory is the *complex Gauss map* of a conformal minimal surface $X \colon M \to \mathbb{R}^n$. This is the Kodaira-type holomorphic map

$G_X \colon M \to \mathbb{CP}^{n-1}$, defined by

$$G_X(p) = [\partial X_1(p) \colon \cdots \colon \partial X_n(p)] \in \mathbb{CP}^{n-1}, \quad p \in M.$$

In view of (1.1) the map G_X assumes values in the complex hyperquadric

$$Q_{n-2} = \{[z_1 : \ldots : z_n] \in \mathbb{CP}^{n-1} : z_1^2 + \cdots + z_n^2 = 0\}.$$

This map is especially interesting in dimension $n = 3$. The quadric Q_1 is the image of a quadratically embedded rational curve $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and hence we may consider G_X as a holomorphic map $g_X \colon M \to \mathbb{CP}^1$, that is, a meromorphic function on M. The complex Gauss map g_X of a minimal surface in $\mathbb{R}^3$ provides crucial information about its geometry. Several important properties of the surface depend only on its Gauss map, in particular, its Gauss curvature and the Jacobi operator (see, for example, [86, 87, 97, 98]). Furthermore, it was shown by Barbosa and do Carmo [27, Theorem 1.2] that the minimal surface $X(M) \subset \mathbb{R}^3$ is stable if the spherical image $g_X(M) \subset \mathbb{CP}^1$ of $X(M)$ has an area less than 2π; this holds, for example, if $g_X(M)$ is a proper subset of the unit disc $\mathbb{D} \subset \mathbb{C}$. Therefore, it is interesting to know the following recent result [18, Corollary 1.2] of the authors with López.

THEOREM 1.5. *Any meromorphic function on an open Riemann surface M is the complex Gauss map of a conformal minimal immersion $X \colon M \to \mathbb{R}^3$. Furthermore, X can be chosen as the real part of a holomorphic null curve $Z = X + \mathrm{i}Y \colon M \to \mathbb{C}^3$.*

The analogous result holds in higher dimensions, see [18, Theorem 1.1]. We refer to Sections 2.6 and 4.3 for more information on this topic.

It is a natural problem to understand the homotopy type of the space $\mathrm{CMI}(M, \mathbb{R}^n)$ of all conformal minimal immersions of a given open Riemann surface M to $\mathbb{R}^n$, endowed with the compact-open topology. A conformal minimal immersion $M \to \mathbb{R}^n$ is called *nonflat* if its image is not contained in any affine plane; the space of all such immersions is denoted by $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n)$. Similarly, $\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n)$ denotes the space of nonflat holomorphic null immersions $M \to \mathbb{C}^n$, and

$$\mathfrak{R}\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n) = \{\mathfrak{R}Z \colon M \to \mathbb{R}^n : Z \in \mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n)\} \subset \mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n).$$

A continuous map $\phi \colon X \to Y$ of topological spaces is a *weak homotopy equivalence* if it induces a bijection of the path components of the two spaces and an isomorphism $\pi_k(X, x_0) \xrightarrow{\cong} \pi_k(Y, \phi(x_0))$ of their homotopy groups for each $k \in \mathbb{N}$ and $x_0 \in X$. The map ϕ is a *homotopy equivalence* if there is a continuous map $\psi \colon Y \to X$ such that $\psi \circ \phi \colon X \to X$ and $\phi \circ \psi \colon Y \to Y$ are homotopic to the identity on the respective spaces. The following result was proved by Forstnerič and Lárusson [55] (see Section 4.4).

THEOREM 1.6. *Let M be an open Riemann surface, and let θ be a holomorphic 1-form without zeros on M. The map*

$$\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) \longrightarrow \mathscr{C}(M, \mathfrak{A}_*^{n-1}), \quad X \longmapsto \partial X/\theta,$$

is a weak homotopy equivalence, and is a homotopy equivalence if M has finite topological type (that is, finite genus and number of ends). Likewise, the inclusion

$$\mathfrak{R}(\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n)) \hookrightarrow \mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n)$$

is a weak homotopy equivalence, and is a homotopy equivalence (indeed, the inclusion of a strong deformation retract) if M has finite topological type.

Besides complex analysis, the proof of Theorem 1.6 strongly relies on Gromov's *convex integration method* which originates in his paper [69] and has been fully developed in his monograph [66] (see also Spring [105]). In the case at hand, this technique provides families of loops with specified integrals in the null quadric. The main interest of Theorem 1.6 is in the fact that the space $\mathscr{C}(M, \mathfrak{A}_*^{n-1})$ is quite easy to understand. When M has finite topology, the second part of Theorem 1.6 may be interpreted as follows.

We can simultaneously continuously deform all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ to those with vanishing flux, keeping the latter ones fixed.

That a single conformal minimal immersion can be deformed to one with zero flux was first shown by Alarcón and Forstnerič in [13]. It was later proved in [18, Corollary 1.6] that such a deformation exists through a family of conformal minimal immersions sharing the same complex Gauss map.

In Theorem 1.6 we have excluded flat conformal minimal immersions and null curves; these present technical difficulties in the analysis of the structure of the respective mapping spaces. When M is a compact bordered Riemann surface, the space $\mathrm{CMI}_{\mathrm{nf}}^r(M, \mathbb{R}^n)$ of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ of class $\mathscr{C}^r$ ($r \in \mathbb{N}$) is a real analytic Banach manifold (see Theorem 3.5), while flat immersions seem to be singular points of $\mathrm{CMI}^r(M, \mathbb{R}^n)$. Nevertheless, in [18, Theorem 7.1] the connected components of $\mathrm{CMI}(M, \mathbb{R}^n)$ were identified as follows.

THEOREM 1.7. *Let M be a connected open Riemann surface. The inclusion $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) \hookrightarrow \mathrm{CMI}(M, \mathbb{R}^n)$ of the space of all nonflat conformal minimal immersions $M \to \mathbb{R}^n$ into the space of all conformal minimal immersions induces a bijection of path components of the two spaces. In particular, the set of path components of $\mathrm{CMI}(M, \mathbb{R}^3)$ is in bijective correspondence with elements of the abelian group $(\mathbb{Z}_2)^l$ where $H_1(M; \mathbb{Z}) = \mathbb{Z}^l$, and $\mathrm{CMI}(M, \mathbb{R}^n)$ is path connected if $n > 3$.*

It was shown by the authors and López [20, 25] that complex analytic methods may also be used in the construction of *nonorientable minimal surfaces* in $\mathbb{R}^n$ by working on their oriented double-sheeted coverings. In [20, Example 6.1] the reader can find the first known example of a properly embedded minimal Möbius strip in $\mathbb{R}^4$ (see also Section 2.3). Space does not permit us to include these results. Another recently developed topic that is not treated in this survey, but which relies on complex analytic tools, is the theory of uniform approximation by complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature, due to López [77, 78].

There are several other important aspects of the classical theory of minimal surfaces in Euclidean spaces where substantial progress has been made in recent years but are not covered in this paper; see in particular the survey by Pérez [100] and the monographs by Meeks and Pérez [86, 87].

We include proofs of the main complex analytic results used in the constructions; an exception is the Riemann–Hilbert boundary value problem for conformal minimal surfaces and null curves (see Section 5.2) whose proofs are too complex to be included. The inductive procedures leading to the proofs of the main results are for the most part only sketched, referring to the original sources for further details.

2. From minimal surfaces to complex analysis and back

In this section we review those classical facts relating minimal surfaces to complex analysis which are indispensable for the subsequent discussion. More complete presentations are available in the books of Osserman [98], Colding and Minicozzi [30, 32], and several others. For the geometry of surfaces we refer to do Carmo [36], and for Riemann surfaces we refer to the monographs by Donaldson [37], Farkas and Kra [43], and Forster [48].

Let $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the real and the complex Euclidean space of dimension $n \in \mathbb{N} = \{1, 2, \ldots\}$, respectively. Set $\mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x \geq 0\}$, $\mathbb{C}_*^n = \mathbb{C}^n \backslash \{0\}$, and $\mathbb{C}_* = \mathbb{C}_*^1$. We denote the coordinates on $\mathbb{R}^n$ by $(x_1, \ldots, x_n)$ and those on $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$, where $z = x + \mathfrak{i}y$ with $x, y \in \mathbb{R}^n$ and $\mathfrak{i} = \sqrt{-1}$. Maps to these spaces will be denoted by the corresponding capital letters, for example, $X \colon M \to \mathbb{R}^n$ and $Z \colon M \to \mathbb{C}^n$. We denote the Euclidean inner product and the norm on $\mathbb{R}^n$ by

$$x \cdot y = \sum_{j=1}^n x_j y_j, \quad |x|^2 = x \cdot x = \sum_{j=1}^n x_j^2.$$

The space $\mathbb{R}^n$ is endowed with the Riemannian metric $ds^2 = dx_1^2 + \cdots + dx_n^2$.

2.1. Riemann surfaces. A *Riemann surface* is a one-dimensional complex manifold. We denote the space of all holomorphic functions on a Riemann surface M by $O(M)$. Since every minimal surface in $\mathbb{R}^n$ is parametrized by a nonconstant harmonic map from a Riemann surface, such a surface cannot be compact and without boundary. Hence we shall mainly consider Riemann surfaces that are either *open* (that is, noncompact and without boundary) or compact with nonempty boundary. On an open Riemann surface M we have the classical Runge–Mergelyan approximation theorem and the Weierstrass interpolation theorem. The former says that, given a *Runge* (also called $O(M)$-*convex*) compact subset $K \subset M$ (that is, such that $M \backslash K$ has no relatively compact components), every continuous function $K \to \mathbb{C}$ that is holomorphic on the interior $\mathring{K}$ of K may be approximated uniformly on K by functions in $O(M)$. The latter says that every map $\Lambda \to \mathbb{C}$ on a closed discrete subset $\Lambda \subset M$ extends to a function in $O(M)$. (Weierstrass's original theorem for planar domains [107] was extended to open Riemann surfaces by Florack [45] in 1948.) An open Riemann

surface is the same thing as a one-dimensional Stein manifold (see Section 3.1), and the aforementioned results extend to any Stein manifold as the Oka–Weil approximation theorem and the Oka–Cartan extension theorem, respectively.

The following classification of open Riemann surfaces has important implications in the theory of minimal surfaces; see Farkas and Kra [43, page 179].

Definition 2.1. An open Riemann surface is *hyperbolic* if it carries nonconstant negative subharmonic functions; otherwise it is *parabolic*.

By Koebe's uniformization theorem, the only simply connected open Riemann surfaces up to biholomorphisms are $\mathbb{C}$, which is parabolic by Liouville's theorem, and the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ which is hyperbolic. Every compact Riemann surface from which finitely many points have been removed is parabolic. We refer to the survey by Grigor'yan [65] for further information on parabolicity and hyperbolicity of Riemannian manifolds.

A *compact bordered Riemann surface* is a compact Riemann surface M with boundary $bM \neq \varnothing$ consisting of finitely many pairwise disjoint Jordan curves. The interior $\mathring{M} = M \setminus bM$ of such M is a *bordered Riemann surface* and is a hyperbolic open Riemann surface. Such $\mathring{M}$ is biholomorphic to a smoothly bounded domain in a compact Riemann surface without boundary.

The only topological invariants of a connected oriented surface M are its genus and number of ends. We say that M has *finite topological type* if both its genus g and the number m of its ends are finite; such M is biholomorphic to a domain in a compact Riemann surface R from which finitely many points and closed discs have been removed (see Stout [106]). Its first homology group equals $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^l$ with $l = 2g + m - 1$. There exist smooth Jordan curves $C_1, \ldots, C_l$ in M representing a basis of $H_1(M; \mathbb{Z})$. If M is either open or compact with nonempty boundary, then these curves can be chosen such that their union $C = \bigcup_{j=1}^l C_j$ is contained in $\mathring{M}$ and is Runge in $\mathring{M}$; furthermore, C is a strong deformation retract of M.

2.2. Immersed surfaces, Riemannian metrics, and isothermal coordinates. Let S be a smooth real surface and $X = (X_1, \ldots, X_n) : S \to \mathbb{R}^n$ be a smooth immersion. The pullback of the Euclidean metric ds^2 on $\mathbb{R}^n$ by the immersion X is a Riemannian metric on S:

$$g = X^*(ds^2) = (dX_1)^2 + \cdots + (dX_n)^2.$$

In any smooth local coordinate (u, v) on S we have that

$$g = A\,du^2 + 2B\,du\,dv + C\,dv^2, \tag{2.1}$$

where $A > 0, B, C > 0$ are smooth functions and $AC - B^2 > 0$; this is called the *first fundamental form* of the immersed surface. If (2.1) holds in coordinates (u, v) on a domain $\Omega \subset S$, then the area of the immersed surface $X : \Omega \to \mathbb{R}^n$ equals

$$\text{Area}\,(X(\Omega)) = \int_\Omega \sqrt{AC - B^2}\,du\,dv.$$

The 2-form

$$dA_{X(\Omega)} = \sqrt{AC - B^2}\, du\, dv \tag{2.2}$$

is called the *area element* of the immersed surface $X|_\Omega \colon \Omega \to \mathbb{R}^n$.

By the celebrated *isometric immersion theorem* of Nash [93, 94], every smooth Riemannian metric on S is induced by a smooth embedding $X \colon S \to \mathbb{R}^n$ to a Euclidean space. See Gromov [66, 68] for more information.

A Riemannian metric g on an orientable surface S determines an almost complex structure operator $J \colon TS \to TS$, with $J^2 = -\mathrm{Id}$, by the condition that for every unit vector $\xi \in T_p S$ the pair $(\xi, J\xi)$ is a positively oriented g-orthonormal basis of the tangent space $T_p S$. Riemannian metrics $g, \tilde{g}$ on S are said to be *conformally equivalent* if $\tilde{g} = \lambda g$ for some function $\lambda > 0$; such metrics determine the same almost complex structure J. Conversely, a choice of J uniquely determines a conformal class of Riemannian metrics. Around any point of S there exist smooth *isothermal coordinates* (u, v) in which the Riemannian metric g has the form

$$g = \lambda(du^2 + dv^2)$$

for some positive function $\lambda > 0$. A change of coordinates which puts the metric in this form is found by solving the *Beltrami equation* (see Ahlfors [3]). In such coordinates, the associated almost complex structure J is the *standard almost complex structure* on $\mathbb{R}^2_{(u,v)} \cong \mathbb{C}$ given by $J_{st}(\partial/\partial u) = \partial/\partial v$, $J_{st}(\partial/\partial v) = -(\partial/\partial u)$. The transition map between any two isothermal coordinates is a conformal isomorphism, hence holomorphic or antiholomorphic with respect to the complex coordinate $\zeta = u + \mathrm{i}v$. If S is orientable, we obtain an atlas $\mathcal{U} = \{(U_j, \phi_j)\}$ consisting of an open covering $\{U_j\}$ of S and positively oriented isothermal coordinates $\phi_j \colon U_j \to \phi_j(U_j) \subset \mathbb{R}^2 \cong \mathbb{C}$ whose transition maps $\phi_{i,j} = \phi_i \circ \phi_j^{-1}$ are biholomorphisms; that is, $\mathcal{U}$ is a complex atlas determining on S the structure of a Riemann surface. If S is connected and nonorientable, it admits a double-sheeted covering map $\pi \colon \widetilde{S} \to S$ with $\widetilde{S}$ orientable, and the same argument applied to the metric $\pi^* g$ on $\widetilde{S}$ shows that $\widetilde{S}$ carries the structure of a Riemann surface such that the projection map π is conformal.

2.3. Minimal surfaces. Assume that S is a smooth orientable surface and $X \colon S \to \mathbb{R}^n$ is a smooth immersion. Let $N \colon S \to \mathbb{R}^n$ be a smooth vector field along X such that for every $p \in S$ the vector $N(p)$ has unit length and is orthogonal to the tangent plane $dX_p(T_p S) \subset \mathbb{R}^n$ of X. Given a smooth function $\psi \colon S \to \mathbb{R}$ with compact support, there is a number $\epsilon > 0$ such that the maps

$$X^t = X + t\psi N \colon S \to \mathbb{R}^n, \quad t \in (-\epsilon, \epsilon),$$

are again smooth immersions. Such a family of immersions is called a *normal variation with compact support* of $X = X^0$. The associated area functional is

$$\mathcal{A}(t) = \mathrm{Area}\,(X^t(S)), \quad t \in (-\epsilon, \epsilon). \tag{2.3}$$

The *first variation of area formula* says that

$$\mathcal{A}'(0) = \frac{d\mathcal{A}(t)}{dt}\bigg|_{t=0} = -2 \int_S \mathbf{H} \cdot N\, \psi\, dA_X,$$

where **H** and dA_X are the mean curvature vector field and the area element (2.2) of X, respectively. This leads to the following observation due to Meusnier.

PROPOSITION 2.2. *Let S be a smooth orientable surface and $X: S \to \mathbb{R}^n$ be a smooth immersion. The following two conditions are equivalent.*

- *X is a critical point of the area functional for all normal variations with compact support.*
- *The mean curvature vector field **H**$: S \to \mathbb{R}^n$ of X vanishes identically.*

An immersed surface $X: S \to \mathbb{R}^n$ is said to be *minimal* if it satisfies the equivalent conditions in Proposition 2.2. It follows from the *second variation of area formula* (see [95, page 95] or [35, pages 83–84]) that every minimal surface in $\mathbb{R}^n$ minimizes the area locally. Surfaces which minimize the area globally are said to be *area minimizing*. Furthermore, for a minimal surface $X: S \to \mathbb{R}^n$ we have that $\mathscr{A}''(0) > 0$ for all normal variations with compact support if and only if every such variation of X strictly increases the area; if this holds then X is said to be *strongly stable*. If, in contrast, for some variation the second derivative is negative, $\mathscr{A}''(0) < 0$, then there are nearby surfaces with a smaller area and X is called *unstable*. Finally, those minimal surfaces for which $\mathscr{A}''(0) \geq 0$ holds for all normal variations with compact support are said to be *stable*. The stability property has important implications in the theory of minimal surfaces.

The simplest example of a minimal surface in $\mathbb{R}^n$ is an affine plane, which is in fact area minimizing. A classical result by Wirtinger [108] says that every holomorphic curve in $\mathbb{C}^n = \mathbb{R}^{2n}$ ($n \geq 2$) is area minimizing, and hence a minimal surface. The following are some of the most classical examples of minimal surfaces in $\mathbb{R}^3$.

- The *catenoids* (Euler 1744) were the first minimal surfaces in $\mathbb{R}^3$ to be discovered, apart from (pieces of) affine planes. Planes and catenoids are the only minimal surfaces of revolution in $\mathbb{R}^3$. Here is a parametrization of a catenoid:

$$X(\rho, \theta) = \left(c \cosh\left(\frac{\rho}{c}\right) \cos \theta, c \cosh\left(\frac{\rho}{c}\right) \sin \theta, \rho \right), \quad \rho \in \mathbb{R}, \ \theta \in [-\pi, \pi),$$

  where $c \in \mathbb{R} \backslash \{0\}$ is a constant. See Figure 2.1.
- The *helicoids*, discovered by Meusnier in 1776, are the only ruled minimal surfaces in $\mathbb{R}^3$ besides planes. Here is a parametrization of a helicoid:

$$X(u, v) = (u \cos(cv), u \sin(cv), v), \quad (u, v) \in \mathbb{R}^2,$$

  where $c \in \mathbb{R} \backslash \{0\}$ is a constant. See Figure 2.1.
- The *Enneper surface*, which has self-intersections, was discovered by Enneper in 1864. Here is a parametrization; see Figure 2.2:

$$X(u, v) = \left(\left(1 - \frac{u^2}{3} + v^2\right)u, -\left(1 - \frac{v^2}{3} + u^2\right)v, u^2 - v^2 \right), \quad (u, v) \in \mathbb{R}^2.$$
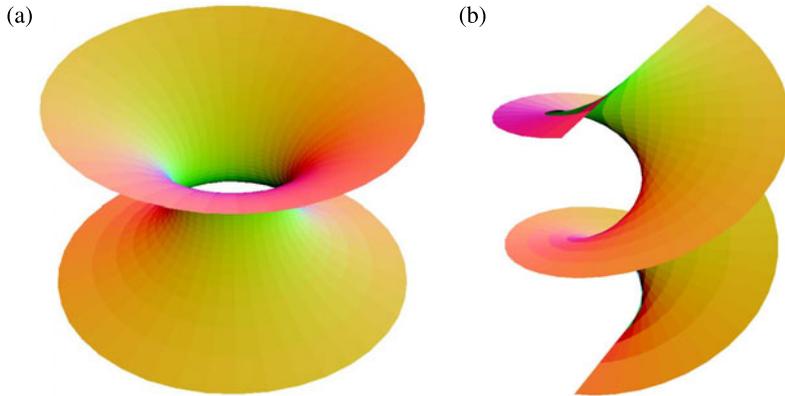
(a)   (b)

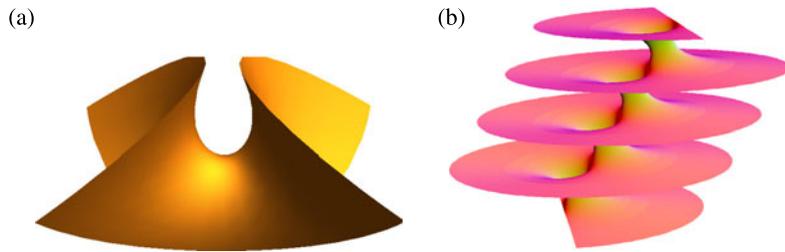FIGURE 2.1. A catenoid (a) and a helicoid (b).

(a)   (b)

FIGURE 2.2. An Enneper surface (a) and a Riemann example (b).

- The *Riemann minimal examples* form a one-parameter family of singly periodic minimal surfaces with infinitely many ends asymptotic to parallel planes (see Figure 2.2). Besides planes, catenoids, and helicoids, these surfaces, described by Riemann in a posthumous paper from 1867, are the only minimal surfaces in $\mathbb{R}^3$ foliated by circles and affine lines in parallel planes.
- *A properly embedded minimal Möbius strip in* $\mathbb{R}^4$ was found by the authors and López [20, Example 6.1]. The harmonic map $X\colon \mathbb{C}_* \to \mathbb{R}^4$ given by

$$X(\zeta) = \Re\left(\mathrm{i}\left(\zeta + \frac{1}{\zeta}\right), \zeta - \frac{1}{\zeta}, \frac{\mathrm{i}}{2}\left(\zeta^2 - \frac{1}{\zeta^2}\right), \frac{1}{2}\left(\zeta^2 + \frac{1}{\zeta^2}\right)\right)$$

is a proper conformal minimal immersion such that $X(\zeta_1) = X(\zeta_2)$ if and only if $\zeta_1 = \zeta_2$ or $\zeta_1 = \Im(\zeta_1)$, where $\Im$ is the fixed-point-free antiholomorphic involution on $\mathbb{CP}^1$ (and on $\mathbb{C}_*$) given by $\Im(\zeta) = -1/\bar{\zeta}$. Since $\mathbb{C}_*/\Im$ is a Möbius strip, the image surface $X(\mathbb{C}_*) \subset \mathbb{R}^4$ is a properly embedded minimal Möbius strip in $\mathbb{R}^4$.

Another famous example is Meeks's immersed Möbius strip in $\mathbb{R}^3$ with finite total curvature -6π (see [85, Theorem 2] and [20, Example 2.6 and Figure 2.3]).

2.4. Conformal immersions and the null quadric. Note that vectors $x, y \in \mathbb{R}^n$ are of the same size and orthogonal to each other if and only if the complex vector $z = x + \mathrm{i} y \in \mathbb{C}^n$ belongs to the *null quadric*

$$\mathfrak{A} = \{z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}. \tag{2.4}$$

Indeed, we have that $\sum_{j=1}^{n}(x_j + \mathrm{i} y_j)^2 = |x|^2 - |y|^2 + \mathrm{i}\, x \cdot y$ from which the claim follows. Elements $z \in \mathfrak{A}$ are called *null vectors*. Note that $\mathfrak{A}$ is a complex cone which is nonsingular except at the vertex $0 \in \mathfrak{A}$. The *punctured null quadric*

$$\mathfrak{A}_* = \mathfrak{A} \backslash \{0\} = \mathfrak{A}_{\mathrm{reg}} \tag{2.5}$$

is a homogeneous space of the complex Lie group $\mathbb{C}_* \oplus \mathrm{O}_n(\mathbb{C})$, where $\mathrm{O}_n(\mathbb{C}) = \{A \in \mathrm{GL}_n(\mathbb{C}) : AA^t = I\}$ is the orthogonal group over $\mathbb{C}$. It follows that maps $M \to \mathfrak{A}_*$ from any Stein manifold (in particular, from any open Riemann surface) satisfy the *Oka principle* (see Theorem 3.1). This is the single most important fact in applications of complex analysis to the theory of minimal surfaces in $\mathbb{R}^n$.

Let M be a Riemann surface. Note that an immersion

$$X = (X_1, X_2, \ldots, X_n) : M \to \mathbb{R}^n \tag{2.6}$$

is conformal if and only if, in any local holomorphic coordinate $\zeta = u + \mathrm{i} v$ on M, the partial derivatives $X_u = \partial X / \partial u = (X_{1,u}, \ldots, X_{n,u}) \in \mathbb{R}^n$ and $X_v = \partial X / \partial v = (X_{1,v}, \ldots, X_{n,v}) \in \mathbb{R}^n$ at any given point have the same length and are orthogonal to each other:

$$|X_u| = |X_v| > 0, \quad X_u \cdot X_v = 0. \tag{2.7}$$

Following the customary notation in complex analysis, we set

$$X_\zeta = \frac{\partial X}{\partial \zeta} = \frac{1}{2}(X_u - \mathrm{i} X_v), \quad X_{\bar{\zeta}} = \frac{\partial X}{\partial \bar{\zeta}} = \frac{1}{2}(X_u + \mathrm{i} X_v).$$

Thus, the equation $X_{\bar{\zeta}} = 0$ characterizes holomorphic functions. In view of what has been said above, condition (2.7) is equivalent to

$$2X_\zeta = X_u - \mathrm{i} X_v \in \mathfrak{A}_*,$$

where $\mathfrak{A}_*$ is given by (2.5). The exterior derivative on M splits into the sum

$$d = \partial + \bar{\partial}$$

of the $(1, 0)$-part ∂ and the $(0, 1)$-part $\bar{\partial}$, where

$$\partial X = X_\zeta \, d\zeta, \quad \bar{\partial} X = X_{\bar{\zeta}} \, d\bar{\zeta}.$$

Hence, an immersion (2.6) is conformal if and only if its $(1, 0)$-differential $\partial X = (\partial X_1, \ldots, \partial X_n)$ satisfies the nullity condition

$$(\partial X_1)^2 + (\partial X_2)^2 + \cdots + (\partial X_n)^2 = 0. \tag{2.8}$$

2.5. The Enneper–Weierstrass representation formula. Assume now that M is an open Riemann surface and $X: M \to \mathbb{R}^n$ is a conformal immersion. In any local holomorphic coordinate $\zeta = u + \mathrm{i}v$ on M the Laplacian equals

$$\Delta = \left(\frac{\partial}{\partial u}\right)^2 + \left(\frac{\partial}{\partial v}\right)^2 = 4\frac{\partial^2}{\partial\bar\zeta\,\partial\zeta}.$$

In particular, X is harmonic if and only if the 1-form ∂X is holomorphic. It is classical (see Osserman [98]) that

$$\Delta X = 2\mu\mathbf{H} \tag{2.9}$$

where $\mathbf{H}$ is the mean curvature vector field of X and $\mu = |X_u|^2 = |X_v|^2$. Taking into account also (2.9) gives the following classical result.

THEOREM 2.3. *Let M be an open Riemann surface and $X: M \to \mathbb{R}^n$ be a smooth conformal immersion for some $n \geq 3$. Then the following conditions are pairwise equivalent.*

- *X is minimal (a stationary point of the area functional (2.3)).*
- *X has vanishing mean curvature vector field: $\mathbf{H} = 0$.*
- *X is harmonic: $\Delta X = 0$.*
- *The $\mathbb{C}^n$-valued 1-form $\partial X = (\partial X_1, \dots, \partial X_n)$ is nowhere vanishing on M, holomorphic, and satisfies the nullity condition (2.8).*
- *Let θ be a nowhere vanishing holomorphic 1-form on M. The map $f = 2\partial X/\theta$: $M \to \mathbb{C}^n$ is holomorphic and assumes values in $\mathfrak{A}_*$ (2.5).*

If these conditions hold then the induced Riemannian metric on M equals

$$X^*(ds^2) = 2(|\partial X_1|^2 + \cdots + |\partial X_n|^2). \tag{2.10}$$

Every open Riemann surface M admits a nowhere vanishing holomorphic 1-form θ by the Oka–Grauert principle (see [52, Theorem 5.3.1(iii)]). If $X: M \to \mathbb{R}^n$ is a conformal minimal immersion then the 1-form $2\partial X = f\theta$ with values in $\mathfrak{A}_*$ has exact real part since $dX = \partial X + \bar\partial X = 2\mathfrak{R}(\partial X)$; equivalently, $\oint_C \mathfrak{R}(f\theta) = 0$ for every closed curve $C \subset M$. Conversely, every holomorphic 1-form $f\theta$ with values in $\mathfrak{A}_*$ and exact real part $\mathfrak{R}(f\theta)$ determines a conformal minimal immersion by integration. We record this observation in the following corollary to Theorem 2.3.

THEOREM 2.4 (The Enneper–Weierstrass representation formula). *Let M be a connected open Riemann surface, θ be a nowhere vanishing holomorphic 1-form on M, and $p_0 \in M$ be an arbitrary point. Every conformal minimal immersion $X: M \to \mathbb{R}^n$ ($n \geq 3$) is of the form*

$$X(p) = X(p_0) + \int_{p_0}^p \mathfrak{R}(f\theta), \quad p \in M, \tag{2.11}$$

where $f: M \to \mathfrak{A}_*$ is a holomorphic map into the punctured null quadric such that the $\mathbb{R}^n$-valued 1-form $\mathfrak{R}(f\theta)$ is exact. We have that $2\partial X = f\theta$.

2.6. The complex Gauss map. Let $X = (X_1, \ldots, X_n) \colon M \to \mathbb{R}^n$ be a conformal minimal immersion. Its differential $\partial X = (\partial X_1, \ldots, \partial X_n)$ determines the Kodaira type holomorphic map

$$G_X \colon M \to \mathbb{CP}^{n-1}, \quad G_X(p) = [\partial X_1(p) \colon \cdots \colon \partial X_n(p)], \quad p \in M, \tag{2.12}$$

called the *generalized Gauss map* of X. In view of Equation (2.8), G_X assumes values in the complex hyperquadric

$$Q_{n-2} = \{[z_1 \colon \ldots \colon z_n] \in \mathbb{CP}^{n-1} \colon z_1^2 + \cdots + z_n^2 = 0\}. \tag{2.13}$$

In the case $n = 3$, the quadric $Q_1 \subset \mathbb{CP}^2$ is the image of a quadratically embedded Riemann sphere $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$, and the *complex Gauss map* of a conformal minimal immersion $X = (X_1, X_2, X_3) \colon M \to \mathbb{R}^3$ is defined to be the holomorphic map

$$g_X = \frac{\partial X_3}{\partial X_1 - \mathrm{i} \partial X_2} = \frac{\partial X_2 - \mathrm{i} \partial X_1}{\mathrm{i} \partial X_3} \colon M \longrightarrow \mathbb{CP}^1. \tag{2.14}$$

The function g_X equals the stereographic projection of the real Gauss map $N = (N_1, N_2, N_3) \colon M \to \mathbb{S}^2 \subset \mathbb{R}^3$ to the Riemann sphere $\mathbb{CP}^1$; explicitly,

$$g_X = \frac{N_1 + \mathrm{i} N_2}{1 - N_3} \colon M \longrightarrow \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1.$$

We can recover the differential $\partial X = (\partial X_1, \partial X_2, \partial X_3)$ from the pair (g_X, ϕ_3), with $\phi_3 = \partial X_3$, by the classical *Weierstrass formula*:

$$\partial X = \Phi = (\phi_1, \phi_2, \phi_3) = \left(\frac{1}{2} \left(\frac{1}{g_X} - g_X \right), \frac{\mathrm{i}}{2} \left(\frac{1}{g_X} + g_X \right), 1 \right) \phi_3. \tag{2.15}$$

(See [98, Lemma 8.1, page 63].) Conversely, given a pair (g, ϕ_3) consisting of a holomorphic map $g \colon M \to \mathbb{CP}^1$ and a holomorphic 1-form ϕ_3 on M, the meromorphic 1-form $\Phi = (\phi_1, \phi_2, \phi_3)$ defined by (2.15) satisfies $\sum_{j=1}^3 \phi_j^2 = 0$; it is the differential ∂X of a conformal minimal immersion $X \colon M \to \mathbb{R}^3$ if and only if it is holomorphic, nowhere vanishing, and its real periods vanish. If this holds then the map X is obtained from Φ (2.15) by integration:

$$X(p) = X(p_0) + 2 \int_{p_0}^{p} \mathfrak{R}(\Phi), \quad p \in M.$$

The generalized Gauss map G_X is of great importance in the theory of minimal surfaces; see Osserman [98] and the papers [18, 58, 60, 72, 79, 99, 102, 103], among many others. In particular, the complex Gauss map $g_X \colon M \to \mathbb{CP}^1$ (2.14) of a minimal surface in $\mathbb{R}^3$ provides crucial information about its geometry since the key quantities such as the Gauss curvature and the Jacobi operator depend only on g_X (see, for example, [86, 87, 97, 98]).

The authors together with López have recently shown in [18, Corollary 1.2] that every meromorphic function on an open Riemann surface M is the complex Gauss map of a conformal minimal immersion $X \colon M \to \mathbb{R}^3$; furthermore, X can be chosen as the real part of a holomorphic null curve $Z = X + \mathrm{i} Y \colon M \to \mathbb{C}^3$.

2.7. Flux, period map, conjugate surfaces, and null curves. The *conjugate differential* of a smooth map $X : M \to \mathbb{R}^n$ is defined by

$$d^c X = \mathrm{i}(\bar{\partial} X - \partial X) = 2 \Im(\partial X).$$

Recall that $d = \partial + \bar{\partial}$. Hence we get

$$2 \partial X = dX + \mathrm{i}\, d^c X, \quad dd^c X = 2 \mathrm{i}\, \partial \bar{\partial} X = \Delta_\zeta X \cdot du \wedge dv,$$

where the last equation holds in any local holomorphic coordinate $\zeta = u + \mathrm{i} v$. Thus, X is harmonic if and only if $d^c X$ is a closed $\mathbb{R}^n$-valued 1-form on M, and in this case $d^c X = dY$ holds for any local harmonic conjugate Y of X.

The *flux map* of a harmonic map $X : M \to \mathbb{R}^n$ is the group homomorphism $\mathrm{Flux}_X : H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ given by

$$\mathrm{Flux}_X([C]) = \oint_C d^c X = \oint_C 2 \Im(\partial X), \quad [C] \in H_1(M; \mathbb{Z}).$$

The integral is independent of the choice of the path in a given homology class, and we shall write $\mathrm{Flux}_X(C)$ for $\mathrm{Flux}_X([C])$ in the sequel.

Fix a nowhere vanishing holomorphic 1-form θ on M. Associated to any holomorphic map $f : M \to \mathbb{C}^n$ is the *period homomorphism*

$$\mathcal{P}(f) : H_1(M; \mathbb{Z}) \to \mathbb{C}^n,$$

defined on any closed oriented curve $C \subset M$ by

$$\mathcal{P}(f)(C) = \oint_C f\theta.$$

The map f corresponds to a conformal minimal immersion $X : M \to \mathbb{R}^n$ as in (2.11) if and only if $f(M) \subset \mathfrak{A}_*$ and $\Re(\mathcal{P}(f)) = 0$; in this case, X is given by (2.11) and

$$\mathrm{Flux}_X = \Im(\mathcal{P}(f)) : H_1(M; \mathbb{Z}) \to \mathbb{R}^n.$$

We have $\mathrm{Flux}_X = 0$ if and only if X admits a harmonic conjugate $Y : M \to \mathbb{R}^n$ (the *conjugate conformal minimal surface*), and in this case the holomorphic immersion $Z = (Z_1, \ldots, Z_n) = X + \mathrm{i} Y : M \to \mathbb{C}^n$ satisfies

$$(dZ_1)^2 + (dZ_2)^2 + \cdots + (dZ_n)^2 = 0.$$

Such Z is called a *null holomorphic immersion* of M into $\mathbb{C}^n$. We see as in Theorem 2.4 that every null holomorphic immersion is of the form

$$Z(p) = Z(p_0) + \int_{p_0}^{p} f\theta, \quad p \in M,$$

where $f : M \to \mathfrak{A}_*$ is a holomorphic map into the punctured null quadric such that the $\mathbb{C}^n$-valued 1-form $f\theta$ is exact. The minimal surfaces

$$X_t = \Re(e^{\mathrm{i}t} Z) : M \to \mathbb{R}^n, \quad t \in \mathbb{R}$$

are called the *associated minimal surfaces* of the null curve Z.

EXAMPLE 2.5. The *catenoid* and the *helicoid* (see Figure 2.1) are conjugate minimal surfaces: the real and the imaginary part of the null curve $Z : \mathbb{C} \to \mathbb{C}^3$ given by

$$Z(\zeta) = (\cos\zeta, \sin\zeta, -\mathrm{i}\zeta), \quad \zeta = x + \mathrm{i}y \in \mathbb{C}.$$

Consider the family of minimal surfaces ($t \in \mathbb{R}$):

$$X_t(\zeta) = \Re(e^{\mathrm{i}t} Z(\zeta))$$
$$= \cos t \begin{pmatrix} \cos x \cdot \cosh y \\ \sin x \cdot \cosh y \\ y \end{pmatrix} + \sin t \begin{pmatrix} \sin x \cdot \sinh y \\ -\cos x \cdot \sinh y \\ x \end{pmatrix}.$$

At $t = 0$ we have a parametrization of a catenoid, and at $t = \pm\pi/2$ we have a (left- or right-handed) helicoid.

2.8. Spaces of mappings. If M is an open Riemann surface, then $O(M)$ denotes the algebra of holomorphic functions $M \to \mathbb{C}$, $O(M, X)$ is the space of all holomorphic mappings $M \to X$ to a complex manifold X,

$$\mathrm{CMI}(M, \mathbb{R}^n)$$

is the space of all conformal minimal immersions $M \to \mathbb{R}^n$, and

$$\mathrm{NC}(M, \mathbb{C}^n)$$

is the space of all null holomorphic immersions $M \to \mathbb{C}^n$. These spaces are endowed with the compact-open topology.

Assume now that M is a compact bordered Riemann surface (see Section 2.1). Given $r \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, we denote by $\mathscr{A}^r(M)$ the space of all functions $M \to \mathbb{C}$ of class $\mathscr{C}^r(M)$ that are holomorphic in $\mathring{M} = M \backslash bM$. More generally, for any complex manifold X we let $\mathscr{A}^r(M, X)$ denote the space of maps $M \to X$ of class $\mathscr{C}^r$ which are holomorphic in $\mathring{M}$. We write $\mathscr{A}^0(M) = \mathscr{A}(M)$ and $\mathscr{A}^0(M, X) = \mathscr{A}(M, X)$. Note that $\mathscr{A}^r(M, \mathbb{C}^n)$ is a complex Banach space, and for any complex manifold X the space $\mathscr{A}^r(M, X)$ is a complex Banach manifold modeled on the Banach space $\mathscr{A}^r(M, \mathbb{C}^n)$ with $n = \dim X$ (see [52, Theorem 8.13.1] or [49, Theorem 1.1]). A compact bordered Riemann surface M can be considered as a smoothly bounded compact domain in an open Riemann surface R. It is classical that each function in $\mathscr{A}^r(M)$ can be approximated in the $\mathscr{C}^r(M)$ norm by functions in $O(M)$, that is, functions holomorphic in a neighborhood of M in R. The same holds for maps to an arbitrary complex manifold (see [40, Theorem 5.1]). For any $r \in \mathbb{N}$ we denote by

$$\mathrm{CMI}^r(M, \mathbb{R}^n)$$

the set of all conformal minimal immersions $M \to \mathbb{R}^n$ of class $\mathscr{C}^r(M)$. More precisely, an immersion $F : M \to \mathbb{R}^n$ of class $\mathscr{C}^r$ belongs to $\mathrm{CMI}^r(M, \mathbb{R}^n)$ if and only if ∂F is a $(1, 0)$-form of class $\mathscr{C}^{r-1}(M)$ that satisfies the nullity condition (2.8) and is holomorphic on the interior $\mathring{M}$. Similarly,

$$\mathrm{NC}^r(M, \mathbb{C}^n)$$

denotes the space of all null holomorphic immersions $M \to \mathbb{C}^n$ of class $\mathscr{A}^r(M)$.

The following notions will play an important role in our analysis.

DEFINITION 2.6. Let M be a connected open or bordered Riemann surface, let θ be a nowhere vanishing holomorphic 1-form on M, and let $\mathfrak{A}$ be the null quadric (2.4).

(1)   A holomorphic map $f : M \to \mathfrak{A}_*$ is *flat* if the image $f(M)$ is contained in a complex ray $\mathbb{C}v \subset \mathfrak{A}$ ($v \in \mathfrak{A}_*$) of the null quadric, and is *nonflat* otherwise.

(2)   A conformal minimal immersion $X : M \to \mathbb{R}^n$ is *nonflat* if the map $f = \partial X/\theta :$ $M \to \mathfrak{A}_*$ is nonflat; equivalently, if the image $X(M) \subset \mathbb{R}^n$ is not contained in an affine plane. A null holomorphic immersion $Z : M \to \mathbb{C}^n$ is *nonflat* if the map $f = dZ/\theta : M \to \mathfrak{A}_*$ is nonflat.

(3)   A holomorphic map $f : M \to \mathfrak{A}_*$ is *full* if the image $f(M)$ is not contained in any complex hyperplane of $\mathbb{C}^n$. A conformal minimal immersion $X : M \to \mathbb{R}^n$ is *full* if the image $X(M)$ is not contained in any affine hyperplane of $\mathbb{R}^n$.

For a conformal minimal immersion $M \to \mathbb{R}^3$, nonflat and full are equivalent conditions. However, in dimensions $n > 3$ we clearly have that full $\implies$ nonflat, but the converse is false. If M is an open Riemann surface, we denote by $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n)$ the open subset of $\mathrm{CMI}(M, \mathbb{R}^n)$ consisting of all immersions which are nonflat on every connected component of M. The analogous notation

$$\mathrm{CMI}_{\mathrm{nf}}^r(M, \mathbb{R}^n) \subset \mathrm{CMI}^r(M, \mathbb{R}^n)$$

is used for a compact bordered Riemann surface M. Likewise, $\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n)$ is the space of all nonflat holomorphic null curves.

The tangent space $T_z\mathfrak{A}$ is the kernel at z of the $(1, 0)$-form $\sum_{j=1}^n z_j \, dz_j$; hence $T_z\mathfrak{A} = T_w\mathfrak{A}$ for $z, w \in \mathbb{C}^n \backslash \{0\}$ if and only if z and w are colinear. This implies the following lemma.

LEMMA 2.7. *A holomorphic map $f : M \to \mathfrak{A}_*$ is nonflat if and only if the linear span of the tangent spaces $T_{f(p)}\mathfrak{A} \subset T_{f(p)}\mathbb{C}^n \cong \mathbb{C}^n$ over all points $p \in M$ equals $\mathbb{C}^n$.*

We now introduce sets in Riemann surfaces that are used in Mergelyan approximation theorems for conformal minimal immersions, and the notion of a generalized conformal minimal immersion on them. Such sets appear naturally in the constructions of conformal minimal immersions.

DEFINITION 2.8. Let M be an open Riemann surface. A compact set $S \subset M$ is *admissible* if it is Runge in M and of the form $S = K \cup \Gamma$, where K is a finite union of pairwise disjoint smoothly bounded compact domains in M and $\Gamma = S \backslash \mathring{K}$ is a finite union of pairwise disjoint smooth Jordan arcs and closed Jordan curves meeting K only in their endpoints (or not at all) and such that their intersections with the boundary bK of K are transverse.

DEFINITION 2.9. Let $S = K \cup \Gamma$ be an admissible set in an open Riemann surface M and let θ be a nowhere vanishing holomorphic 1-form on M. A *generalized conformal minimal immersion* $S \to \mathbb{R}^n$ is a pair $(X, f\theta)$, where $X : S \to \mathbb{R}^n$ is a smooth map that is a conformal minimal immersion on an open neighborhood of K and $f : S \to \mathfrak{A}_*$ is a smooth map that is holomorphic on a neighborhood of K, such that the following conditions are satisfied:

- $f\theta = 2\partial X$ holds on an open neighborhood of K; and
- for any smooth path α in M parametrizing a connected component of Γ we have
$\mathfrak{R}(\alpha^*(f\theta)) = \alpha^*(dX) = d(X \circ \alpha)$.

The space of all generalized conformal minimal immersions $S \to \mathbb{R}^n$ is denoted by

$$\mathrm{GCMI}(S, \mathbb{R}^n).$$

3. Oka theory, period dominating sprays, and loops with given periods in the null quadric

Oka theory concerns the existence, approximation, and extension theorems for holomorphic maps $f \colon S \to O$ from Stein manifolds S to Oka manifolds O. In this section we recall the main results of Oka theory which are used in the study of minimal surfaces. For Stein manifolds, see any of the monographs [64, 70, 73] or [52, Ch. 2]. For Oka theory, see [52, Chs 5–7] and the surveys [50, 54]. A recent survey of holomorphic approximation theory is available in [46].

3.1. Stein manifolds. A complex manifold S is said to be a *Stein manifold* (named after Karl Stein who introduced this important class of complex manifolds in 1951) if it satisfies the following two conditions:

(1)   holomorphic functions on S separate any pair of distinct points;
(2)   if K is a compact subset of S, then so is its $O(S)$-convex hull

$$\widehat{K} = \{x \in S : |f(x)| \le \sup_K |f| \ \forall f \in O(S)\}.$$

A compact set $K \subset S$ is called $O(S)$-*convex* if $K = \widehat{K}$. If $S = \mathbb{C}^n$ then $\widehat{K}$ is the *polynomial hull* of K. Clearly, no manifold containing a compact complex submanifold of positive dimension is Stein.

The main example for the purposes of this paper is when $\dim S = 1$, that is, S is a Riemann surface. Every open Riemann surface is a Stein manifold according to Behnke and Stein (1949), and in this case the hull $\widehat{K}$ of any compact set $K \subset S$ is the union of K with all relatively compact connected components of its complement $S \setminus K$. Furthermore, the Cartesian product $S_1 \times S_2$ of a pair of Stein manifolds is Stein, and the total space E of any holomorphic vector bundle $E \to S$ over a Stein base S is Stein. A domain $\Omega \subset \mathbb{C}^n$ is Stein if and only if it is a domain of holomorphy, which holds if and only if it is pseudoconvex. In particular, every domain in $\mathbb{C}$ is Stein, and every convex domain in $\mathbb{C}^n$ for any $n \ge 1$ is Stein.

There are several other characterizations of the class of Stein manifolds. One is that a Stein n-manifold S embeds properly holomorphically into the Euclidean space $\mathbb{C}^{2n+1}$ (Remmert 1956, Bishop 1960, Narasimhan 1961; see [52, Theorem 2.4.1]); the converse is easily seen by restricting holomorphic polynomials to the embedded submanifold. Another characterization of Stein manifolds is by the existence of strongly plurisubharmonic exhaustion functions (see [52, Section 2.5]).

Axioms (1) and (2) say that a Stein manifold admits many holomorphic functions. More explicit manifestations of this phenomenon are the *Oka–Weil approximation theorem* and the *Oka–Cartan extension theorem*. The first one says that, given an $O(S)$-convex compact set K in a Stein manifold S and a holomorphic function f on a neighborhood of K, we can approximate f as closely as desired uniformly on K by global holomorphic functions on S. This generalizes the classical Runge theorem for functions on $\mathbb{C}$. (See the survey [46] for more information.) The second one says that for any closed complex subvariety S' of a Stein manifold S and holomorphic function $f : S' \to \mathbb{C}$ there exists a holomorphic function $F : S \to \mathbb{C}$ extending f, that is, $F|_{S'} = f$. If S is an open Riemann surface and S' is a discrete subset of S, this is the classical Weierstrass interpolation theorem [107] (see also [45]). One may combine the approximation and the interpolation statement, including also jet interpolation on a subvariety and continuous dependence on parameters; see [52, Theorem 2.8.4]. The same results hold for sections of any holomorphic vector bundle over a Stein manifold. These classical results, along with Cartan's theorems A and B (see [52, Section 2.6]), form the basis for analysis on Stein manifolds.

3.2. Oka theory. We may consider holomorphic functions on Stein manifolds as holomorphic maps $S \to \mathbb{C}$. Applying the above mentioned approximation and interpolation results componentwise, we can extend them to maps $S \to \mathbb{C}^N$ for any $N \in \mathbb{N}$. A completely different picture emerges for maps $S \to X$ to more general complex manifolds. For example, *Picard's theorem* says that there are no nonconstant holomorphic maps $\mathbb{C} \to \mathbb{C} \backslash \{0, 1\}$. On the other hand, Grauert proved in 1957–58 [62, 63] that the approximation and interpolation results still hold in the absence of topological obstructions for maps to complex homogeneous manifolds; the case when $X = \mathbb{C}_*$ is Oka's theorem from 1939.

THEOREM 3.1 (The Oka–Grauert theorem). *Assume that S is a Stein manifold, K is an $O(S)$-convex compact subset of S, S' is a closed complex subvariety of S, X is a complex homogeneous manifold, and $f : S \to X$ is a continuous map that is holomorphic on an open neighborhood of K and whose restriction $f|_{S'} : S' \to X$ is holomorphic. Then, f can be approximated uniformly on K by holomorphic maps $F : S \to X$ satisfying $F|_{S'} = f$. If in addition f is holomorphic on a neighborhood of S', then F can be chosen to agree with f to any given finite order along S'. The analogous result holds for sections $S \to E$ of any principal fiber bundle $\pi : E \to S$ over a Stein manifold S.*

In the theory of minimal surfaces, Theorem 3.1 is mainly used with X either the punctured null quadric $\mathfrak{A}_* \subset \mathbb{C}^n$, the intersection of $\mathfrak{A}_*$ with an affine complex hyperplane in $\mathbb{C}^n$, the punctured Euclidean space $\mathbb{C}^n_*$, or a projective space $\mathbb{CP}^n$. All these manifolds are complex homogeneous.

A complex manifold X satisfying the conclusion of Theorem 3.1 is called an *Oka manifold*. The class of Oka manifolds also contains many nonhomogeneous manifolds; see [52, Section 5.6 and Ch. 7]. The most general Oka principle for maps

from Stein manifolds to Oka manifolds is given by [52, Theorem 5.4.4] which also includes the parametric case, that is, families of maps depending continuously on a parameter in a compact Hausdorff space. It follows in particular that for every Stein manifold S and Oka manifold O, the natural inclusion $O(S, O) \hookrightarrow \mathscr{C}(S, O)$ is a weak homotopy equivalence (see [52, Corollary 5.5.6]), and is the inclusion of a strong deformation retract (hence a homotopy equivalence) if S is of finite analytic type in the sense that it admits a strongly plurisubharmonic exhaustion function with only finitely many critical points (see [52, Theorem 5.5.9] due to Lárusson). Note that an open Riemann surface S is of finite analytic type if and only if it is of finite topological type, that is, the homology group $H_1(S; \mathbb{Z})$ is finitely generated.

Theorem 3.1 and its extension to Oka manifolds also hold with Mergelyan type approximation; see [52, Corollaries 5.4.6 and 5.4.7] and [46].

A useful sufficient condition for a manifold X to be Oka is the existence of finitely many $\mathbb{C}$-complete holomorphic vector fields $V_1, \ldots, V_N$ on X which span the tangent space of X at any point. (If $X = G/H$ is a homogeneous manifold of a complex Lie group G, this holds for G-invariant holomorphic vector fields on X which are always complete.) The composition of their flows ϕ_t^j for complex values of t gives the map $\sigma \colon X \times \mathbb{C}^N \to X$, defined by

$$\sigma(x, t_1, \ldots, t_N) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_N}^N(x) \in X \qquad (3.1)$$

for $x \in X$ and $t = (t_1, \ldots, t_N) \in \mathbb{C}^N$, satisfying the domination condition

$$\left. \frac{\partial \sigma(x, t)}{\partial t} \right|_{t=0} : \mathbb{C}^N \to T_x X \quad \text{is surjective for every } x \in X. \qquad (3.2)$$

A holomorphic map $\sigma \colon X \times \mathbb{C}^N \to X$ satisfying $\sigma(x, 0) = x$ for all $x \in X$ and the domination condition (3.2) is called a *dominating spray* on X. More generally, we may take as the domain of the spray the total space E of any holomorphic vector bundle $\pi \colon E \to X$. Gromov proved in [67] that every complex manifold admitting a dominating spray is an Oka manifold. For more on this subject see [52, Ch. 6].

A *(holomorphic) dominating spray of maps* $S \to X$ is a holomorphic map $F \colon S \times V \to X$, where $V \subset \mathbb{C}^N$ is an open neighborhood of the origin in a complex Euclidean space, such that

$$\left. \frac{\partial F(s, t)}{\partial t} \right|_{t=0} : \mathbb{C}^N \longrightarrow T_{F(s,0)} X \quad \text{is surjective for every } s \in S.$$

The map $F_0 = F(\cdot, 0) \colon S \to X$ is called the *core map*, or simply the *core*, of F. If X admits a dominating spray $\sigma \colon X \times \mathbb{C}^N \to X$, then for any holomorphic map $f \colon S \to X$, the map $F \colon S \times \mathbb{C}^N \to X$ given by

$$F(s, t) = \sigma(f(s), t) \in X, \quad s \in S, \ t \in \mathbb{C}^N,$$

is a dominating spray of maps with the core $F_0 = f$. For instance, if σ is of the type (3.1) defined by flows ϕ_t^j of complete holomorphic vector fields, then

$$F(s, t_1, \ldots, t_N) = \phi_{t_1}^1 \circ \cdots \circ \phi_{t_N}^N(f(s)) \in X, \quad s \in S, \ t \in \mathbb{C}^N. \qquad (3.3)$$

In general, globally defined dominating sprays with a given core do not exist unless X is an Oka manifold. However, for every holomorphic map $f: S \to X$ from a Stein manifold S to an arbitrary complex manifold X and for any compact subset $K \subset S$ there exist a Stein neighborhood $U \Subset S$ of K and a dominating spray $F: U \times V \to X$ of the form (3.3) with $F(\cdot, 0) = f|_U$, where V is a neighborhood of the origin in some $\mathbb{C}^N$. Such F is obtained by composing flows of (not necessarily complete) holomorphic vector fields on X defined on a neighborhood $\Omega \subset S \times X$ of the graph $G_f(U) = \{(s, f(s)) : s \in U\} \subset S \times X$ of $f|_U$. Note that $G_f(U)$ admits an open Stein neighborhood in $S \times X$ by Siu's theorem (see [52, Theorem 3.1.1]), and the rest follows from Cartan's theorem A on Stein manifolds.

3.3. Period dominating sprays of maps into the null quadric.

Let M be a compact connected bordered Riemann surface with boundary bM. Denote by $g \geq 0$ the genus of M and by $m \geq 1$ the number of its boundary components; hence $H_1(M; \mathbb{Z}) \cong \mathbb{Z}^l$ with $l = 2g + m - 1$. We may assume that M is a smoothly bounded domain in an open Riemann surface R. For a fixed choice of a nowhere vanishing holomorphic 1-form θ on R and of a basis $\{C_j\}_{j=1}^l$ of $H_1(M; \mathbb{Z})$ we let

$$\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l) : \mathscr{A}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l = \mathbb{C}^{ln} \tag{3.4}$$

be the *period map* whose jth component equals

$$\mathcal{P}_j(f) = \oint_{C_j} f\theta \in \mathbb{C}^n, \quad f \in \mathscr{A}(M, \mathbb{C}^n). \tag{3.5}$$

Note that the holomorphic 1-form $f\theta$ on M is exact if and only if $\mathcal{P}(f) = 0$; this condition is clearly independent of the choice of a period basis.

Recall that $\mathfrak{A}_*$ denotes the punctured null quadric (2.5). The following lemma provides one of our main technical tools (see [14, Lemma 5.1] and [19, Lemma 3.2]).

LEMMA 3.2. *Given a nonflat map $f \in \mathscr{A}(M, \mathfrak{A}_*)$ (see Definition 2.6), there exist an open neighborhood V of the origin in $\mathbb{C}^{ln}$ and a map $\Phi_f: M \times V \to \mathfrak{A}_*$ of class $\mathscr{A}(M \times V, \mathfrak{A}_*)$ such that $\Phi_f(\cdot, 0) = f$ and*

$$\left.\frac{\partial}{\partial t}\right|_{t=0} \mathcal{P}(\Phi_f(\cdot, t)) : (\mathbb{C}^n)^l \to (\mathbb{C}^n)^l \quad \text{is an isomorphism.} \tag{3.6}$$

Furthermore, given a finite set $P \subset M$ and an integer $r \in \mathbb{N}$, we may choose Φ_f such that

$$\Phi_f(\cdot, t): M \to \mathfrak{A}_* \quad \text{agrees with } f \text{ to order } r \text{ at each } p \in P \text{ for all } t \in V. \tag{3.7}$$

Furthermore, there is a neighborhood Ω_f of f in $\mathscr{A}(M, \mathfrak{A}_*)$ such that the map $\Omega_f \ni g \mapsto \Phi_g$ depends holomorphically on g.

A map Φ_f satisfying Lemma 3.2 is called a *period dominating spray* of maps $M \to \mathfrak{A}_*$ with the core $\Phi_f(\cdot, 0) = f$.

PROOF. We first consider the case without paying attention to (3.7); the modification to ensure this matching condition will be explained at the end.

Let $C_1, \ldots, C_l \subset \mathring{M}$ be smooth oriented Jordan curves providing a homology basis for $H_1(M; \mathbb{Z})$ such that $C = \bigcup_{j=1}^{l} C_j$ is Runge in M (see Section 2.1). We may assume that the curves C_j have a single common point $p_0 \in M$, that is, $C_i \cap C_j = \{p_0\}$ for any $i \neq j$. Let $\mathcal{P} = (\mathcal{P}_1, \ldots, \mathcal{P}_l)$ be the associated period map (3.4), (3.5). Since f is nonflat, Lemma 2.7 and the identity principle show that for every $j = 1, \ldots, l$ there are points $p_{j,k} \in C_j \backslash \{p_0\}$ and holomorphic vector fields $V_{j,k}$ ($k = 1, \ldots, n$) on $\mathbb{C}^n$, tangent to the null quadric $\mathfrak{A}$, such that

$$\mathrm{span}\{V_{j,k}(x_{j,k}) : k = 1, \ldots, n\} = \mathbb{C}^n \quad \text{where } x_{j,k} = f(p_{j,k}). \tag{3.8}$$

Let $\phi_t^{j,k}$ denote the local holomorphic flow of $V_{j,k}$ for a complex time variable t. Write $t = (t_1, \ldots, t_l) \in (\mathbb{C}^n)^l$ where $t_j = (t_{j,1}, \ldots, t_{j,n}) \in \mathbb{C}^n$. For every $j = 1, \ldots, l$ and $k = 1, \ldots, n$ we pick a smooth function $h_{j,k} \colon C \to \mathbb{C}$, supported on a short arc around the point $p_{j,k} \in C_j$, and consider the map

$$\Phi(p, t) = \phi_{h_{1,1}(p)t_{1,1}}^{1,1} \circ \cdots \circ \phi_{h_{l,n}(p)t_{l,n}}^{l,n}(f(p)) \in \mathfrak{A}_*, \quad p \in C. \tag{3.9}$$

(We take the composition of flows $\phi_{h_{j,k}(p)t_{j,k}}^{j,k}$ for all $j = 1, \ldots, l$ and $k = 1, \ldots, n$.) Note that Φ is well-defined for all $t \in \mathbb{C}^{ln}$ sufficiently close to the origin and $\Phi(\cdot, 0) = f$. Clearly we have that

$$\left. \frac{\partial \Phi(p, t)}{\partial t_{j,k}} \right|_{t=0} = h_{j,k}(p) V_{j,k}(f(p)), \quad p \in C,$$

and hence

$$\left. \frac{\partial \mathcal{P}_i(\Phi(\cdot, t))}{\partial t_{j,k}} \right|_{t=0} = \oint_{C_i} h_{j,k}(V_{j,k} \circ f)\theta$$

for all $i, j = 1, \ldots, l$ and $k = 1, \ldots, n$. A suitable choice of the functions $h_{j,k}$ ensures that the above expression is as close as desired to $V_{j,k}(x_{j,k})$ if $i = j$, and it equals zero otherwise. In view of (3.8) it follows that the differential $\partial/\partial t|_{t=0}\mathcal{P}(\Phi(\cdot, t)) \colon \mathbb{C}^{ln} \to \mathbb{C}^{ln}$ has a block structure with vanishing off-diagonal $n \times n$ blocks and with invertible diagonal blocks; hence it is invertible. By Mergelyan's theorem we can approximate each function $h_{j,k}$ uniformly on C by a holomorphic function $\tilde{h}_{j,k} \in O(M)$. Inserting these new functions into the definition of Φ (3.9) we obtain a spray Φ_f of maps $M \to \mathfrak{A}_*$ satisfying the conclusion of the lemma. Indeed, (3.6) holds provided that the approximation of the functions $h_{j,k}$ by $\tilde{h}_{j,k}$ is close enough, and the other properties are obvious.

In order to ensure the condition (3.7), we choose the curves C_i in the homology basis such that they do not intersect the finite set P. Choose a funtion $g \in O(M)$ that vanishes to order $r + 1$ at each of the points in P and has no other zeros. We replace each of the functions $h_{j,k}$ in the spray (3.9) by the product $gh_{j,k}$. Proceeding as before, we obtain a new spray of the same type with $h_{j,k} \in O(M)$. It is elementary to see that the map $(p, t) \to \phi_{g(p)h_{j,k}(p)t}^{j,k}(f(p))$ is tangent to f to order r at every point $p \in P$ (see [5, Lemma 2.2]); hence the same holds for their composition Φ_f. $\square$

REMARK 3.3. By using additional flows in the definition of Φ_f (3.9) we can ensure that the spray Φ_{f_q} is period dominating for a given continuous family $\{f_q : q \in Q\}$ of holomorphic maps $f_q : M \to \mathfrak{A}_*$ with the parameter in a compact Hausdorff space Q. In this case, condition (3.6) is replaced by asking that the t-differential of the period map is surjective at $t = 0$. On the other hand, we are unable to find a period dominating spray whose core is a flat map since the tangent spaces to $\mathfrak{A}_*$ are constant along a complex ray of $\mathfrak{A}_*$, and hence they do not span $\mathbb{C}^n$.

REMARK 3.4. Proofs of Lemmas 3.2 and 3.7 extend in an obvious way to the case when $M = K \cup \Gamma$ is an admissible set in an open Riemann surface R; see Definition 2.8. A map $f : M \to \mathfrak{A}_*$ of class $\mathscr{A}(M)$ is said to be nonflat or full if the restriction of f to K and to each connected component of Γ is nonflat or full, respectively. Such f typically arises as the derivative map $f = 2\partial\Phi/\theta : M \to \mathfrak{A}_*$ of a generalized conformal minimal immersion on an admissible set; see Definition 2.9.

The following result (see [19, Theorem 3.1]) is a straightforward application of Lemma 3.2. The notation has been established in Section 2.8.

THEOREM 3.5. *Let M be a compact bordered Riemann surface with nonempty boundary bM, and let $n \geq 3$ and $r \geq 1$ be integers. Then the following hold.*

(a)   *The space $\mathrm{CMI}^r_{\mathrm{nf}}(M, \mathbb{R}^n)$ is a real analytic Banach manifold.*
(b)   *The space $\mathrm{NC}^r_{\mathrm{nf}}(M, \mathbb{C}^n)$ is a complex Banach manifold.*

We do not know whether the spaces $\mathrm{CMI}^r(M, \mathbb{R}^n)$ and $\mathrm{NC}^r(M, \mathbb{C}^n)$ are also Banach manifolds. In fact, it seems that flat conformal minimal immersions and holomorphic null curves are singular points of these spaces.

PROOF. By [49, Theorem 1.1] the space $\mathscr{A}^{r-1}(M, \mathfrak{A}_*)$ is a complex Banach manifold modeled on the complex Banach space $\mathscr{A}^{r-1}(M, \mathbb{C}^{n-1})$, where $\dim \mathfrak{A}_* = n - 1$. Let $\mathcal{P} : \mathscr{A}^{r-1}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ denote the holomorphic period map (3.4). Set

$$\mathscr{A}^{r-1}_0(M, \mathfrak{A}_*) = \{f \in \mathscr{A}^{r-1}(M, \mathfrak{A}_*) : \mathfrak{R}(\mathcal{P}(f)) = 0\},$$

and let $\mathscr{A}^{r-1}_{0,\mathrm{nf}}(M, \mathfrak{A}_*)$ denote the open subset of $\mathscr{A}^{r-1}_0(M, \mathfrak{A}_*)$ consisting of all nonflat maps (see Definition 2.6). Lemma 3.2 implies that the differential $d\mathcal{P}_{f_0}$ at any point $f_0 \in \mathscr{A}^{r-1}_{0,\mathrm{nf}}(M, \mathfrak{A}_*)$ has maximal rank equal to ln. By the implicit function theorem, f_0 admits an open neighborhood $\Omega \subset \mathscr{A}^{r-1}(M, \mathfrak{A}_*)$ such that $\Omega \cap \mathscr{A}^{r-1}_0(M, \mathfrak{A}_*) = \Omega \cap \mathscr{A}^{r-1}_{0,\mathrm{nf}}(M, \mathfrak{A}_*)$ is a real analytic Banach submanifold of Ω, parametrized by the kernel of the real part $\mathfrak{R}(d\mathcal{P}_{f_0})$ of the differential of $\mathcal{P}$ at f_0; this is a real codimension ln subspace of the complex Banach space $\mathscr{A}^{r-1}(M, \mathbb{C}^{n-1})$ (the tangent space of the complex Banach manifold $\mathscr{A}^{r-1}(M, \mathfrak{A}_*)$). This shows that $\mathscr{A}^{r-1}_{0,\mathrm{nf}}(M, \mathfrak{A}^*)$ is a real analytic Banach manifold. The integration $p \mapsto v + \int_{p_0}^p \mathfrak{R}(f\theta)$ ($p \in M$), with an arbitrary choice of the value $v \in \mathbb{R}^n$ at a base point $p_0 \in M$, provides an isomorphism between the Banach manifold $\mathscr{A}^{r-1}_{0,\mathrm{nf}}(M, \mathfrak{A}^*) \times \mathbb{R}^n$ and $\mathrm{CMI}^r_{\mathrm{nf}}(M, \mathbb{R}^n)$, so the latter is also a real analytic Banach manifold. This proves (a). Essentially the same argument applies in case (b).   □

We now give another version of Lemma 3.2 in which a period dominating spray is obtained by multiplying the given core map $f \colon M \to \mathfrak{A}_*$ by a nonvanishing holomorphic function, called a *multiplier*.

A path $f \colon I = [0, 1] \to \mathbb{C}^n$ is said to be *full* if the $\mathbb{C}$-linear span of its image equals $\mathbb{C}^n$. Let $\mathcal{P} \colon \mathscr{C}(I, \mathbb{C}^n) \to \mathbb{C}^n$ denote the map

$$\mathcal{P}(f) = \int_0^1 f(s) \, ds \in \mathbb{C}^n, \quad f \in \mathscr{C}(I, \mathbb{C}^n).$$

The following result is [18, Lemma 2.1].

LEMMA 3.6. *Assume that I' is a nontrivial closed subinterval of $I = [0, 1]$ and Q is a compact Hausdorff space. Given a continuous map $f \colon I \times Q \to \mathbb{C}^n$ such that $f(\cdot, q)$ is full on I' for every $q \in Q$, there exist finitely many continuous functions $g_1, \ldots, g_N \colon I \to \mathbb{C}$, supported on I', such that the function $h \colon I \times \mathbb{C}^N \to \mathbb{C}$ given by*

$$h(s, t) = 1 + \sum_{i=1}^N t_i g_i(s), \quad s \in I, \ t = (t_1, \ldots, t_N) \in \mathbb{C}^N, \tag{3.10}$$

is a period dominating multiplier *of f, in the sense that*

$$\frac{\partial}{\partial t} \mathcal{P}(h(\cdot, t) f(\cdot, q)) \big|_{t=0} : \mathbb{C}^N \to \mathbb{C}^n \text{ is surjective for every } q \in Q. \tag{3.11}$$

PROOF. Let $N \geq n$ be an integer and, for each $i \in \{1, \ldots, N\}$, let $g_i \colon I \to \mathbb{C}$ be a continuous function; both the number N and the functions g_i will be specified later. Let h be defined by (3.10). Note that $\partial h(s, t)/\partial t_i \big|_{t=0} = g_i(s)$ and hence

$$\frac{\partial}{\partial t_i} \mathcal{P}(h(\cdot, t) f(\cdot q)) \big|_{t=0} = \int_0^1 \frac{\partial h(s, t)}{\partial t_i} \big|_{t=0} f(s, q) \, ds = \int_0^1 g_i(s) f(s, q) \, ds. \tag{3.12}$$

Since $f(\cdot, q)$ is full on I' for every $q \in Q$, compactness of Q and continuity of f ensure that there are points $s_1, \ldots, s_N \in \mathring{I}'$ for a big $N \in \mathbb{N}$ such that

$$\mathrm{span}\{f(s_1, q), \ldots, f(s_N, q)\} = \mathbb{C}^n \quad \text{for all } q \in Q. \tag{3.13}$$

Pick a small $\epsilon > 0$ and for every $i = 1, \ldots, N$ a continuous function $g_i \colon I \to \mathbb{C}$ supported on $(s_i - \epsilon, s_i + \epsilon) \subset I$ such that

$$\int_0^1 g_i(s) \, ds = \int_{s_i - \epsilon}^{s_i + \epsilon} g_i(s) \, ds = 1. \tag{3.14}$$

For small $\epsilon > 0$ we have in view of (3.12) and (3.14) that

$$\frac{\partial \mathcal{P}(h(\cdot, t) f(\cdot q))}{\partial t_i} \big|_{t=0} = \int_0^1 g_i(s) f(s, q) \, ds \approx f(s_i, q)$$

for all $q \in Q$ and $i \in \{1, \ldots, N\}$. Assuming as we may that the approximations are close enough, it follows from (3.13) that (3.11) holds. $\square$

By using Lemma 3.6 we easily obtain the following result which is essentially [18, Lemma 3.2]. Here, $\mathcal{P}\colon \mathscr{A}(M, \mathbb{C}^n) \to (\mathbb{C}^n)^l$ again denotes the period map (3.4), (3.5). The same result holds if M is a compact admissible subset of an open Riemann surface; see Definition 2.8 and Remark 3.4.

LEMMA 3.7. *Let M be a compact bordered Riemann surface with the homology group $H_1(M; \mathbb{Z}) = \mathbb{Z}^l$, let θ be a nowhere vanishing holomorphic 1-form on M, and let Q be a compact Hausdorff space. Given a continuous map $f\colon M \times Q \to \mathbb{C}^n$ such that $f(\cdot, q)\colon M \to \mathbb{C}^n$ is a full map of class $\mathscr{A}(M)$ for every $q \in Q$, there exist finitely many holomorphic functions $g_1, \ldots, g_N \in O(M)$ such that the function $h\colon M \times \mathbb{C}^N \to \mathbb{C}$, given by*

$$h(p, t) = 1 + \sum_{i=1}^{N} t_i g_i(p), \quad t = (t_1, \ldots, t_N) \in \mathbb{C}^N, \ p \in M,$$

is a period dominating multiplier *of f, meaning that*

$$\frac{\partial}{\partial t} \mathcal{P}(h(\cdot, t) f(\cdot, q)) \Big|_{t=0} \colon \mathbb{C}^N \to (\mathbb{C}^n)^l \quad \text{is surjective for every } q \in Q. \tag{3.15}$$

3.4. Paths with prescribed periods in the null quadric. In this section we present a construction of paths with prescribed integrals in the punctured null quadric. An elementary result concerning a single path is [14, Lemma 7.3]. The parametric version [55, Lemma 3.1] is needed in the investigation of the homotopy structure of the spaces $\mathrm{CMI}(M, \mathbb{R}^n)$ and $\mathrm{NC}(M, \mathbb{C}^n)$; see Section 4.4. Here we present a one-parametric version, [18, Lemma 2.3], which has the advantage of preserving the Gauss map, so it can be used to construct conformal minimal immersions with prescribed complex Gauss map (see Section 2.6).

LEMMA 3.8. *Set $I = [0, 1]$. Given continuous maps $\alpha\colon I \to \mathbb{C}^n$ and $f\colon I \times I \to \mathbb{C}^n$ such that the path $f_t := f(\cdot, t)\colon I \to \mathbb{C}^n$ is full for every $t \in I$, there exists a continuous function $h\colon I \times I \to \mathbb{C}_*$ such that $h(s, t) = 1$ for all $t \in I$ and $s \in \{0, 1\}$, and*

$$\int_0^1 h(s, t) f(s, t)\, ds = \alpha(t), \quad t \in [0, 1]. \tag{3.16}$$

If in addition we have that $\int_0^1 f(s, 0)\, ds = \alpha(0)$, then h can be chosen such that $h(s, 0) = 1$ for all $s \in [0, 1]$.

REMARK 3.9. If the map f in Lemma 3.8 has a range in the punctured null quadric $\mathfrak{A}_*$, then the same holds for the map hf for any nowhere vanishing function h. The analogous conclusion holds when f has a range in any conical complex subvariety of $\mathbb{C}^n$.

PROOF. We begin by explaining a reduction to the case when the exact condition (3.16) is replaced by an approximate condition

$$\left| \int_0^1 h(s, t) f(s, t)\, ds - \alpha(t) \right| < \epsilon, \quad t \in [0, 1], \tag{3.17}$$

where $\epsilon > 0$ is any given number. Indeed, since the path f_t is full for every $t \in I$, we can divide the t-interval I into finitely many subintervals $I_1, \ldots, I_m$ such that for every $i = 1, \ldots, m$ there is a closed subinterval $J_i \subset I$ such that the restricted path $f_t \colon J_i \to \mathbb{C}^n$ is full for every $t \in I_i$. Clearly it suffices to consider the problem separately on each I_i. Hence, replacing I by I_i, we may assume that there is a closed subinterval $J \subset I$ such that $f_t \colon J \to \mathbb{C}^n$ is full for every $t \in I$. Choose nontrivial disjoint subintervals $J_1, J_2 \subset J$. Replacing the s-interval I by J_1, it suffices to prove that for any given $\epsilon > 0$ there is a function $h \colon I \times I \to \mathbb{C}_*$ satisfying (3.17). Choosing ϵ small enough, we can then correct the small error and obtain (3.16) by applying the period dominating argument, furnished by Lemma 3.6, on the subinterval J_2.

It remains to explain the construction of a function h satisfying (3.17). Since f_t is full for each $t \in I$, there is a division $0 = s_0 < s_1 < \cdots < s_N = 1$ of I such that

$$\mathrm{span}\{f_t(s_1), \ldots, f_t(s_N)\} = \mathbb{C}^n \quad \text{for all } t \in I.$$

Set

$$V_j(t) = \int_{s_{j-1}}^{s_j} f_t(s)\, ds, \quad j = 1, \ldots, N.$$

Note that $V_j(t)$ is close to $f_t(s_j)(s_j - s_{j-1})$ if the intervals $[s_{j-1}, s_j]$ are short. Passing to a finer division if necessary we may therefore assume that

$$\mathrm{span}\{V_1(t), \ldots, V_N(t)\} = \mathbb{C}^n, \quad t \in I.$$

For each $t \in I$ let $\Sigma_t \subset \mathbb{C}^N$ denote the affine complex hyperplane defined by

$$\Sigma_t = \left\{ (g_1, \ldots, g_N) \in \mathbb{C}^N : \sum_{j=1}^{N} g_j V_j(t) = \alpha(t) \right\}.$$

Clearly, there exists a continuous map $g = (g_1, \ldots, g_N) \colon I \to \mathbb{C}^N$ such that $g(t) \in \Sigma_t$ for every $t \in I$. (We may view g as a section of the affine bundle over I whose fiber over the point t equals Σ_t.) This can be written as follows:

$$\sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} g_j(t) f_t(s)\, ds = \alpha(t), \quad t \in I. \tag{3.18}$$

Note that $\sum_{j=1}^{N} V_j(t) = \int_0^1 f_t(s)\, ds$. Hence, if $\int_0^1 f(0, s)\, ds = \alpha(0)$ then g can be chosen such that $g(0) = (1, \ldots, 1) \in \mathbb{C}^N$. We assume in the following that this holds since the proof is even simpler otherwise.

By a small perturbation we may assume that $g_j(t) \in \mathbb{C}_*$ for every $t \in I$ and $j = 1, \ldots, N$. This changes the exact condition (3.18) to the approximate condition

$$\left| \sum_{j=1}^{N} \int_{s_{j-1}}^{s_j} g_j(t) f_t(s)\, ds - \alpha(t) \right| < \frac{\epsilon}{2}, \quad t \in I. \tag{3.19}$$

For a fixed $t \in I$ we consider the vector $g(t) = (g_j(t))_j \in \mathbb{C}^N$ as a step function of $s \in I$ which equals the constant $g_j(t)$ on $s \in [s_{j-1}, s_j)$ for every $j = 1, \ldots, N$. We now approximate this step function by a continuous function $h_t = h(\cdot, t) \colon I \to \mathbb{C}_*$ which agrees with the step function, except near the points $s_0, s_1, \ldots, s_N$, ensuring also that $h_t(0) = h_t(1) = 1$. Here are the details. Let $C > 1$ be chosen such that

$$\max_{(s,t) \in I \times I} |f(s,t)| \leq C, \qquad \max_{t \in I, \, j=1,\ldots,N} |g_j(t)| \leq C.$$

Pick a number $\eta > 0$ such that

$$4C(C+1)N\eta < \epsilon. \tag{3.20}$$

For each $t \in I$ and $j = 1, \ldots N$ we define the function $h(\cdot, t) \colon [s_{j-1}, s_j] \to \mathbb{C}_*$ by

$$h(s,t) = \begin{cases} g_j((s - s_{j-1})t/\eta), & s \in [s_{j-1}, s_{j-1} + \eta], \\ g_j(t), & s \in [s_{j-1} + \eta, s_j - \eta], \\ g_j((s_j - s)t/\eta), & s \in [s_j - \eta, s_j]. \end{cases}$$

Thus, $h(s,t)$ spends most of its time (for $s \in [s_{j-1} + \eta, s_j - \eta]$) at the point $g_j(t)$, and it travels between the point $1 \in \mathbb{C}_*$ (where it is at the endpoints $s = s_{j-1}$ and $s = s_j$) and the point $g_j(t)$ along the trace of the path $\tau \mapsto g_j(\tau t) \in \mathbb{C}_*$. This defines a continuous function $h \colon I \times I \to \mathbb{C}_*$ satisfying

$$|h(s,t)| \leq C \quad \text{for all } (s,t) \in I \times I. \tag{3.21}$$

It follows easily from (3.19), (3.20), and (3.21) that the replacement of the step function $g(t) = (g_j(t))_j$ by $h(s,t)$ causes an error of size $< \epsilon/2$. This yields the estimate (3.17). □

3.5. Transversality methods for conformal minimal surfaces.

In this section we indicate how the techniques of Section 3.3, especially Lemma 3.2, can be used to prove the following general position theorem for conformal minimal immersions of bordered Riemann surfaces. The original reference is [19, Theorem 4.1].

THEOREM 3.10. *Let M be a compact bordered Riemann surface and $r \in \mathbb{N}$. Every conformal minimal immersion $X \in \mathrm{CMI}^r(M, \mathbb{R}^n)$ for $n \geq 5$ can be approximated arbitrarily closely in the $\mathscr{C}^r(M)$ norm by a conformal minimal embedding $\widetilde{X} \in \mathrm{CMI}^r(M, \mathbb{R}^n)$ satisfying $\mathrm{Flux}_{\widetilde{X}} = \mathrm{Flux}_X$. If $n = 4$ then X can be approximated by conformal minimal immersions with simple (transverse) double points.*

Since the set of embeddings $M \to \mathbb{R}^n$ is clearly open in the set of immersions of class $\mathscr{C}^r(M)$ for any $r \geq 1$ and $\mathrm{CMI}^r(M, \mathbb{R}^n)$ is a closed subset of the Banach space $\mathscr{C}^r(M, \mathbb{R}^n)$ (hence a Baire space), the following corollary is immediate.

COROLLARY 3.11. *Let M be a compact bordered Riemann surface. For every pair of integers $n \geq 5$ and $r \geq 1$ the set of conformal minimal embeddings $M \hookrightarrow \mathbb{R}^n$ of class $\mathscr{C}^r(M)$ is residual (of the second category) in the Baire space $\mathrm{CMI}^r(M, \mathbb{R}^n)$. The same holds for the set of conformal minimal immersions $M \to \mathbb{R}^4$ with simple double points.*

SKETCH OF PROOF OF THEOREM 3.10. We may assume that M is a smoothly bounded domain in an open Riemann surface R and X is a nonflat conformal minimal immersion in an open neighborhood of M in R. We associate to X the *difference map* $\delta X \colon M \times M \to \mathbb{R}^n$, defined by

$$\delta X(p, q) = X(q) - X(p), \quad p, q \in M.$$

Clearly, X is injective if and only if $(\delta X)^{-1}(0) = D_M := \{(p, p) : p \in M\}$. Since X is an immersion, it is locally injective, and hence there is an open neighborhood $U \subset M \times M$ of the diagonal D_M such that δX does not assume the value $0 \in \mathbb{R}^n$ on $\overline{U} \setminus D_M$. To prove the theorem, it suffices to find arbitrarily close to X a conformal minimal immersion $\widetilde{X} \colon M \to \mathbb{R}^n$ whose difference map $\delta \widetilde{X}$ is transverse to the origin $0 \in \mathbb{R}^n$ on $M \times M \setminus U$. (Since $\dim_{\mathbb{R}} M \times M = 4 < n$, this will imply that $\delta \widetilde{X}$ does not assume the value zero on $M \times M \setminus U$, so $\widetilde{X}(p) \neq \widetilde{X}(q)$ if $(p, q) \in M \times M \setminus U$. If on the other hand $(p, q) \in U \setminus D_M$ then $\widetilde{X}(p) \neq \widetilde{X}(q)$ provided that $\widetilde{X}$ is close enough to X, so $\widetilde{X}$ is an embedding.) To obtain such $\widetilde{X}$, we find a neighborhood $\Omega \subset \mathbb{R}^N$ of the origin in a Euclidean space and a real analytic map $H \colon \Omega \times M \to \mathbb{R}^n$ satisfying the following conditions:

(a) $H(0, \cdot) = X$;
(b) $H(\xi, \cdot) \in \mathrm{CMI}^r(M, \mathbb{R}^n)$ for every $\xi \in \Omega$; and
(c) the difference map $\delta H \colon \Omega \times M \times M \to \mathbb{R}^n$, defined by

$$\delta H(\xi, p, q) = H(\xi, q) - H(\xi, p), \quad \xi \in \Omega, \quad p, q \in M,$$

is a submersive family on $M \times M \setminus U$, in the sense that the partial differential

$$\partial_\xi |_{\xi=0} \, \delta H(\xi, p, q) \colon \mathbb{R}^N \to \mathbb{R}^n \tag{3.22}$$

is surjective for every $(p, q) \in M \times M \setminus U$.

For the details of the construction of H, see [19, Theorem 4.1]; one uses Lemma 3.2 and the implicit function theorem. Assume now that such H exists. By compactness of $M \times M \setminus U$, the partial differential $\partial_\xi (\delta H)$ (3.22) is surjective for all ξ in a neighborhood $\Omega' \subset \Omega$ of the origin in $\mathbb{R}^N$. Hence, the map $\delta H \colon M \times M \setminus U \to \mathbb{R}^n$ is transverse to any submanifold of $\mathbb{R}^n$, in particular, to the origin $0 \in \mathbb{R}^n$. The transversality argument due to Abraham [1] (see also [52, Section 8.8]) shows that for a generic choice of $\xi \in \Omega'$, the difference map $\delta H(\xi, \cdot, \cdot)$ is transverse to $0 \in \mathbb{R}^n$ on $M \times M \setminus U$, and hence it omits the value 0 by dimension reasons. Choosing ξ close enough to $0 \in \mathbb{R}^N$ we thus obtain a conformal minimal embedding $\widetilde{X} = H(\xi, \cdot) \colon M \to \mathbb{R}^n$ close to X, thereby proving the theorem. $\square$

4. Conformal minimal immersions: approximation, interpolation, embeddings, and isotopies

At the dawn of the 21st century, not much was known about how to deform a given minimal surface in $\mathbb{R}^n$ into another one with more desirable properties. At that time we only counted on a few techniques which had been created ad hoc in order to settle

specific problems. This is the case for instance of the *López–Ros deformation* for minimal surfaces $X\colon M \to \mathbb{R}^3$ (see [80]) which amounts to multiplying the complex Gauss map g_X by a nowhere vanishing holomorphic function, subject to suitable period vanishing conditions on the Weierstrass data. Its main shortcoming is that one needs the initial conformal minimal immersion X already defined everywhere on M; let us point out that, at that time, few open Riemann surfaces were known to be the underlying complex structure of a minimal surface in $\mathbb{R}^3$.

The implementation of the complex analytic tools from Sections 3.2–3.5, and also those to be explained in Section 5, gave rise to the birth and development of the theories of approximation, interpolation, and isotopies for conformal minimal immersions $M \to \mathbb{R}^n$ from any given open Riemann surface M, leading to an array of new results. In this section we discuss both the foundations of the aforementioned theories and some of their applications. Results depending on the Riemann–Hilbert boundary value problem (see, for example, Theorem 1.3) are treated in the following section.

4.1. Runge approximation with jet interpolation for conformal minimal immersions.

The following is one of the main new tools for the construction of minimal surfaces in $\mathbb{R}^n$ for any $n \geq 3$ with interesting global properties and arbitrary conformal structure. It is analogous in spirit to the combination of the Runge approximation theorem and the Weierstrass interpolation theorem for holomorphic maps from open Riemann surfaces to $\mathbb{C}^n$.

THEOREM 4.1 (Runge approximation with jet interpolation for conformal minimal surfaces). *Let M be an open Riemann surface, $\Lambda \subset M$ be a closed discrete subset, and $K \subset M$ be a smoothly bounded compact Runge domain. For each $p \in \Lambda$ let $\Omega_p \subset M$ be a neighborhood of p such that $\Omega_p \cap \Omega_q = \varnothing$ for all $p \neq q \in \Lambda$, and set $\Omega := \bigcup_{p \in \Lambda} \Omega_p$. Given a function $r\colon \Lambda \to \mathbb{N}$, every conformal minimal immersion $X\colon K \cup \Omega \to \mathbb{R}^n$ ($n \geq 3$) can be approximated uniformly on K by conformal minimal immersions $\widetilde{X}\colon M \to \mathbb{R}^n$ having a contact of order $r(p)$ with X at every point $p \in \Lambda$.*

REMARK 4.2. In fact, more is true: the conformal minimal immersions $\widetilde{X}\colon M \to \mathbb{R}^n$ in Theorem 4.1 can be chosen complete (see Theorem 4.5), and if the map $X\colon \Lambda \to \mathbb{R}^n$ is proper (this holds in particular if Λ is finite) then $\widetilde{X}$ can also be chosen proper (see Theorems 4.8 and 4.9). As we shall see in the proof, one can also obtain Mergelyan approximation on admissible sets (see Definition 2.8). By using the general position argument in Theorem 3.10, one easily sees that the immersions $\widetilde{X} \in \mathrm{CMI}(M, \mathbb{R}^n)$ can be chosen embeddings if $n \geq 5$, immersions with simple double points if $n = 4$, and to have prescribed flux compatible with the flux of the initial immersion X for any loop in K.

REMARK 4.3. The analogous Runge approximation theorems with jet interpolation holds for holomorphic null curves and, more generally, for holomorphic immersions $M \to \mathbb{C}^n$ directed by any conical complex subvariety $A \subset \mathbb{C}^n$ such that $A \setminus \{0\}$ is an Oka manifold (see [14, Theorems 7.2 and 7.7] and [5, Theorem 1.3]). Here we say that a holomorphic immersion $Z\colon M \to \mathbb{C}^n$ is *directed by A*, or an *A-immersion*, if

$(dZ/\theta)(M) \subset A\backslash\{0\}$, where θ is any nowhere vanishing holomorphic 1-form on M. Null holomorphic immersions correspond to the case when $A = \mathfrak{A}$ (2.4).

Theorem 4.1 is a compilation of results from the paper [23] by Alarcón and López where the existence and approximation was proved for conformal minimal immersions into $\mathbb{R}^3$, the paper [19] by the authors and López where the same was done in any dimension $n \geq 3$, and the paper [5] by Alarcón and Castro-Infantes where interpolation was added.

SKETCH OF PROOF OF THEOREM 4.1. We assume that the function $r: \Lambda \to \mathbb{N}$ is constant; the general case is obtained by an obvious modification. Pick a smooth strongly subharmonic Morse exhaustion function $\rho: M \to \mathbb{R}$ and exhaust M by an increasing sequence

$$K = M_1 \Subset M_2 \Subset \cdots \Subset \bigcup_{i=1}^{\infty} M_i = M \tag{4.1}$$

of compact smoothly bounded domains $M_i = \{p \in M : \rho(p) \leq c_i\}$, where $c_1 < c_2 < \cdots$ is an increasing sequence of regular values of ρ with $\lim_{i\to\infty} c_i = +\infty$. Thus, each M_i is a possibly disconnected compact bordered Riemann surface. For convenience of exposition we also assume that ρ has at most one critical point p_i in each difference $\mathring{M}_{i+1}\backslash M_i$ and no point of Λ is a critical point of ρ. It follows that M_i is Runge in M for every $i \in \mathbb{N}$. Set $\Lambda_i = \Lambda \cap M_i$ for $i \in \mathbb{N}$; this is a finite set since $\Lambda \subset M$ is closed and discrete. Up to enlarging K and Λ if necessary, we may assume that ρ is chosen such that $\Lambda \cap bM_i = \varnothing$ and $\Lambda_{i+1}\backslash\Lambda_i$ consists of a single point for all $i \in \mathbb{N}$.

Set $X_1 = X|_{M_1}$ and assume as we may that X_1 is nonflat. We shall inductively construct a sequence of nonflat conformal minimal immersions $\{X_i \in \mathrm{CMI}(M_i)\}_{i\geq 2}$ satisfying the following conditions.

(a) X_i is as close to X_{i-1} as desired in the $\mathscr{C}^1(M_{i-1})$ topology for all $i \geq 2$.
(b) X_i and X have a contact of order r at every point in Λ_i.

It is clear that if the approximations in (a) are close enough then the limit $\widetilde{X} = \lim_{i\to\infty} X_i : M \to \mathbb{R}^n$ satisfies the conclusion of the theorem.

The basis of the induction is given by the map X_1. Assume that we already have the immersion X_i for some $i \in \mathbb{N}$. We consider two different cases depending on the topology of $M_{i+1}\backslash M_i$.

The noncritical case: ρ has no critical value in $[c_i, c_{i+1}]$. In this case M_i is a strong deformation retract of M_{i+1}. We may assume that M_i is connected; otherwise we apply the same argument in each connected component. Set $f_i = 2\partial X_i/\theta : M_i \to \mathfrak{A}_*$, write $\Lambda_i = \{q_1, \ldots, q_k\}$, and denote by q_0 the only point in $\Lambda_{i+1}\backslash\Lambda_i$. Pick a point $p_0 \in \mathring{M}_i\backslash\Lambda$ and choose a family of smooth Jordan arcs $\alpha_0, \alpha_1, \ldots, \alpha_k$ in $\mathring{M}_{i+1}$ and smooth Jordan curves $\alpha_{k+1}, \ldots, \alpha_{k+l}$ in $\mathring{M}_i$ ($l = \dim H_1(M_i; \mathbb{Z})$) satisfying the following conditions.

• $\alpha_a \cap \alpha_b = \{p_0\}$ for all $a \neq b \in \{0, \ldots, k + l\}$.

- The endpoints of α_a are p_0 and q_a for all $a \in \{0, \ldots, k\}$. We orient each α_a so that p_0 is its initial point and q_a is its final point.
- The curves $\alpha_{k+1}, \ldots, \alpha_{k+l}$ determine a homology basis of M_i.
- $\Upsilon = \bigcup_{a=0}^{k+l} \alpha_a$ is a Runge set in M.
- The set $S = M_i \cup \Upsilon = M_i \cup \Gamma$, where $\Gamma = \bigcup_{a=0}^{k} \alpha_a$, is admissible in M (see Definition 2.8).

By Lemma 3.8 we can extend $X_i \colon M_i \to \mathbb{R}^n$ to a generalized conformal minimal immersion $(\widetilde{X}_i, f_i \theta) \colon S \to \mathbb{R}^n$ (see Definition 2.9) such that $\widetilde{X}_i = X$ on Λ_{i+1} and on a neighborhood of q_0; this is possible by condition (b) for the index i. (Here, θ is a nowhere vanishing holomorphic 1-form on M.) Consider the period map

$$\mathcal{P}(f) = \left(\int_{\alpha_a} f\theta \right)_{a=0}^{k+l}, \quad f \in \mathscr{A}(S, \mathbb{C}^n).$$

Lemma 3.2 and Remark 3.4 furnish a period dominating spray of maps $f_{i;w} \colon S \to \mathfrak{A}_*$ of class $\mathscr{A}(S)$ with core $f_{i;0} = f_i$, depending holomorphically on a parameter w in a ball $B \subset \mathbb{C}^N$ for some $N \in \mathbb{N}$, such that $f_{i;w}$ and f_i have a contact of order r at every point in Λ_{i+1}. Since $\mathfrak{A}_*$ is a complex homogeneous manifold and S is Runge in M and a deformation retract of M_{i+1}, we may apply Theorem 3.1 to approximate $f_{i;w}$ uniformly on M_i and uniformly with respect to $w \in B$ (shrinking B slightly if necessary) by a holomorphic spray of holomorphic maps $g_w \colon M_{i+1} \to \mathfrak{A}_*$ having a contact of order r with f_i at every point in Λ_{i+1}. Assuming that the approximation is close enough, the period domination condition of $f_{i;w}$ and the implicit function theorem give a point $w_0 \in B$ close to $0 \in \mathbb{C}^N$ such that $\mathcal{P}(g_{w_0}) = \mathcal{P}(f_i)$. The conformal minimal immersion defined by

$$X_{i+1}(p) = X_i(p_0) + \int_{p_0}^{p} \mathfrak{R}(g_{w_0}\theta), \quad p \in M_{i+1},$$

then satisfies conditions (a) and (b) for the index $i + 1$.

The critical case: ρ has a unique (Morse) critical point $p_{i+1} \in M_{i+1} \backslash M_i$. Since ρ is strongly subharmonic, p_{i+1} has Morse index either 0 or 1.

If the Morse index is 0, a new simply connected component of the sublevel set $\{\rho \leq c\}$ appears at p_{i+1} when c passes the value $\rho(p_{i+1})$. We define X_{i+1} on this new component as any conformal minimal immersion, thereby reducing the proof to the noncritical case.

If the Morse index of p_{i+1} is 1, the change of topology at p_{i+1} is described by attaching to M_i a smooth arc $E \subset \mathring{M}_{i+1} \backslash (M_i \cup \Lambda)$ such that $M_i \cup E$ is a compact admissible Runge set (see Definition 2.8) which is a strong deformation retract of M_{i+1}. Let θ be a nowhere vanishing holomorphic 1-form on M. Consider the smooth map $f_i = 2\partial X_i / \theta \colon M_i \to \mathfrak{A}_*$ which is holomorphic in $\mathring{M}_i$. We can extend f_i to a smooth map $\tilde{f}_i \colon M_i \cup E \to \mathfrak{A}_*$. We orient E and let $p, q \in bM_i$ denote the beginning and the endpoint of E, respectively. Lemma 3.8 applied on E furnishes a smooth function $h \colon E \to \mathbb{C}_*$ which equals 1 near both endpoints such that

$$\int_E h\tilde{f}_i\theta = X_i(q) - X_i(p).$$

We extend h to a smooth function on $M_i \cup E$ by setting $h|_{M_i} = 1$. Let $\hat{f}_i = h\tilde{f}_i$. By integrating $\hat{f}_i\theta$ from any initial point $p_0 \in M_i$ we obtain a generalized conformal minimal immersion $(\widehat{X}_i, \hat{f}_i\theta) \in \mathrm{GCMI}(M_i \cup E, \mathbb{R}^n)$ (see Definition 2.9) such that $\widehat{X}_i = X_i$ on M_i. We finish as in the noncritical case considered above, applying the method of period dominating sprays on the admissible set $M_i \cup E$. $\square$

The López–Ros deformation [80] for minimal surfaces in $\mathbb{R}^3$ enables one to perturb a given conformal minimal immersion by preserving one of its component functions; this is crucial in all applications of this technique in the literature. Theorem 4.1 also admits the following version in which all but two components of the initial immersion are preserved; this is a compilation of results from [5, 11, 19, 23].

THEOREM 4.4 (Assumptions as in Theorem 4.1). *If in addition $X = (X_1, \ldots, X_n)$ is nonflat and $X_3, \ldots, X_n$ extend to harmonic functions on M, then the conformal minimal immersions $\widetilde{X} = (\widetilde{X}_1, \ldots, \widetilde{X}_n) \in \mathrm{CMI}(M, \mathbb{R}^n)$ in Theorem 4.1 can be chosen such that $\widetilde{X}_k = X_k$ for $k = 3, \ldots, n$.*

The following extension of Theorem 4.1 requires some additional work.

THEOREM 4.5. *The conformal minimal immersions $\widetilde{X} \colon M \to \mathbb{R}^n$ in Theorem 4.1 can be chosen complete.*

If one ignores the interpolation, then Theorem 4.5 follows from Theorem 4.8 to the effect that a conformal minimal immersion $X \colon K \to \mathbb{R}^n$ for $n \geq 3$ from a Runge set K in an open Riemann surface M can be approximated by *proper* (hence complete) conformal minimal immersions $\widetilde{X} \colon M \to \mathbb{R}^n$. Assuming in addition that $X \colon \Lambda \to \mathbb{R}^n$ is a proper map, it is also possible to match the interpolation condition in Theorem 4.1 by a proper conformal minimal immersions $\widetilde{X} \colon M \to \mathbb{R}^n$ (see Theorem 4.9).

SKETCH OF PROOF OF THEOREM 4.5. Following the noncritical case in the proof of Theorem 4.1, we assume without loss of generality that M_i is connected and, for simplicity of exposition, that bM_i is connected as well; hence $A = M_{i+1}\backslash \mathring{M}_i$ is a smoothly bounded compact annulus with $bA = bM_{i+1} \cup bM_i$. Write $X_i = (X_{i,1}, \ldots, X_{i,n})$. We split A into two annuli A_0 and A_1 such that $A_0 \cap A_1$ is a boundary component of both A_0 and A_1, $bM_i \subset bA_0$, $bM_{i+1} \subset bA_1$, and the only point q_0 in $\Lambda_{i+1}\backslash \Lambda_i$ lies in $\mathring{A}_0$. By the proof of Theorem 4.1 we may assume that X_i extends to M_{i+1} having a contact of order k with X at every point in Λ_{i+1}, and that $\partial X_{i,1}$ does not vanish anywhere on A_1. We then consider a labyrinth of compact sets Υ in $\mathring{A}_1$ as in Jorge–Xavier [75], that is, Υ is a finite union of pairwise disjoint compact sets in $\mathring{A}_1$ such that, if $\gamma \colon [0, 1] \to A_1\backslash\Upsilon$ is any path connecting the two boundary components of A_1, then

$$\int_\gamma |\partial X_{i,1}| > \tau \tag{4.2}$$

for a given number $\tau > 0$. By Theorem 4.4 we obtain an immersion $X_{i+1} = (X_{i+1,1}, \ldots, X_{i+1,n}) \in \mathrm{CMI}(M_{i+1}, \mathbb{R}^n)$ which is close to X_i in $\mathscr{C}^1(M_i \cup A_0)$, has a contact of order k with X at every point of $\Lambda_{i+1} \subset M_i \cup A_0$, it satisfies $X_{i+1,1} = X_{i,1}$ on M_{i+1},

and $|X_{i+1,2}(p) - X_{i,2}(q)| > \tau$ holds for all points $p \in \Upsilon$ and $q \in A_0$. Together with (2.10), this and (4.2) guarantee that, if the approximation of X_i by X_{i+1} is close enough, the intrinsic distance between the two boundary curves of the annulus A_1 with respect to the metric $X_{i+1}^*(ds^2)$ on M_{i+1} is greater than τ. Since $\tau > 0$ is arbitrary, this shows that we may arbitrarily enlarge the intrinsic diameter of the surface in every step of the inductive construction in the proof of Theorem 4.1, thereby ensuring completeness of the limit map.   □

4.2. On Sullivan's and Schoen–Yau's conjectures and the embedding problem.
As mentioned in the introduction, as late as in the 1990s hyperbolic Riemann surfaces were thought to play only a marginal role in the global theory of minimal surfaces as seen from the following well-known conjectures.

CONJECTURE 4.6 (Sullivan). *Every properly immersed minimal surface in $\mathbb{R}^3$ with finite topology is parabolic.*

CONJECTURE 4.7 (Schoen–Yau [104, page 18]). *No hyperbolic open Riemann surface M carries a proper harmonic map $M \to \mathbb{R}^2$. In particular, every minimal surface in $\mathbb{R}^3$ with a proper projection to $\mathbb{R}^2$ is parabolic.*

The first and more ambitious part of Schoen–Yau's conjecture was refuted in 1999 by Božin [29] who constructed in a very explicit manner a proper harmonic map $\mathbb{D} \to \mathbb{R}^2$. Another counterexample was given in 2001 by Forstnerič and Globevnik [53] who constructed a proper holomorphic map $f = (f_1, f_2) \colon \mathbb{D} \to \mathbb{C}^2$ with $f(\mathbb{D}) \subset (\mathbb{C}_*)^2$; hence, $(\log|f_1|, \log|f_2|) \colon \mathbb{D} \to \mathbb{R}^2$ is a proper harmonic map. However, the second part of the conjecture concerning minimal surfaces remained open at that time.

Sullivan's conjecture was refuted in 2003 by Morales [91] who constructed a proper conformal minimal immersion $\mathbb{D} \to \mathbb{R}^3$ by using the López–Ros deformation and the Runge theorem in a highly intricate way. Morales' result was later extended to the existence of proper hyperbolic minimal surfaces in $\mathbb{R}^3$ with arbitrary topology; see Ferrer *et al.* [44].

Finally, Alarcón and López [23] proved in 2012 that every open Riemann surface admits a conformal minimal immersion into $\mathbb{R}^3$ properly projecting to a plane; this gave a counterexample to the second part of Schoen–Yau's conjecture and provided an optimal solution to the two problems. The following more precise result in this direction is due to the authors and López (see [19, Theorem 7.1]).

THEOREM 4.8 (Conformal minimal immersions with proper projections to $\mathbb{R}^2$). *Let M be an open Riemann surface and $K \subset M$ be a Runge compact set. Every conformal minimal immersion $U \to \mathbb{R}^n$ ($n \geq 3$) from an open neighborhood $U \subset M$ of K can be approximated uniformly on K by proper conformal minimal immersions $M \to \mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ properly projecting into $\mathbb{R}^2 \times \{0\}^{n-2} \cong \mathbb{R}^2$. The approximating immersions can be chosen with prescribed flux compatible with the flux of the initial immersion, with simple double points if $n = 4$, and embeddings if $n \geq 5$.*

Concerning the analogue of the Schoen–Yau conjecture in higher dimension, it was recently shown by Forstnerič [51, Corollary 3.5] that every Stein manifold X of complex dimension $n \geq 1$ admits a proper pluriharmonic map into $\mathbb{R}^{2n}$.

Although Theorem 4.8 contributes to the aforementioned conjectures, its main relevance concerns the problem of determining the minimal dimension d for which every open Riemann surface properly embeds into $\mathbb{R}^d$ as a conformal minimal surface; compare with Theorem 1.1 and Conjecture 1.2 in the introduction.

SKETCH OF PROOF OF THEOREM 4.8. We may assume that K is a smoothly bounded compact Runge domain in M. Let $X_0 \in \mathrm{CMI}(K, \mathbb{R}^n)$. Choose an exhaustion $K = M_0 \Subset M_1 \Subset \cdots$ of M as in (4.1) and inductively construct a sequence $X_i = (X_{i,1}, X_{i,2}, \ldots, X_{i,n}) \in \mathrm{CMI}(M_i, \mathbb{R}^n)$ $(i \in \mathbb{N})$ satisfying the following conditions:

(a) $\max\{X_{i,1}, X_{i,2}\} > i$ everywhere on bM_i;
(b) $\max\{X_{i,1}, X_{i,2}\} > i - 1$ everywhere on $M_i \backslash \mathring{M}_{i-1}$;
(c) X_i is as close to X_{i-1} as desired in the $\mathscr{C}^1(M_{i-1})$ norm;
(d) X_i has simple double points if $n = 4$ and is an embedding if $n \geq 5$.

The way to prescribe the flux map is the standard one; we shall omit it. If the approximation in (c) is close enough then the limit conformal minimal immersion $\lim_{i \to \infty} X_i \colon M \to \mathbb{R}^n$ satisfies the conclusion of the theorem.

We begin the induction with $X_0 \in \mathrm{CMI}(K, \mathbb{R}^n)$ which, up to composing with a translation and using Theorem 3.10, satisfies (a) and (d), while conditions (b) and (c) are vacuous.

We now explain the *noncritical case* in the inductive step. Assume that M_{i-1} is a strong deformation retract of M_i for some $i \geq 1$ and that we already have $X_{i-1} \in \mathrm{CMI}(M_{i-1}, \mathbb{R}^n)$ with the desired properties. Note that $M_i \backslash \mathring{M}_{i-1}$ is a union of finitely many pairwise disjoint compact annuli. For simplicity of exposition we assume that there is only one annulus $A = M_i \backslash \mathring{M}_{i-1}$ since the same argument can be applied separately to each of them. Note that $bA = bM_i \cup bM_{i-1}$. By Theorem 3.10 it suffices to find $X_i \in \mathrm{CMI}(M_i, \mathbb{R}^n)$ satisfying (a), (b), and (c). In view of condition (a) for the index $i - 1$, bM_{i-1} splits into $l \geq 3$ compact subarcs α_k, $k \in \mathbb{Z}_l = \mathbb{Z}/l\mathbb{Z}$, lying end to end, for which there are complementary subsets I_1 and $I_2 = \mathbb{Z}_l \backslash I_1$ of $\mathbb{Z}_l$ satisfying that $X_{i-1,\sigma} > i - 1$ everywhere on α_k for all $k \in I_\sigma$, $\sigma = 1, 2$. For every $k \in \mathbb{Z}_l$ we denote by $p_k \in bM_{i-1}$ the only point in $\alpha_k \cap \alpha_{k+1}$. Choose a family γ_k $(k \in \mathbb{Z}_l)$ of pairwise disjoint smooth Jordan arcs in A such that γ_k connects p_k with a point $q_k \in bM_i$ and is otherwise disjoint from bA, and the set $S = M_{i-1} \cup \bigcup_{k \in \mathbb{Z}_l} \gamma_k \subset M$ is admissible (see Definition 2.8). Denote by β_k the Jordan arc in bM_i connecting q_{k-1} and q_k, and let $\Omega_k \subset A$ be the closed disc bounded by $\gamma_{k-1} \cup \alpha_k \cup \gamma_k \cup \beta_k$ for $k \in \mathbb{Z}_l$. Theorem 4.1, applied to a suitable generalized conformal minimal immersion on S extending X_{i-1}, furnishes $Y = (Y_1, Y_2, \ldots, Y_n) \in \mathrm{CMI}(M_i, \mathbb{R}^n)$ as close as desired to X_{i-1} in the $\mathscr{C}^1(M_{i-1})$ norm and smoothly bounded compact discs $D_k \subset \Omega_k \backslash (\gamma_{k-1} \cup \alpha_k \cup \gamma_k)$, $k \in \mathbb{Z}_l$, such that $D_k \cap \beta_k \neq \varnothing$ is a Jordan arc in $\beta_k \backslash \{q_{k-1}, q_k\}$:

(P1) $Y_\sigma > i$ everywhere on $\overline{\beta_k \backslash D_k}$ for all $k \in I_\sigma$, $\sigma = 1, 2$; and
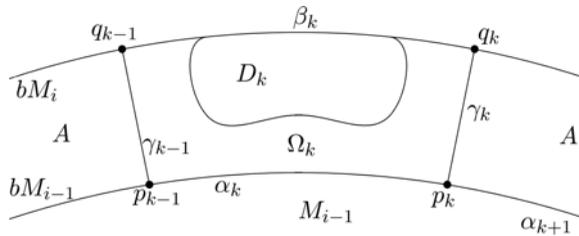
FIGURE 4.1. Sets in the proof of Theorem 4.8.

(P2) $Y_\sigma > i - 1$ everywhere on $\overline{\Omega_k \backslash D_k}$ for all $k \in I_\sigma$, $\sigma = 1, 2$.

(See Figure 4.1.)

Note that Y already satisfies conditions (a) and (b) but only on $\bigcup_{k \in \mathbb{Z}_l} \overline{\Omega_k \backslash D_k}$. Now, since Y is defined everywhere on M_i and it may be assumed nonflat in view of Theorem 4.1, by Theorem 4.4 (the Mergelyan approximation with fixed components) we may approximate Y on $M_{i-1} \cup \bigcup_{k \in I_2} \Omega_k$ by a conformal minimal immersion $Y' = (Y'_1, Y'_2, \ldots, Y'_n) \in \text{CMI}(M_i, \mathbb{R}^n)$ such that the following hold:

(P3) $Y'_1 = Y_1$ everywhere on M_i; and
(P4) $Y'_2 > i$ everywhere on $\bigcup_{k \in I_1} D_k$.

Indeed, it suffices to apply Theorem 4.4, keeping the first component fixed, with a conformal minimal immersion $\widetilde{Y} \in \text{CMI}(M_{i-1} \cup \bigcup_{k \in I_2} \Omega_k \cup \bigcup_{k \in I_1} D_k)$ of the form

$$\widetilde{Y} = \begin{cases} Y & \text{on } M_{i-1} \cup \bigcup_{k \in I_2} \Omega_k, \\ (0, C, 0, \ldots, 0) + Y & \text{on } \bigcup_{k \in I_1} D_k, \end{cases}$$

where $C > 0$ is a large enough constant.

Note that, by (P3), if the approximation of Y by Y' is close enough then (P1) and (P2) hold with Y' in place of Y. Observe that Y' already satisfies conditions (a) and (b) on $\bigcup_{k \in \mathbb{Z}_l} \overline{\Omega_k \backslash D_k} \cup \bigcup_{k \in I_1} D_k$; we will now deform it to meet these requirements also on $\bigcup_{k \in I_2} D_k$ and this will finish the proof. Indeed, proceeding in a symmetric way, we may approximate Y' on $M_{i-1} \cup \bigcup_{k \in I_1} \Omega_k$ by a conformal minimal $Y'' = (Y''_1, Y''_2, \ldots, Y''_n) \in \text{CMI}(M_i, \mathbb{R}^n)$ such that the following hold:

(P5) $Y''_2 = Y'_1$ everywhere on M_i; and
(P6) $Y''_2 > i$ everywhere on $\bigcup_{k \in I_2} D_k$.

As above, by (P5), if the approximation of Y' by Y'' is close enough then Y'' formally satisfies (P1), (P2), and (P4). This and (P6) shows that $X_i := Y''$ meets conditions (a) and (b). Finally, (c) also holds provided that the approximations of X_{i-1} by Y, of Y by Y', and of Y' by Y'' are sufficiently close. This concludes the proof. $\qquad\square$

By joining the ideas in the above proof with those in Theorem 4.1 we obtain the following extension of Theorem 4.1, due to Alarcón and Castro-Infantes [5].

THEOREM 4.9 [5, Theorem 1.2]. *In the assumptions of Theorem 4.1, if in addition* $X|_\Lambda \colon \Lambda \to \mathbb{R}^n$ *is a proper map then the conformal minimal immersions* $\widetilde{X} \colon M \to \mathbb{R}^n$ *in Theorem 4.1 can be chosen proper.*

4.3. On the Gauss map. The Gauss map of a minimal surface in $\mathbb{R}^3$ and, more generally, the generalized Gauss map $G_X \colon M \to \mathbb{CP}^{n-1}$ (see (2.12)) of a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ ($n \geq 3$), is a fundamental object in the theory. It is classical that G_X is a holomorphic map assuming values in the null quadric (see Section 2.6). Somewhat surprisingly, the following converse was proved only very recently by the authors and López (see [18, Theorem 1.1 and Corollary 1.2]).

THEOREM 4.10. *Let M be an open Riemann surface and $n \geq 3$ be an integer. For any holomorphic map $\mathscr{G} \colon M \to Q_{n-2} \subset \mathbb{CP}^{n-1}$ into the quadric (2.13) there is a conformal minimal immersion $X \colon M \to \mathbb{R}^n$ with the generalized Gauss map $G_X = \mathscr{G}$ and vanishing flux. If in addition the map $\mathscr{G}$ is full (that is, its image is not contained in any proper projective subspace), then X can be chosen to have arbitrary flux and to be an immersion with simple double points if $n = 4$ and an embedding if $n \geq 5$.*

In particular, every holomorphic map $g \colon M \to \mathbb{CP}^1$ is the complex Gauss map (2.14) of a conformal minimal immersion $X \colon M \to \mathbb{R}^3$ with vanishing flux. If g is nonconstant, we can find X with arbitrary given flux.

SKETCH OF PROOF. By the Oka–Grauert principle (see Theorem 3.1) we can lift the map $\mathscr{G} \colon M \to \mathbb{CP}^{n-1}$ to a holomorphic map $G \colon M \to \mathbb{C}^n_*$ such that $\pi \circ G = \mathscr{G}$, where $\pi \colon \mathbb{C}^n_* \to \mathbb{CP}^{n-1}$ is the canonical projection. Obviously, G assumes values in the punctured null quadric $\mathfrak{A}_* \subset \mathbb{C}^n$ (2.4), (2.5). Fix a nowhere vanishing holomorphic 1-form θ on M. To complete the proof of the first part of the theorem, it then suffices to find a holomorphic function $\varphi \colon M \to \mathbb{C}_*$ such that the real part of the 1-form $\Phi = \varphi G \theta$ is exact on M. If such φ exists then given $p_0 \in M$ the Enneper–Weierstrass formula (Theorem 2.4) shows that the map $X \colon M \to \mathbb{R}^n$ given by $X(p) = \int_{p_0}^p \Re(\Phi)$, $p \in M$, is a conformal minimal immersion with the generalized Gauss map $G_X = [\partial X] = [\Phi] = \pi \circ (\varphi G) = \pi \circ G = \mathscr{G}$.

The construction of the function φ follows the proof of Theorem 4.1, but using Lemma 3.7 instead of Lemma 3.2. Let us focus on the case of vanishing flux, that is, we look for φ such that $\varphi G \theta$ is exact. Choose an exhaustion $M_1 \Subset M_2 \Subset \cdots$ of M as in (4.1) such that M_1 is simply connected. We inductively construct a sequence of holomorphic functions $\varphi_i \colon M_i \to \mathbb{C}_*$ ($i \in \mathbb{N}$) such that the following hold:

(a) $\varphi_i G \theta$ is exact on M_i; and
(b) φ_i is as close to φ_{i-1} as desired in the $\mathscr{C}(M_{i-1})$ norm for all $i \geq 2$.

The limit function $\varphi = \lim_{i \to \infty} \varphi_i \colon M \to \mathbb{C}_*$ clearly meets the requirements if all approximations in (b) are close enough.

Since M_1 is simply connected, the basis of the induction is given by any holomorphic function $\varphi_1 \colon M_1 \to \mathbb{C}_*$. For the inductive step we assume that we already have φ_{i-1} for some $i \geq 2$. For simplicity of exposition we assume that M_{i-1} is connected

and a strong deformation retract of M_i (that is, we only discuss the noncritical case). Lemma 3.7 provides a period dominating multiplier $h: M_{i-1} \times \mathbb{C}^N \to \mathbb{C}$ of φ_{i-1}. Next, we approximate φ_{i-1} and h on M_{i-1} by a holomorphic function $f: M_i \to \mathbb{C}_*$ and a spray of holomorphic functions $h': M_i \times \mathbb{C}^N \to \mathbb{C}$, respectively, such that h' is a period dominating multiplier of f. If the approximations of φ_{i-1} by f, and of h by h', are close enough, there is a point $\zeta_0 \in \mathbb{C}^N$ close to the origin such that the function $\varphi_i(p) = f(p)h(p, \zeta_0)$, $p \in M_i$, does not vanish anywhere and it meets conditions (a) and (b).

For the second assertion concerning the cases $n = 4$ and $n \geq 5$, we adapt the transversality method, described in Section 3.5, to the current framework. By using period dominating multipliers, given by Lemma 3.7, we may improve Theorem 3.10 by ensuring that the approximating immersion $\widetilde{X}$ has the same generalized Gauss map as X. (See [18, Proof of Theorem 1.1] for the details.) This enables us to find the function φ_i in the inductive step such that the conformal minimal immersion $X_i: M_i \to \mathbb{R}^n$ given by $X_i(p) = \int_{p_0}^{p} \mathfrak{R}(\varphi_i G\theta)$, $p \in M_i$, has simple double points if $n = 4$, and is an embedding if $n \geq 5$. The same holds for the limit $X = \lim_{i \to \infty} X_i: M \to \mathbb{R}^n$ provided the approximations in (b) are sufficiently close. $\square$

The size of the spherical image of the Gauss map of a minimal surface in $\mathbb{R}^3$ has important implications. For instance, Barbosa and do Carmo [27] proved that if the area of the spherical image is smaller than 2π (half of the area of the sphere) then the surface is stable. Thus, Theorem 4.10 leads to the following corollary.

COROLLARY 4.11. *If M is an open Riemann surface and $g: M \to \mathbb{CP}^1$ is a holomorphic map whose image $g(M)$ has a spherical area less than 2π, then there is a stable conformal minimal immersion $M \to \mathbb{R}^3$ with the complex Gauss map g.*

Another important direction in the study of the Gauss map of conformal minimal surfaces in $\mathbb{R}^3$ is to understand how many points it can omit when the surface is complete. A seminal result of Fujimoto [59] says that the Gauss map of a complete nonflat minimal surface in $\mathbb{R}^3$ can omit at most four points of $\mathbb{CP}^1$; there are examples with four omitted points, for instance, the classical Sherk's doubly periodic surface. In higher dimensions, Ru [103] proved that if $X: M \to \mathbb{R}^n$ is a complete nonflat conformal minimal immersion then its generalized Gauss map G_X can omit at most $\frac{1}{2}n(n + 1)$ hyperplanes in general position in $\mathbb{CP}^{n-1}$. (As pointed out in [58, page 280], this is equivalent to Fujimoto's theorem for $n = 3$.) However, the number of exceptional hyperplanes depends on the complex structure of the surface. Ahlfors [2] proved that any holomorphic map $\mathbb{C} \to \mathbb{CP}^{n-1}$ avoiding $n + 1$ hyperplanes of $\mathbb{CP}^{n-1}$ in general position is degenerate (for $n = 2$ this is Picard's theorem). This shows that the following result of Alarcón *et al.* [10, 11] is the best possible for all minimal surfaces with a nondegenerate Gauss map.

THEOREM 4.12. *Let M be an open Riemann surface. For any group homomorphism $\mathfrak{p}: H_1(M; \mathbb{Z}) \to \mathbb{R}^n$ $(n \geq 3)$ there is a complete conformal minimal immersion $M \to \mathbb{R}^n$ with the flux map $\mathfrak{p}$ whose generalized Gauss map is nondegenerate and omits n*

hyperplanes of $\mathbb{CP}^{n-1}$ in general position. In particular, every open Riemann surface admits a complete nonflat conformal minimal immersion into $\mathbb{R}^3$ whose complex Gauss map omits two points of $\mathbb{CP}^1$.

On the other hand, Osserman proved in 1964 [96] that the Gauss map of a complete nonflat minimal surface with finite total curvature in $\mathbb{R}^3$ can omit at most three points of $\mathbb{CP}^1$ (see also [98, page 89]). His question, whether there is an example of this kind whose Gauss map omits three points, is still open.

SKETCH OF PROOF. We explain the case $n = 3$ without taking care of the flux. It suffices to find a complete nonflat conformal minimal immersion $X = (X_1, X_2, X_3) \colon M \to \mathbb{R}^3$ such that ∂X_3 does not vanish anywhere on M. Indeed, by (2.14), this implies that the complex Gauss map $g_X \colon M \to \mathbb{C}$ of X is holomorphic and nowhere vanishing, and hence the Gauss map of X assumes neither the north nor the south poles of $\mathbb{S}^2$. Note that Theorem 4.4 (the Runge theorem with fixed components for minimal surfaces) ensures that every nonconstant harmonic function $X_3 \colon M \to \mathbb{R}$ is a component function of a nonflat conformal minimal immersion $X = (X_1, X_2, X_3) \colon M \to \mathbb{R}^3$; choosing X_3 with no critical points we have that ∂X_3 vanishes nowhere on M. To complete the proof, it remains to show that such an immersion X may be chosen to be complete. As in the proof of Theorem 4.1, the map $(X_1, X_2) \colon M \to \mathbb{R}^2$ is constructed inductively: $(X_1, X_2) = \lim_{i \to \infty} (\widetilde{X}_{i,1}, \widetilde{X}_{i,2})$ for suitable harmonic maps $(\widetilde{X}_{i,1}, \widetilde{X}_{i,2}) \colon M_i \to \mathbb{R}^2$. (Here $M_1 \Subset M_2 \Subset \cdots$ is an exhaustion of M as in (4.1).) To ensure completeness of $X = \lim_{i \to \infty} \widetilde{X}_i = (\widetilde{X}_{i,1}, \widetilde{X}_{i,2}, X_3) \colon M \to \mathbb{R}^3$ we suitably enlarge the intrinsic diameter of each immersion $\widetilde{X}_i \colon M_i \to \mathbb{R}^3$ by using a Jorge–Xavier type labyrinth of compact sets in $\mathring{M}_i \backslash M_{i-1}$ as in the proof of Theorem 4.5. $\qquad\square$

4.4. Rough shape of the space of conformal minimal immersions. Several of the results already stated in the paper may be extended to continuous families of conformal minimal immersions by exploiting the parametric Oka property of the punctured null quadric $\mathfrak{A}_* \subset \mathbb{C}^n$. For instance, for an open Riemann surface M, *every conformal minimal immersion $X_0 \colon M \to \mathbb{R}^n$ is isotopic through conformal minimal immersions $X_t \colon M \to \mathbb{R}^n$ ($t \in [0, 1]$) to the following:*

(a) *a complete conformal minimal immersion [13];*
(b) *a complete conformal minimal immersion with arbitrary flux if the generalized Gauss map of X_0 is nondegenerate [13]; and*
(c) *a complete conformal minimal immersion with vanishing flux such that all maps X_t have the same generalized Gauss map $M \to \mathbb{CP}^{n-1}$ (see [18, Corollary 1.4]).*

Fix a nowhere vanishing holomorphic 1-form θ on M and consider the following commuting diagram of spaces and maps:

$$
\begin{array}{ccccccc}
\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n) & \xrightarrow{\ \phi\ } & O_{\mathrm{nf}}(M, \mathfrak{A}_*) & \overset{i}{\hookrightarrow} & O(M, \mathfrak{A}_*) & \overset{j}{\hookrightarrow} & \mathscr{C}(M, \mathfrak{A}_*) \\
{\scriptstyle \mathfrak{R}}\big\downarrow & & {\scriptstyle \psi}\big\uparrow & & & & \\
\mathfrak{R}\,\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n) & \overset{\alpha}{\hookrightarrow} & \mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) & \overset{\beta}{\hookrightarrow} & \mathrm{CMI}(M, \mathbb{R}^n) & &
\end{array}
$$

The left-hand side map $\mathfrak{R}$ (the real part projection) is a homotopy equivalence by continuity of the conjugate map transform. Here, $\mathfrak{R} \, \mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n)$ denotes the space of conformal minimal immersions $M \to \mathbb{R}^n$ with zero flux. The maps ϕ and ψ are defined by $\phi(Z) = dZ/\theta$ and $\psi(X) = 2\partial X/\theta$, respectively. The space $O_{\mathrm{nf}}(M, \mathfrak{A}_*)$ consists of all nonflat holomorphic maps $M \to \mathfrak{A}_*$ (see Definition 2.6).

Forstnerič and Lárusson proved in [55] that all maps in the above diagram, with the only possible exception of the inclusion $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) \hookrightarrow \mathrm{CMI}(M, \mathbb{R}^n)$, are weak homotopy equivalences, and are homotopy equivalences if M has finite topological type. (See [55, Theorem 1.1] for the inclusion $\alpha \colon \mathfrak{R}\mathrm{NC}_{\mathrm{nf}}(M, \mathbb{C}^n) \hookrightarrow \mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n)$, [55, Theorem 1.2] for maps ϕ and ψ, and [55, Theorem 5.4] for the inclusion $i \colon O_{\mathrm{nf}}(M, \mathfrak{A}_*) \hookrightarrow O(M, \mathfrak{A}_*)$. The natural inclusion $O(M, \mathfrak{A}_*) \hookrightarrow \mathscr{C}(M, \mathfrak{A}_*)$ is a weak homotopy equivalence by the Oka–Grauert Theorem 3.1. For the proof of strong homotopy equivalences, see [55, Section 6].) Subsequently, Alarcón and Lárusson [22] used the methods from [55] to show that the map $\pi \colon O_{\mathrm{nf}}(M, \mathfrak{A}_*) \to H^1(M; \mathbb{C}^n)$, sending a nonflat holomorphic map $g \colon M \to \mathfrak{A}_*$ to the cohomology class of $g\theta$, is a Serre fibration; this also implies the aforementioned results from [55].

The only map in the above diagram which is not completely understood is the inclusion $\mathrm{CMI}_{\mathrm{nf}}(M, \mathbb{R}^n) \hookrightarrow \mathrm{CMI}(M, \mathbb{R}^n)$ of the space of nonflat conformal minimal immersions into the space of all conformal minimal immersions. The authors together with López showed in [18, Theorem 7.1] that this inclusion induces a bijection of path components of the two spaces. In particular, we have the following result.

COROLLARY 4.13. *Let M be an open Riemann surface. If $n > 3$ then the space $\mathrm{CMI}(M, \mathbb{R}^n)$ is path connected, whereas the set of path components of $\mathrm{CMI}(M, \mathbb{R}^3)$ is in bijective correspondence with the elements of the abelian group $(\mathbb{Z}_2)^l$ where $H_1(M; \mathbb{Z}) = \mathbb{Z}^l$.*

5. The Riemann–Hilbert method for minimal surfaces

The Riemann–Hilbert problem is a classical boundary value problem for holomorphic functions and maps. Its basic form was mentioned by Riemann in his dissertation in 1851. A brief history, references, and a partial list of applications can be found in [16, Section 3]. In Section 5.1 we describe the original complex analytic setting. In Section 5.2 we state without proof a version of the Riemann–Hilbert problem for conformal minimal immersions from bordered Riemann surfaces to $\mathbb{R}^n$. This serves as the basis for the construction of complete conformal minimal surfaces in $\mathbb{R}^n$ bounded by Jordan curves and normalized by any given bordered Riemann surface (see Theorems 1.3 and 5.4), the construction of proper complete conformal minimal immersions of such surfaces to minimally convex domains in $\mathbb{R}^n$ (see Section 5.4), and the description of the minimal hull of a compact set in $\mathbb{R}^n$ by sequences of minimal discs (see [42] for $n = 3$ and [8] for $n > 3$). Due to space limitations we shall not discuss minimal hulls in this survey.

5.1. The Riemann–Hilbert problem in complex analysis. Let X be a complex manifold. We are given a holomorphic map $f \colon \overline{\mathbb{D}} \to X$ (an *analytic disc* in X) and for each point $z \in \mathbb{T} = b\mathbb{D}$ a holomorphic map $g_z \colon \overline{\mathbb{D}} \to X$ with $g_z(0) = f(z)$ and depending continuously on $z \in \mathbb{T}$. Set $T_z = g_z(\mathbb{T}) \subset X$ and $S_z = g_z(\overline{\mathbb{D}}) \subset X$ for $z \in \mathbb{T}$. Fix a distance function dist on X. Given numbers $0 < r < 1$ and $\epsilon > 0$, the *approximate Riemann–Hilbert problem* asks for a holomorphic map $F \colon \overline{\mathbb{D}} \to X$ satisfying the following conditions for some $r' \in [r, 1)$:

(a) $\mathrm{dist}(F(z), T_z) < \epsilon$ for $z \in \mathbb{T}$;
(b) $\mathrm{dist}(F(\rho z), S_z) < \epsilon$ for $z \in \mathbb{T}$ and $r' \le \rho \le 1$; and
(c) $\mathrm{dist}(F(z), f(z)) < \epsilon$ for $|z| \le r'$.

(The domain of f may in fact be any bordered Riemann surface M, but the domain of the maps g_z is always the closed disc $\overline{\mathbb{D}}$.) This implies the following:

- $F(\overline{\mathbb{D}})$ lies in the ϵ-neighborhood of the set $\Sigma = f(\overline{\mathbb{D}}) \cup \bigcup_{z \in \mathbb{T}} g_z(\overline{\mathbb{D}})$; and
- its boundary $F(\mathbb{T})$ is contained in the ϵ-neighborhood of the torus $T = \bigcup_{z \in \mathbb{T}} g_z(\mathbb{T})$.

This shows that the placement of the image curve $F(\overline{\mathbb{D}})$ in X is well-controlled, a very important point in most applications. (The exact problem, asking for F satisfying $F(z) \in T_z$ for every $z \in b\mathbb{D}$, is only rarely solvable.)

For $X = \mathbb{C}^n$ the problem is solved as follows (see [41] for the details). The map

$$\mathbb{T} \times \overline{\mathbb{D}} \ni (z, w) \mapsto g_z(w) - f(z) \in \mathbb{C}^n$$

is continuous in z, holomorphic in w, and it vanishes at $w = 0$ for any $z \in \mathbb{T}$. We can approximate it arbitrarily closely by a rational map

$$G(z, w) = z^{-m} \sum_{j=1}^{N} A_j(z) w^j \in \mathbb{C}^n,$$

where the A_j's are $\mathbb{C}^n$-valued holomorphic polynomials. Pick $k \in \mathbb{N}$ and set

$$F(z) = f(z) + G(z, z^k) = f(z) + z^{k-m} \sum_{j=1}^{N} A_j(z) z^{k(j-1)}, \quad z \in \overline{\mathbb{D}}. \tag{5.1}$$

The pole at $z = 0$ cancels if $k > m$, and one easily verifies that F satisfies conditions (a)–(c) if the integer k is chosen big enough.

Consider now the case when the domain of f is a bordered Riemann surface M and the target manifold X is arbitrary. In most applications it suffices to solve the following restricted problem. Pick a pair of arcs $I_0, I_1 \subset bM$ with I_0 contained in the relative interior of I_1, and a smooth function $\chi \colon bM \to [0, 1]$ such that $\chi = 1$ on I_0 and $\chi = 0$ on $bM \backslash I_1$. Set $\tilde{g}_z(w) = g_z(\chi(z)w)$ for $z \in bM$ and $w \in \overline{\mathbb{D}}$. Note that $\tilde{g}_z$ agrees with g_z for $z \in I_0$ and is the constant disc $w \to f(z)$ for any point $z \in bM \backslash I_1$. Let $D \subset M$ be a smoothly bounded simply connected domain (a disc) such that I_1 is contained in the relative interior of $bD \cap bM$. We define $\tilde{g}_z$ as the constant dics $f(z)$

for points $z \in bD \setminus I_1$ (this is consistent with the previous choices). Let $\widetilde{F} : \overline{D} \to X$ be an approximate solution of the Riemann–Hilbert problem with the data $f|_{\overline{D}}$ and $\tilde{g}_z$, $z \in bD$. (Such $\widetilde{F}$ is found by reducing this local problem to the Euclidean case via suitable Stein neighborhoods of the graphs of our maps.) By choosing the integer k in (5.1) big enough, $\widetilde{F}$ satisfies condition (a) for $z \in I_0$, it satisfies (b) for $z \in bD$, and is uniformly close to f on $\overline{D} \setminus U$ where $U \subset \overline{M}$ is any given neighborhood of the arc I_1. Write $\overline{M} = A \cup B$ where $A, B \subset \overline{M}$ are smoothly bounded compact domains such that A is the complement of a small neighborhood of the arc I_1, $B \subset \overline{D}$ contains a relative neighborhood of I_1, and we have that $\overline{A \setminus B} \cap \overline{B \setminus A} = \varnothing$. By what has been said above, the map $\widetilde{F}$ can then be chosen uniformly as close as desired to f on $C = A \cap B$. Next, we glue f and $\widetilde{F}$ into a solution $F : M \to X$ with the desired properties by the method of *gluing holomorphic sprays*. An outline of this gluing technique can be found in [16, Section 3], and it is fully explained in [52, Ch. 5]. This method lies at the heart of the proof of the Oka principle for maps from Stein manifolds to Oka manifolds (see [52, Theorem 5.4.4]).

5.2. The Riemann–Hilbert method for null curves and minimal surfaces.

The Riemann–Hilbert boundary value problem has been adapted to null holomorphic curves in $\mathbb{C}^n$ and conformal minimal surfaces in $\mathbb{R}^n$ for any $n \geq 3$ in the papers [7, 15]. A special case for null curves in $\mathbb{C}^3$ was first obtained by the authors in [15] by using the double sheeted spinor parametrization $\pi : \mathbb{C}^2_* \to \mathfrak{A}^2_*$ of the null quadric, lifting the derivative maps from $\mathfrak{A}^2_*$ to $\mathbb{C}^2_*$, applying the Riemann–Hilbert method in $\mathbb{C}^2_*$ and then pushing the resulting maps back to $\mathfrak{A}^2_*$.

When replacing the disc by a bordered Riemann surface M, one must glue local solutions on small discs abutting bM by using the method of gluing sprays as described in the previous subsection. Since the Riemann–Hilbert problem is not used directly for null curves but for their derivatives, one must pay attention to the period vanishing conditions to ensure that the approximating maps integrate to null curves. The results from [15] were extended to any dimension $n \geq 3$ and adapted to conformal minimal immersions by the authors together with Drinovec Drnovšek and López [7]. The following result is [7, Theorem 3.6].

THEOREM 5.1 (Riemann–Hilbert method for conformal minimal surfaces in $\mathbb{R}^n$). *Let M be a compact bordered Riemann surface with boundary $bM \neq \varnothing$, and let $I_1, \ldots, I_k$ be pairwise disjoint compact arcs in bM which are not connected components of bM. Let $r : bM \to \mathbb{R}_+$ be a continuous nonnegative function supported on $I := \bigcup_{i=1}^k I_i$. Also, let $\sigma : I \times \overline{\mathbb{D}} \to \mathbb{C}$ be a function of class $\mathscr{C}^1$ such that for every $\zeta \in I$ the function $\overline{\mathbb{D}} \ni \xi \mapsto \sigma(\zeta, \xi)$ is holomorphic on $\mathbb{D}$, $\sigma(\zeta, 0) = 0$, and the partial derivative $\partial \sigma(\zeta, \xi) / \partial \xi$ is nonvanishing on $I \times \overline{\mathbb{D}}$. Choose a thin annular neighborhood $A \subset M$ of bM and a smooth retraction $\rho : A \to bM$. For each $i = 1, \ldots, k$ let $\mathbf{u}_i, \mathbf{v}_i \in \mathbb{R}^n$ be a pair of orthogonal vectors satisfying $|\mathbf{u}_i| = |\mathbf{v}_i| > 0$. Given a conformal minimal immersion $X \in \mathrm{CMI}^1(M, \mathbb{R}^n)$ ($n \geq 3$), consider the continuous map $\varkappa : bM \times \overline{\mathbb{D}} \to \mathbb{R}^n$ given by*

$$\varkappa(\zeta, \xi) = \begin{cases} X(\zeta), & \zeta \in bM \setminus I, \\ X(\zeta) + r(\zeta)(\mathfrak{R}\sigma(\zeta, \xi)\mathbf{u}_i + \mathfrak{I}\sigma(\zeta, \xi)\mathbf{v}_i), & \zeta \in I_i, \ i \in \{1, \ldots, k\}. \end{cases} \quad (5.2)$$

Given $\epsilon > 0$ there exist an arbitrarily small open neighborhood $\Omega \subset M$ of I and a conformal minimal immersion $Y \in \mathrm{CMI}^1(M, \mathbb{R}^n)$ satisfying the following conditions:

(i) $\mathrm{dist}(Y(\zeta), \varkappa(\zeta, \mathbb{T})) < \epsilon$ *for all* $\zeta \in bM$;

(ii) $\mathrm{dist}(Y(\zeta), \varkappa(\rho(\zeta), \overline{\mathbb{D}})) < \epsilon$ *for all* $\zeta \in \Omega$;

(iii)   *Y is ϵ-close to X in the $\mathscr{C}^1$ norm on $M \backslash \Omega$;*

(iv) $\mathrm{Flux}(Y) = \mathrm{Flux}(X)$.

The proof will not be reproduced here due to its complexity and space limitations. Note that the boundary discs $\varkappa(\zeta, \cdot)$ ($\zeta \in bM$) lie in affine 2-planes. A more precise result is available in dimension $n = 3$; see [8, Theorem 3.2]. In that case, the map $\varkappa$ (5.2) may be chosen of the form

$$\varkappa(\zeta, \xi) = X(\zeta) + \alpha(\zeta, r(\zeta)\,\xi), \tag{5.3}$$

where $\alpha \colon I \times \overline{\mathbb{D}} \to \mathbb{R}^3$ is a map of class $\mathscr{C}^1$ such that for every $\zeta \in I$ the map $\overline{\mathbb{D}} \ni \xi \mapsto \alpha(\zeta, \xi) \in \mathbb{R}^3$ is a conformal minimal immersion with $\alpha(\zeta, 0) = 0$ and we take $\alpha(\zeta, r(\zeta)\,\xi) = 0$ for $\zeta \in bM \backslash I$. The advantage is that the conformal minimal discs $\overline{\mathbb{D}} \ni \xi \mapsto \alpha(\zeta, r(\zeta)\,\xi) \in \mathbb{R}^3$ are now arbitrary, and not necessarily flat as before.

The Riemann–Hilbert method has also been adapted by the authors and López [21] to the construction of bounded complete holomorphic Legendrian curves in the standard complex contact structure on $\mathbb{C}^{2n+1}$ ($n \in \mathbb{N}$). Combining this result with the Darboux-type holomorphic contact neighborhood theorem, proved by the authors in [17, Theorem 1.1], it follows that every holomorphic Legendrian immersion from a compact bordered Riemann surface to an arbitrary complex contact manifold can be uniformly approximated by complete holomorphic Legendrian curves [17, Theorem 1.3].

5.3. On the Calabi–Yau problem.   The Calabi–Yau problem for minimal surfaces originates in the following conjecture of Calabi from 1965.

CONJECTURE 5.2 (Calabi [76, page 170]). *A complete minimal hypersurface in $\mathbb{R}^n$ for $n \geq 3$ is unbounded. Even more, its projection to every $(n - 2)$-dimensional affine subspace of $\mathbb{R}^n$ is unbounded.*

Nothing seems known about this problem for $n \geq 4$. For $n = 3$, the latter assertion in Calabi's conjecture was refuted by Jorge and Xavier [75] in 1980, and the former by Nadirashvili [92] in 1996. In both cases there is a counterexample normalized by the disc, and the proof combines the López–Ros deformation for minimal surfaces [80] with an inventive use of the Runge approximation theorem for holomorphic functions.

In his *2000 Millennium Lecture* [110], Yau revisited Calabi's conjectures and proposed several questions concerning topology, complex structure, and asymptotic behavior of complete bounded minimal surfaces in $\mathbb{R}^3$. Ferrer *et al.* [44] showed in 2012 that there are no restrictions on their topological type. On the other hand, controlling the complex structure is a much more challenging task. The second part of Conjecture 5.2 was settled in 2012 by Alarcón *et al.* [10] who proved the following result. (A special case was obtained beforehand in [9].)

THEOREM 5.3. *Given an open Riemann surface M and a nonconstant harmonic function $h\colon M \to \mathbb{R}$, there is a complete conformal minimal immersion $X\colon M \to \mathbb{R}^3$ whose third coordinate function equals h. In particular, M admits a complete nonflat conformal minimal immersion $M \to \mathbb{R}^3$ with a bounded component function if and only if there exists a bounded nonconstant harmonic function $M \to \mathbb{R}$.*

In the subsequent paper [11] of the same authors this result was extended to conformal minimal surfaces in $\mathbb{R}^n$ for $n > 3$, where now $n - 2$ of the coordinate functions can be prescribed.

On the other hand, open Riemann surfaces normalizing complete bounded minimal surfaces in $\mathbb{R}^3$ are far from classified. By introducing the Riemann–Hilbert method into the picture (see Theorem 5.1), the authors with Drinovec Drnovšek and López [7] proved that every bordered Riemann surface normalizes a complete bounded minimal surface with Jordan boundary.

THEOREM 5.4. *Let M be a compact bordered Riemann surface. Every conformal minimal immersion $X\colon M \to \mathbb{R}^n$ ($n \geq 3$) of class $\mathscr{C}^1(M)$ may be approximated uniformly on M by continuous maps $\widetilde{X}\colon M \to \mathbb{R}^n$ such that $\widetilde{X}|_{bM}\colon bM \hookrightarrow \mathbb{R}^n$ is a topological embedding and $\widetilde{X}|_{\overset{\circ}{M}}\colon \overset{\circ}{M} \to \mathbb{R}^n$ is a complete conformal minimal immersion. If $n \geq 5$ there are embeddings $\widetilde{X}\colon M \hookrightarrow \mathbb{R}^n$ with these properties. The flux map of $\widetilde{X}$ can also be prescribed.*

Theorem 5.4 follows by an inductive application of the following approximation result (see [7, Lemma 4.1]), together with Theorem 3.10 and a transversality argument which provides injectivity on the boundary.

LEMMA 5.5. *In the assumptions of Theorem 5.4, given a point $p_0 \in \overset{\circ}{M}$ and a number $\lambda > 0$, X may be approximated arbitrarily closely uniformly on M by a conformal minimal immersion $Y\colon M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ such that $\mathrm{dist}_Y(p_0, bM) > \lambda$.*

Lemma 5.5 in turn follows from the maximum principle, the divergence of the sequence $d_j = d_{j-1} + (1/j)(d_0 > 0)$, the convergence of the sequence $\delta_j = \sqrt{\delta_{j-1}^2 + (1/j)^2}$ ($\delta_0 > 0$), and an inductive application of the following result (see [7, Lemma 4.2]).

LEMMA 5.6. *In the assumptions of Theorem 5.4, let $\mathfrak{Y}\colon bM \to \mathbb{R}^n$ be a smooth map and choose $\delta > 0$ such that $|X(p) - \mathfrak{Y}(p)| < \delta$ for all $p \in bM$. Also let $p_0 \in \overset{\circ}{M}$ and choose $d > 0$ such that $0 < d < \mathrm{dist}_X(p_0, bM)$. For any $\eta > 0$ the map X may be approximated uniformly on compacts in $\overset{\circ}{M}$ by conformal minimal immersions $Y\colon M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ satisfying $\mathrm{dist}_Y(p_0, bM) > d + \eta$ and $|Y(p) - \mathfrak{Y}(p)| < \sqrt{\delta^2 + \eta^2}$ for all $p \in bM$.*

SKETCH OF PROOF OF LEMMA 5.6. We assume that M is a smoothly bounded compact domain in an open Riemann surface $\widetilde{M}$ and, for simplicity of exposition, that bM is connected. Choose a smoothly bounded compact domain $K \subset \overset{\circ}{M}$ which is a strong deformation retract of M and such that $\mathrm{dist}_X(p_0, bK) > d$. We assume without loss

of generality that $X - \mathfrak{Y} \neq 0$ on bM. Given $\epsilon > 0$ we look for a conformal minimal immersion $Y \colon M \to \mathbb{R}^n$ with $|Y - X| < \epsilon$ on K and satisfying the lemma.

Fix a number $\epsilon_0 > 0$ to be specified later. By continuity of X and $\mathfrak{Y}$, bM splits into $l \geq 3$ compact arcs α_k, $k \in \mathbb{Z}_l$, lying end to end and such that for all pairs of points $p, q \in \alpha_k$ we have $|\mathfrak{Y}(p) - \mathfrak{Y}(q)| < \epsilon_0$, $|X(p) - \mathfrak{Y}(q)| < \delta$, and $|X(p) - X(q)| < \epsilon_0$. Denote by p_k the only point in $\alpha_k \cap \alpha_{k+1}$ and by $\pi_k \colon \mathbb{R}^n \to span\{X(p_k) - \mathfrak{Y}(p_k)\} \subset \mathbb{R}^n$ the orthogonal projection onto the real line $span\{X(p_k) - \mathfrak{Y}(p_k)\}$. The first step consists of perturbing X near the points $\{p_k \colon k \in \mathbb{Z}_l\}$ in order to find a conformal minimal immersion $X_0 \colon M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ which is close to X in the $\mathscr{C}^1(K)$ norm and such that the distance between p_0 and $\{p_k \colon k \in \mathbb{Z}_l\}$ in the induced metric $X_0^*(ds^2)$ is large in a suitable way. To be precise, we ask X_0 to keep satisfying

(i) $|X_0(p) - \mathfrak{Y}(q)| < \delta$ and $|X_0(p) - X(q)| < \epsilon_0$ for all $\{p, q\} \in \alpha_k$, $k \in \mathbb{Z}_l$,

and to meet also the following condition:

(ii) for each $k \in \mathbb{Z}_l$ there is a small open neighborhood U_k of p_k in M, with $\overline{U}_k \cap K = \varnothing$, such that if $\gamma \subset M$ is an arc with the initial point in K and the final point in $\overline{U}_k$ and $\{J_a\}_{a \in \mathbb{Z}_l}$ is a partition of γ by Borel measurable subsets, then $\sum_{a \in \mathbb{Z}_l} length \, \pi_a(X_0(J_a)) > \eta$.

To find X_0, choose pairwise disjoint Jordan arcs $\{\gamma_k \subset \widetilde{M} \colon k \in \mathbb{Z}_l\}$ such that each γ_k contains p_k as an endpoint, is attached transversely to M at p_k and is otherwise disjoint from M, and the set $S := M \cup \bigcup_{k \in \mathbb{Z}_l} \gamma_k \subset \widetilde{M}$ is admissible (see Definition 2.8). We then extend X to a generalized conformal minimal immersion $(X, f\theta) \in GCMI(S, \mathbb{R}^n)$ such that the following analogues to conditions (i) and (ii) hold.

* $|X(x) - \mathfrak{Y}(q)| < \delta$ and $|X(x) - X(q)| < \epsilon_0$ for all $(x, q) \in (\gamma_{k-1} \cup \alpha_k \cup \gamma_k) \times \alpha_k$.
* If $\{J_a\}_{a \in \mathbb{Z}_l}$ is a partition of γ_k by Borel measurable subsets, then $\sum_{a \in \mathbb{Z}_l} length \, \pi_a(X(J_a)) > 2\eta$.

This means that on each arc γ_k, X is highly oscillating in the direction of $F(p_a) - \mathfrak{Y}(p_a)$ for all $a \in \mathbb{Z}_l$, but with very small extrinsic diameter. By Theorem 4.1 we can approximate $(X, f\theta)$ uniformly on S by a conformal minimal immersion $\widetilde{X} \colon \widetilde{M} \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$. Let q_k denote the endpoint of γ_k different from p_k. If the approximation is close enough then [56, Theorem 2.3] provides a smooth diffeomorphism $\phi \colon M \to \phi(M)$ satisfying the following:

* $\phi \colon \mathring{M} \to \phi(\mathring{M})$ is a biholomorphism;
* ϕ is as close as desired to the identity in the $\mathscr{C}^1$ norm on the complement in M of a small neighborhood of $\{p_k \colon k \in \mathbb{Z}_l\}$; and
* $\phi(p_k) = q_k \in b \, \phi(M)$ and ϕ maps a suitably chosen neighborhood of $\{p_k \colon k \in \mathbb{Z}_l\}$ in M to a small neighborhood of $\bigcup_{k \in \mathbb{Z}_l} \gamma_k$ in $\widetilde{M}$.

Thus, doing things in the right way, when composing ϕ with $\widetilde{X}$ we are merging the arcs $X(\gamma_k)$ into $\widetilde{X}(M)$ without modifying M itself. It follows that the $\mathscr{C}^1(M)$ conformal minimal immersion $X_0 = \widetilde{X} \circ \phi \colon M \to \mathbb{R}^n$ satisfies conditions (i) and (ii).

We may assume that the sets $\overline{U}_k$, $k \in \mathbb{Z}_l$, are simply connected, smoothly bounded, and pairwise disjoint. Roughly speaking, X_0 meets the requirements in the lemma, except that $\mathrm{dist}_{X_0}(p_0, p) > d + \delta$ only holds for the points $p \in bM$ which lie in one of the sets $\overline{U}_k$. To conclude the proof, we perturb X_0 outside $\bigcup_{k \in \mathbb{Z}_l} U_k$, preserving what has already been achieved. At this point the Riemann–Hilbert method is invoked. Fix $\epsilon_1 > 0$ to be specified later, choose an annular neighborhood $A \subset M \backslash K$ of bM and a smooth retraction $\rho \colon A \to bM$. By (i) there is a family of pairwise disjoint, smoothly bounded closed discs $\overline{D}_k$ in $M \backslash K$, $k \in \mathbb{Z}_l$, such that $\bigcup_{k \in \mathbb{Z}_l} \overline{D}_k \subset A$, $\overline{D}_k \cap bM$ is a compact connected Jordan arc in $\alpha_k \backslash \{p_{k-1}, p_k\}$ with an endpoint in U_{k-1} and the other endpoint in U_k, and the following conditions hold:

(iii) $|X_0(p) - \mathfrak{Y}(q)| < \delta$ for all $(p, q) \in \overline{D}_k \times \alpha_k$; and
(iv) $\rho(\overline{D}_k) \subset \alpha_k \backslash \{p_{k-1}, p_k\}$ and $|X_0(\rho(x)) - X_0(x)| < \epsilon_1$ for all $x \in \overline{D}_k$, $k \in \mathbb{Z}_l$.

For each $k \in \mathbb{Z}_l$ we choose a pair of compact Jordan arcs $\beta_k \Subset I_k \Subset \overline{D}_k \cap \alpha_k$ with an endpoint in U_{k-1} and the other endpoint in U_k, and a pair of vectors $\mathbf{u}_k, \mathbf{v}_k \in \mathbb{R}^n$, such that $|\mathbf{u}_k| = 1 = |\mathbf{v}_k|$ and $\mathbf{u}_k, \mathbf{v}_k$, and $X(p_k) - \mathfrak{Y}(p_k)$ are pairwise orthogonal. Choose a continuous function $\mu \colon bM \to \mathbb{R}_+$ such that

$$0 \le \mu \le \eta, \quad \mu = \eta \text{ on } \bigcup_{k \in \mathbb{Z}_l} \beta_k, \quad \mu = 0 \text{ on } bM \backslash \bigcup_{k \in \mathbb{Z}_l} I_k.$$

Consider the continuous map $\varkappa \colon bM \times \overline{\mathbb{D}} \to \mathbb{R}^n$ given by

$$\varkappa(x, \xi) = \begin{cases} X_0(x), & x \in bM \backslash \bigcup_{k \in \mathbb{Z}_l} I_k, \\ X_0(x) + \mu(x)(\mathfrak{R}\xi \mathbf{u}_k + \mathfrak{I}\xi \mathbf{v}_k), & x \in I_k, \ k \in \mathbb{Z}_l. \end{cases}$$

These are the boundary data of a Riemann–Hilbert problem on X_0 for which the boundary disc $\varkappa(x, \cdot)$ at each point $x \in \beta_k$ is a round planar disc of radius η that is orthogonal to $X(p_k) - \mathfrak{Y}(p_k)$. Theorem 5.1 provides for every $k \in \mathbb{Z}_l$ an arbitrarily small open neighborhood $\Omega_k \subset \overline{D}_k$ of I_k and a conformal minimal immersion $Y \colon M \to \mathbb{R}^n$ of class $\mathscr{C}^1(M)$ satisfying the following conditions:

(v) $\mathrm{dist}(Y(x), \varkappa(x, \mathbb{T})) < \epsilon_1$ for all $x \in bM$;
(vi) $\mathrm{dist}(Y(x), \varkappa(\rho(x), \overline{\mathbb{D}})) < \epsilon_1$ for all $x \in \Omega := \bigcup_{k \in \mathbb{Z}_l} \Omega_k$;
(vii) Y is ϵ_1-close to X_0 in the $\mathscr{C}^1$ norm on $M \backslash \Omega$.

Using conditions (i)–(vii) and Pythagoras' theorem, it is not hard to see that Y satisfies the conclusion of the lemma provided that $\epsilon_0 > 0$ and $\epsilon_1 > 0$ are chosen sufficiently small. Very briefly, by (vi), (iv), and the definition of $\varkappa$ we have that $\pi_k \circ Y$ is $2\epsilon_1$-close to $\pi_k \circ X_0$ in the $\mathscr{C}^0(\Omega_k)$ topology for all $k \in \mathbb{Z}_l$, and so a weaker version of condition (ii) is preserved by the second deformation procedure. This enables us to ensure that $\mathrm{dist}_Y(p_0, p) > d + \eta$ for all $p \in bM \cap \bigcup_{k \in \mathbb{Z}_l} \overline{U}_k$. Taking into account (i), (v), and that $\mu = \eta$ on $\bigcup_{k \in \mathbb{Z}_l} \beta_k$, we infer the same inequality for all points $p \in bM \backslash \bigcup_{k \in \mathbb{Z}_l} \overline{U}_k$. On the other hand, (i), (v), (vii), the facts that $\mu = \eta$ on $\bigcup_{k \in \mathbb{Z}_l} \beta_k$ and that $\mathbf{u}_k, \mathbf{v}_k$,

and $X(p_k) - \mathfrak{Y}(p_k)$ are pairwise orthogonal, and Pythagoras' theorem guarantee that $|Y(p) - \mathfrak{Y}(p)| < \sqrt{\delta^2 + \eta^2}$ for all $p \in bM$. □

Another recent application of the Riemann–Hilbert method is the construction of complete minimal surfaces lying densely in arbitrary domains of $\mathbb{R}^n$. The following result is due to Alarcón and Castro-Infantes [6].

THEOREM 5.7. *Let $D \subset \mathbb{R}^n$ ($n \geq 3$) be an open connected set. Every bordered Riemann surface M admits a complete conformal minimal immersion $X \colon M \to \mathbb{R}^n$ such that $X(M)$ is a dense subset of D. If $n \geq 5$ then X may be chosen injective.*

5.4. Proper conformal minimal surfaces in minimally convex domains. A major problem in minimal surface theory is to understand which domains in $\mathbb{R}^n$ admit complete properly immersed minimal surfaces, and how the geometry of the domain influences the conformal properties of such surfaces. For a background on this topic we refer to Meeks and Perez [86, Section 3]. In dimension $n = 3$ this subject is connected with the Calabi–Yau problem. In view of Nadirashvili's example [92], Yau [110] asked whether there exist complete minimal surfaces properly immersed in the ball of $\mathbb{R}^3$. An affirmative answer for any either convex or bounded and smoothly bounded domain in $\mathbb{R}^3$ was given by Martín and Morales in [82–84]. In the opposite direction, Martín *et al.* [81] provided examples of bounded domains in $\mathbb{R}^3$ which do not admit any complete proper minimal surfaces of finite topology.

If M is a bordered Riemann surface for which one is able to construct in a standard inductive way a proper conformal minimal immersion into a given domain $D \subset \mathbb{R}^n$, then one can also construct a *complete* proper one by using the procedure in Lemma 5.5 which enlarges the intrinsic boundary distance within the surface as much as desired by an arbitrarily uniformly small displacement of the surface in D. Hence it suffices to focus on the existence of proper conformal minimal immersions.

Recent examples by Alarcón *et al.* [8] show that some geometric assumptions on the domain are necessary to obtain positive results. Indeed, there is a bounded simply connected domain $D \subset \mathbb{R}^3$ carrying no proper conformal minimal disc $\mathbb{D} \to D$ passing through a certain point in D, and a bounded domain $D \subset \mathbb{R}^3$ admitting no proper minimal surfaces with finite topology and a single end (see [8, Examples 1.13 and 1.14]). Much earlier, Dor [38] found a bounded domain $\Omega \subset \mathbb{C}^m$ for any $m \geq 2$ which does not admit any proper holomorphic discs $\mathbb{D} \to \Omega$. It remains an open problem whether there is a domain $D \subset \mathbb{R}^n$ for $n > 3$ without any proper minimal discs.

A $\mathscr{C}^2$ function $\rho \colon D \to \mathbb{R}$ on a domain $D \subset \mathbb{R}^n$ is said to be *strongly minimal plurisubharmonic* if $\mathrm{tr}_L \mathrm{Hess}_\rho(x) > 0$ for every affine two-dimensional linear subspace L and every point $x \in D \cap L$; equivalently, if $\lambda_1(x) + \lambda_2(x) > 0$ for all $x \in D$ where $\lambda_1(x) \leq \lambda_2(x)$ are the two smallest eigenvalues of the Hessian $\mathrm{Hess}_\rho(x)$. The domain D is said to be *minimally convex* if it admits a strongly minimal plurisubharmonic exhaustion function (see [8, Section 2]). The following result was obtained as an application of Theorem 5.1 (the Riemann–Hilbert method) with functions $\varkappa$ of the form (5.3); see [8, Theorem 1.1 and Remark 3.8].

THEOREM 5.8. *Let $D \subset \mathbb{R}^n$ ($n \geq 3$) be a minimally convex domain and M be a compact bordered Riemann surface. Every conformal minimal immersion $M \to D$ may be approximated uniformly on compacts in $\mathring{M} = M \backslash bM$ by proper (and complete if so desired) conformal minimal immersions $\mathring{M} \to D$ (embeddings if $n \geq 5$).*

In particular, every *mean-convex* domain in $\mathbb{R}^3$ admits complete proper minimal surfaces normalized by any bordered Riemann surface. We refer to [7, 8] for more precise results including infinite topologies, control of the flux, and continuous extendibility up to the boundary.

Since the complement of a properly embedded minimal surface in $\mathbb{R}^3$ is a minimally convex domain [8, Corollary 1.3], we have the following corollary to Theorem 5.8.

COROLLARY 5.9. *Let $S \subset \mathbb{R}^3$ be a properly embedded minimal surface and D be a connected component of $\mathbb{R}^3 \backslash S$. Every bordered Riemann surface admits a proper conformal minimal immersion into D.*

SKETCH OF PROOF OF THEOREM 5.8. We discuss the case $n = 3$. Let $\rho : D \to \mathbb{R}_+$ be a smooth strongly minimally plurisubharmonic Morse exhaustion function with the (discrete) critical locus P. We extend ρ to a function on the tube $D \times i\mathbb{R}^3 \subset \mathbb{C}^3$ which is independent of the imaginary coordinates. A simple calculation shows that ρ is strongly minimally plurisubharmonic if and only if the Levi form of the extended function at any point of $D \times i\mathbb{R}^3$ is positive on every null vector $w \in \mathfrak{A}_*$ (see [42, Lemma 4.3]; such functions are called *strongly null plurisubharmonic*). It is then not hard to see (see [8, Lemma 3.1]) that for any compact set $L \subset D \backslash P$ there are a constant $c = c_L > 0$ and families of embedded null holomorphic discs $\sigma_x^j = \alpha_x^j + i\beta_x^j : \overline{\mathbb{D}} \to \mathbb{C}^3$ ($x \in L$, $j = 1, 2$), depending locally $\mathscr{C}^1$ smoothly on the point $x \in L$ and satisfying the following conditions:

(a) $\sigma_x^j(0) = 0$;
(b) $\{x + \alpha_x^j(\xi) : \xi \in \overline{\mathbb{D}}\} \subset D$;
(c) the function $\overline{\mathbb{D}} \ni \xi \mapsto \rho(x + \alpha_x^j(\xi))$ is strongly convex and satisfies $\rho(x + \alpha_x^j(\xi)) \geq \rho(x) + c|\xi|^2$ for $\xi \in \overline{\mathbb{D}}$.

Indeed, it suffices to choose σ_x^j in the quadratic complex hypersurface

$$\Sigma_x = \left\{ w = (w_1, w_2, w_3) \in \mathbb{C}^3 : \sum_{j=1}^{3} \frac{\partial \rho}{\partial x_j}(x) w_j + \sum_{j,k=1}^{3} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(x) w_j w_k = 0 \right\}.$$

The tangent plane $T_0 \Sigma_x \subset \mathbb{C}^3$ contains precisely two null directions, which leads to two families of null discs as above. The restriction of ρ to the affine hypersurface $x + \Sigma_x$ has the Taylor expansion

$$\rho(x + w) = \rho(x) + \mathcal{L}_\rho(x; w) + o(|w|^2),$$

where $\mathcal{L}_\rho$ denotes the Levi form of ρ. Since $\mathcal{L}_\rho(x; w) > 0$ on null vectors $w \in \mathfrak{A}_*$, we get the estimates in condition (c).

Using the conformal minimal discs α_x^j as above and a conformal minimal immersion $X \colon M \to D$, we consider the Riemann–Hilbert problem in Theorem 5.1, but with the function $\varkappa$ of the form (5.3). This shows that for any compact set $L \subset D \backslash P$ there are constants $\epsilon_0 > 0$ and $C_0 > 0$ such that the following holds. Let M be a compact bordered Riemann surface and $X \colon M \to D$ be a conformal minimal immersion of class $\mathscr{C}^1(M)$ with $X(bM) \subset L$. Given a continuous function $\epsilon \colon bM \to [0, \epsilon_0]$ supported on the set $J = \{\zeta \in bM : X(\zeta) \in L\}$, an open set $U \subset M$ containing $\mathrm{supp}(\epsilon)$ in its relative interior, and a constant $\delta > 0$, there exists a conformal minimal immersion $Y \colon M \to D$ satisfying the following conditions:

(i) $|\rho(Y(\zeta)) - \rho(X(\zeta)) - \epsilon(\zeta)| < \delta$ for every $\zeta \in bM$;
(ii) $\rho(Y(\zeta)) \geq \rho(X(\zeta)) - \delta$ for every $\zeta \in M$;
(iii) Y is δ-close to X in the $\mathscr{C}^1$ norm in $M \backslash U$;
(iv) Y is $C_0 \sqrt{\epsilon_0}$-close to X in the $\mathscr{C}^0$ norm in M.

The theorem is proved by a inductive application of this procedure, together with a well-known method of avoiding critical points of ρ. By using also Lemma 5.5 we can obtain complete proper conformal minimal immersions $\mathring{M} \to D$. □

There is a variety of results and open questions in the literature as to which domains in $\mathbb{R}^3$ may or may not contain minimal surfaces (possibly with additional properties) which are proper in $\mathbb{R}^3$. One of the main examples is the theorem of Hoffman and Meeks from 1990 [71] to the effect that a properly immersed minimal surface $M \subset \mathbb{R}^3$ is never contained in a half space, unless M is a plane. Since minimally convex domains are not necessarily convex and they may be quite big, it is a natural question whether they may contain nonflat minimal surfaces which are proper in $\mathbb{R}^3$. Although we do not know a definitive answer to this question, we have the following rigidity result [8, Theorem 1.16] for properly immersed minimal surfaces of finite total curvature in $\mathbb{R}^3$ lying in minimally convex domains.

THEOREM 5.10. *Let S be a complete connected properly immersed minimal surface with finite total curvature in $\mathbb{R}^3$, possibly with (compact) boundary, and let $D \subset \mathbb{R}^3$ be a connected minimally convex domain containing S. If S is not a plane then $D = \mathbb{R}^3$. If S is a plane then D is a slab (a domain bounded by two parallel planes), a halfspace, or $\mathbb{R}^3$.*

Acknowledgements

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