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RESOLVABLE (r, λ)-DESIGNS AND THE FISHER INEQUALITY

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Abstract

It is well known that in any (v, b, r, k, λ) resolvable balanced incomplete block design that $b \geqslant v+r-1$ with equality if and only if the design is affine resolvable. In this paper, we show that a similar inequality holds for resolvable regular pairwise balanced designs ((r, λ)-designs) and we characterize those designs for which equality holds. From this characterization, we deduce certain results about block intersections in (r, λ)-designs.

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1. Introduction

An (r, λ)-design D is a collection B of nonempty subsets (called blocks) of a finite set V (called varieties) such that (i) every variety is contained in precisely r blocks and (ii) every pair of distinct varieties is contained in exactly λ blocks.

If every block of B has cardinality k then D is called a balanced incomplete block design (BIBD). Any block of D, which contains all of the varieties, is called a complete block.

If the blocks of D can be partitioned into classes such that every variety is contained in precisely one block of each class, then D is called a resolvable (r, λ)-design. The classes are called resolution classes. A resolvable (r, λ)-design which is a BIBD is denoted RBIBD. An RBIBD having the property that any two blocks from distinct resolution classes intersect in a constant m number of varieties is termed affine.

In 1940, Fisher showed that in any BIBD having v varieties and b blocks, $b \geqslant v$. This inequality was later shown to hold for (r, λ)-designs and, in fact, for a more

Solving (4) for $\sum_{j=1}^{v} x_j$ and substituting into (3), we see that y_i is a linear combination of the vectors, $\mathbf{B}^* \cup \{\mathbf{B}_{ij}\colon 1 \leqslant j \leqslant t_i,\ 1 \leqslant i \leqslant r\}$. Also, if we substitute $\sum_{i=1}^{v} x_i$ into (2) and $\sum_{i=1}^{r} y_i = \mathbf{B}^*$, then x_1 is a linear combination of $\mathbf{B}^* \cup \{\mathbf{B}_{ij}\colon 1 \leqslant j \leqslant t_i,\ 1 \leqslant i \leqslant r\}$. Clearly, the same can be done for any x_i, $1 \leqslant i \leqslant v$. Since it is possible to write the basis $x_1, x_2, \ldots, x_v, y_1, \ldots, y_r$ as linear combinations of $\mathbf{B}^* \cup \{\mathbf{B}_{ij}\}$, this set must be a spanning set of S. Hence, the number of vectors in this set must be greater than or equal to $v+r$. That is,

$$b+1 \geqslant v+r \quad \text{or} \quad b \geqslant v+r-1.$$

This completes the proof.

THEOREM 2.2. *Let D be a resolvable (r, λ)-design having v varieties and b blocks. If $b = v+r-1$ then (i) the blocks of any given resolution class are equicardinal, (ii) $\lambda(v-1) \geqslant r(n-1)$ with equality if and only if D is an affine resolvable BIBD.*

PROOF. As in the proof of Theorem 2.1, consider the vector space S. Let k_{ij} be the number of elements in B_{ij}. Summing the blocks which contain x_l gives

$$\sum_{\mathbf{B},\, x_l \in \mathbf{B}} \mathbf{B} = n x_l + \lambda \sum_{i=1}^{v} x_i + \sum_{i=1}^{r} y_i.$$

Now, summing over all varieties yields

$$\sum_{i=1}^{r} \sum_{j=1}^{t_i} k_{ij} \mathbf{B}_{ij} = (n+\lambda v) \sum_{i=1}^{v} x_i + v\mathbf{B}^*. \tag{5}$$

From the proof of Theorem 2.1,

$$\sum_{i=1}^{v} x_i = \frac{1}{L}\left[\sum_{i=1}^{r} \frac{1}{t_i}\left(\sum_{j=1}^{t_i} B_{ij}\right) - \mathbf{B}^*\right]$$

where $L = \sum_{i=1}^{r}(1/t_i)$. Substituting this into (5) and rearranging,

$$\sum_{i=1}^{r} \sum_{j=1}^{t_i} \left[k_{ij} - \frac{1}{Lt_i}(n+\lambda v)\right] \mathbf{B}_{ij} + \left[\frac{(n+\lambda v)}{L} - v\right] \mathbf{B}^* = 0.$$

Since $b = v+r-1$, $\{\mathbf{B}^*\} \cup \{\mathbf{B}_{ij}\colon 1 \leqslant j \leqslant t_i,\ 1 \leqslant i \leqslant r\}$ is a basis for S and, hence,

$$k_{ij} - \frac{(n+\lambda v)}{Lt_i} = 0, \quad 1 \leqslant j \leqslant t_i, \quad 1 \leqslant i \leqslant r. \tag{6}$$

and

$$\frac{n+\lambda v}{L} - v = 0. \tag{7}$$

For fixed i, k_{ij} is independent of j for all j, $1 \leqslant j \leqslant t_i$. Therefore, the blocks of R_i

are equicardinal and we can let

$$k_i = k_{ij} = \frac{n+\lambda v}{Lt_i}, \quad 1 \leqslant i \leqslant r.$$

This proves (i) of the theorem.

Let $M = \sum_{i=1}^{r} t_i$. Clearly, $M = v+r-1$.

From (7),

$$L = \frac{n+\lambda v}{v}.$$

Since $t_i \geqslant 1$, $1 \leqslant i \leqslant r$, the arithmetic–geometric mean inequality can be applied to the series $L = \sum_{i=1}^{r}(1/t_i)$ and $M = \sum_{i=1}^{r} t_i$ to produce

$$LM \geqslant r^2 \tag{8}$$

with equality if and only if all of the t_i are equal. Hence,

$$\frac{(n+\lambda v)}{v}(v+r-1) \geqslant r^2$$

or

$$\lambda(v-1) \geqslant r(n-1). \tag{9}$$

If equality holds in (9) then we have equality in (8) and, hence, D is a resolvable BIBD with $b = v+r-1$. By the result of Bose (1942), D is an affine resolvable BIBD and the proof is complete.

If D is a resolvable (r, λ)-design which is a BIBD with block size k, then $bk = rv$, $\lambda(v-1) = r(k-1)$ and from the Bose inequality it follows that $k \leqslant n$. Using this fact, it is readily deduced that for any resolvable BIBD with parameters (v, b, r, k, λ),

$$\lambda(v-1) \leqslant r(n-1),$$

which reverses the inequality given in (ii) of Theorem 2.2.

As an example of a resolvable (r, λ)-design having

$$b = v+r-1 \quad \text{and} \quad \lambda(v-1) > r(n-1),$$

we give the following: $B_1 = \{1,2,3,4\}$, $B_2 = \{1,2\}$, $B_3 = \{3,4\}$, $B_4 = \{1,3\}$, $B_5 = \{2,4\}$, $B_6 = \{1,4\}$, $B_7 = \{2,3\}$, which is a resolvable (4, 2)-design having 4 varieties and 7 blocks.

3. Mutually disjoint blocks

Let D be an (r, λ)-design having v varieties, and b blocks, t of which are mutually disjoint and of size k. Let A be the $v \times b$ incidence matrix of D where the first t columns correspond to the t mutually disjoint blocks and the first column has ones in the first k rows, the second column has ones in the next k rows and so on for the first t columns. This is, of course, always possible. Now define a new matrix N as follows.

Let $E_{t\times b}$ be a $t\times b$ matrix having a one in position (i,i), $1\leqslant i\leqslant t$, and zeros elsewhere. Let $I_{1\times b}$ be the all ones vector. Then,

$$N=\begin{bmatrix} A_{v\times b}\\ E_{t\times b}\\ I_{1\times b}\end{bmatrix}_{(v+t+1)\times b}$$

$$NN^{\mathrm{T}}=\left[\begin{array}{ccccccccccc|cccccc}
 & & & & & & & & & & & 1 & & & & & r\\
 & & & & & & & & & & & \vdots & & & & & r\\
 & & & & & & & & & & & & 1 & & & & \cdot\\
 & & & & & & & & & & & & \vdots & & & & \cdot\\
 & & & & & & & & & & & & 1 & \cdots & & & \cdot\\
 & & & & & \lambda J+(r-\lambda)I & & & & & & & & & 1 & & \cdot\\
 & & & & & & & & & & & & & & \vdots & & \cdot\\
 & & & & & & & & & & & & & & 1 & & \cdot\\
 & & & & & & & & & & & & & & 0 & & \cdot\\
 & & & & & & & & & & & & & & \vdots & & r\\
 & & & & & & & & & & & & & & 0 & & r\\
\hline
1 & 1 & \cdots & 1 & & & & & & & & 1 & & & & & 1\\
 & & & & 1 & \cdots & 1 & & & & & & 1 & & & & \vdots\\
 & & & & & & \vdots & & & & & & & \ddots & & & 1\\
 & & & & & & 1 & \cdots & 10 & \cdots & 0\ \ 0 & & 1 & 1 & \cdots & 1 & b\\
r & r & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & r\ \ r & & & & & &
\end{array}\right]_{(v+t+1)\times(v+t+1)}$$

where J is a $v\times v$ matrix of all ones and I is the $v\times v$ identity matrix. Using Lemma 3.1 of Connor (1952),

$$\text{(10)}\qquad \det NN^{\mathrm{T}}=a^{-t}n^{v-t-2}\begin{vmatrix} F & H & \cdots & H & A\\ H & F & \cdots & \cdots & \cdots\\ \vdots & & \ddots & \vdots & \vdots\\ \cdots & \cdots & \cdots & H & \cdots\\ H & \cdots & H & F & A\\ A & \cdots & \cdots & A & G\end{vmatrix}_{(t+1)\times(t+1)}$$

where

$$a = n+\lambda v,\quad F = an-ak+\lambda k^2,\quad H = \lambda k^2,$$

$$A = an-akr+\lambda vrk,\quad G = anb-avr^2+\lambda v^2 r^2.$$

Now, if $t+1<b-v$, then $\det NN^{\mathrm{T}}\geqslant 0$. Since $a>0$ and $n>0$, the $(t+1)\times(t+1)$ determinant in (10) must also be non-negative. Hence,

$$\text{(11)}\qquad \begin{vmatrix} F & H & \cdots & H & A \\ H & F & \cdots & \cdots & \cdots \\ \vdots & & \ddots & & \vdots \\ \cdots & \cdots & \cdots & H & \cdots \\ H & \cdots & H & F & A \\ A & \cdots & \cdots & A & G \end{vmatrix} = a^t(n-k)^{t-1}n\{s(n-k)+tX\}\geqslant 0$$

where $s = ab-vr^2$ and $X = (b\lambda-r^2)k^2+2nrk-an$. If we have t mutually disjoint blocks of size k, then there must be t' mutually disjoint blocks of size k for all t', $1\leqslant t'\leqslant t$. Therefore, the inequality in (11) must hold if we replace t by t' for all t', $1\leqslant t'\leqslant t$. If we set $t = 1$ in (11), we obtain an inequality on the block sizes in any (r,λ)-design. This inequality is

$$(b\lambda-r^2)k^2+(vr^2-ab+2nr)k+n(ab-vr^2-a)\geqslant 0$$

and it was first proven in McCarthy and Vanstone (1979). We now state and prove a few consequences of (11) which will be useful in the characterization of resolvable (r,λ)-designs having $b = v+r-1$. Let s and X be as defined above.

THEOREM 3.1. *Let D be any (r,λ)-design having v varieties and b blocks and such that $s\neq 0$ and $X\neq 0$. If D contains t mutually disjoint blocks of size $k>n$, then $t\leqslant 2$.*

PROOF. From (10), we have that

$$\text{(12)}\qquad (n-k)^{l-1}\{s(n-k)+lX\}\geqslant 0$$

for all l, $1\leqslant l\leqslant t$. Assume $t\geqslant 3$, in which case there exists an integer i, $1<i<t$. Since k is a fixed integer, $s(n-k)+iX$ is either positive or negative. If it is positive and X is negative and $(n-k)^{i-1}$ is positive, then replacing i by $i-1$ in (11) makes $(n-k)^{i-2}$ negative, and $s(n-k)+iX$ remains positive which contradicts (11). Suppose $s(n-k)+iX$ is negative, X is negative and $(n-k)^{i-1}$ is negative. If we replace i by $i+1$ in (12), $(n-k)^i$ is positive and $s(n-k)+(i+1)X$ is negative which contradicts (12). There are several other cases to consider but they all produce contradictions in a similar manner and so are omitted. Therefore, $t\leqslant 2$ and the proof is complete.

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THEOREM 3.3. *Let D be any (r, λ)-design having v varieties, b blocks and such that $s>0$. Then, for $k>n$, any two blocks of size k intersect.*

PROOF. If $t=2$ in (12), $(n-k)^{t-1}$ is negative for $k>n$. Hence, if $T=s(n-k)+tX$ is positive, we get a contradiction. Suppose T is negative. Since $s(n-k)<0$, then

$$s(n-k)+iX<0, \quad \text{for } i=1 \text{ or } 2,$$

since it is negative for $i=2$. Hence, if T is negative, replace t by $t-1$; then, $(n-k)^{t-2}$ is positive, T is negative and we have a contradiction. Therefore, D cannot contain a pair of disjoint blocks of size $k>n$. This completes the proof.

4. Characterization of resolvable (r, λ)-designs having $b=v+r-1$

We now apply the results of Sections 2 and 3 to characterize all resolvable (r, λ)-designs having v varieties and $b=v+r-1$ blocks. First, we require the following lemma.

LEMMA 4.1. *Let D be a resolvable (r, λ)-design having v varieties and $b=v+r-1$ blocks. Then*

$$s=ab-vr^2 \geqslant 0, \quad a=n+\lambda v,$$

with equality if and only if D is affine resolvable.

PROOF.

$$\begin{aligned} s &= ab-vr^2 \\ &= (n+\lambda v)(v+r-1)-vr^2 \\ &= (n+\lambda v)(v-1)+nr+r^2v-rnv-vr^2 \\ &= (v-1)(n+\lambda v-rn). \end{aligned}$$

But Theorem 2.1 gives $\lambda(v-1) \geqslant r(n-1)$ with equality if and only if D is affine resolvable which implies that $n+\lambda v-rn \geqslant 0$ and the proof is complete.

THEOREM 4.2. *Let D be a resolvable (r, λ)-design having v varieties and $b=v+r-1$ blocks. Then D is either (i) an affie resolvable BIBD or (ii) an affine resolvable BIBD with complete blocks adjoined.*

PROOF. By Theorem 2.1, the blocks in a given resolution class of D are equicardinal. Suppose some resolution class of D contains t blocks of size $k>n$. Then Lemma 4.1 and Theorems 3.1 and 3.3 imply that $t=1$, and hence the resolution class consists of a single complete block. Therefore, the blocks of D either are complete or have cardinality less than or equal to n. Form a new resolvable (r', λ')-design D' by deleting all complete blocks of D. In D', $b' = v + r' - 1$ and all blocks have size less than or equal to n. By counting the number of pairs which contain a particular variety, we get $\lambda'(v-1) \leqslant r'(n-1)$. But Theorem 2.2, gives $\lambda'(v-1) \geqslant r'(n-1)$ for D'. Hence,

$$\lambda'(v-1) = r'(n-1)$$

and by Theorem 2.2, D' is an affine resolvable BIBD. This completes the proof.

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