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BIFURCATION OF POSITIVE ENTIRE SOLUTIONS FOR A SEMILINEAR ELLIPTIC EQUATION

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In this paper, we consider the nonhomogeneous semilinear elliptic equation

$$(*)_\lambda \qquad -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \mathbb{R}^N, u > 0 \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N),$$

where $\lambda \geqslant 0$, $1 < p < (N+2)/(N-2)$, if $N \geqslant 3$, $1 < p < \infty$, if $N = 2$, $h(x) \in H^{-1}(\mathbb{R}^N)$, $0 \not\equiv h(x) \geqslant 0$ in $\mathbb{R}^N$, $K(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N$. We prove that if $K(x) \geqslant K_\infty > 0$ in $\mathbb{R}^N$, and $\lim\limits_{|x|\to\infty} K(x) = K_\infty$, then there exists a positive constant λ^* such that $(*)_\lambda$ has at least two solutions if $\lambda \in (0, \lambda^*)$ and no solution if $\lambda > \lambda^*$. Furthermore, $(*)_\lambda$ has a unique solution for $\lambda = \lambda^*$ provided that $h(x)$ satisfies some suitable conditions. We also obtain some further properties and bifurcation results of the solutions of $(1.1)_\lambda$ at $\lambda = \lambda^*$.

1. INTRODUCTION

In this paper, we consider the semilinear elliptic equation

$$(1.1)_\lambda \qquad \begin{cases} -\Delta u + u = \lambda K(x)u^p + h(x) \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N, u \in H^1(\mathbb{R}^N), \end{cases}$$

where $\lambda \geqslant 0$, $1 < p < (N+2)/(N-2)$, if $N \geqslant 3$, $1 < p < \infty$, if $N = 2$, $h(x) \in H^{-1}(\mathbb{R}^N)$, $0 \not\equiv h(x) \geqslant 0$ in $\mathbb{R}^N$, $K(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N$. Moreover, $h(x)$ and $K(x)$ satisfy the following conditions:

$(h1)$ $h(x) \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > N/2$ if $N \geqslant 3$, $q = 2$ if $N = 2$.

$(k1)$ $K(x) \geqslant K_\infty > 0$ in $\mathbb{R}^N$, and $\lim\limits_{|x|\to\infty} K(x) = K_\infty$.

The homogeneous case, that is, $h(x) \equiv 0$, the equation $(1.1)_\lambda$ has been studied by many authors (see **[5, 8, 13, 14, 15]**.) For the nonhomogeneous case ($h(x) \not\equiv 0$), Zhu **[16]**, Zhu and Zhou **[18]** and Cao and Zhou **[6]**, established the existence of multiple positive solutions of equations with structure unlike that here.

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The main aim of this paper is concerned with the existence and nonexistence of multiple positive solutions of $(1.1)_\lambda$ for the full $\lambda \in [0, \infty)$. We also obtain some properties of solutions and some bifurcation results of solutions at $\lambda = 0$ and $\lambda = \lambda^*$, where λ^* is given in Theorem 1.1 below.

Throughout this paper, we always assume that $h(x) \geqslant 0$, $h(x) \not\equiv 0$ in $\mathbb{R}^N$, $K(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N$ and u_0 is the unique solution of $(1.1)_0$, unless otherwise specified and we set

$$\begin{aligned}
\|u\| &= \left(\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx\right)^{1/2}, \\
\|u\|_q &= \left(\int_{\mathbb{R}^N} |u|^q \, dx\right)^{1/q}, \quad 2 \leqslant q < \infty, \\
\|u\|_\infty &= \sup_{x \in \mathbb{R}^N} |u(x)|, \\
M &= \inf\left\{\int_{\mathbb{R}^N} \left(|\nabla u|^2 + |u|^2\right) dx : \int_{\mathbb{R}^N} |u|^{p+1} \, dx = 1\right\}.
\end{aligned}$$

Now, we state our main results in the following.

THEOREM 1.1. *If $h(x) \geqslant 0$ and $h(x) \not\equiv 0$ in $\mathbb{R}^N$, $K(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N$ and $K(x)$ satisfies $(k1)$. Then there is λ^*, $0 < \lambda^* < \infty$, such that:*

(i) *$(1.1)_\lambda$ has at least two solutions u_λ, U_λ and $u_\lambda < U_\lambda$ if $\lambda \in (0, \lambda^*)$;*
(ii) *$(1.1)_{\lambda^*}$ has a unique solution u_{λ^*} provided that $h(x)$ satisfies $(h1)$;*
(iii) *$(1.1)_\lambda$ has no positive solutions if $\lambda > \lambda^*$.*

Furthermore,

$$\begin{aligned}
\lambda_1 \equiv\ & \frac{(p+1)(p-1)^{p-1} M^{(p+1)/2}}{(2p)^p \|K\|_\infty \|h\|_{H^{-1}}^{p-1}} \\
\leqslant\ & \lambda^* \leqslant \inf_{w \in H^1(\mathbb{R}^N)\setminus\{0\}} \left(\frac{\|w\|^2}{p\int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx}\right) \equiv \lambda_2 \qquad (1.2) \\
\leqslant\ & \frac{p\|h\|_{H^{-1}}^2}{(p-1)^2 \int_{\mathbb{R}^N} K u_0^{p+1} \, dx} \equiv \lambda_3
\end{aligned}$$

where u_λ is the minimal solution of $(1.1)_\lambda$, U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4 and u_0 is the unique positive solution of $(1.1)_0$.

THEOREM 1.2. *If $(h1)$, $(k1)$ hold, $h(x) \geqslant 0$, $h(x) \not\equiv 0$ in $\mathbb{R}^N$ and $K(x)$ is a positive, bounded and continuous function on $\mathbb{R}^N$. Then*

(i) *u_λ is strictly increasing with respect to λ , u_λ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for all $\lambda \in [0, \lambda^*]$ and*

$$u_\lambda \to u_0 \text{ in } L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \text{ as } \lambda \to 0,$$

where u_0 is the unique positive solution of $(1.1)_0$.

(ii) *U_λ is strictly decreasing with respect to λ and U_λ is unbounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, that is*

$$\lim_{\lambda\to 0}\|U_\lambda\| = \lim_{\lambda\to 0}\|U_\lambda\|_\infty = \infty.$$

(iii) *Moreover, we assume that $K(x)$ and $h(x)$ are in $C^\alpha(\mathbb{R}^N)\cap L^2(\mathbb{R}^N)$, then all solutions of $(1.1)_\lambda$ are in $C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N)$, and $(\lambda^*, u_{\lambda^*})$ is a bifurcation point for $(1.1)_\lambda$ and*

$$u_\lambda \to u_0 \text{ in } C^{2,\alpha}(\mathbb{R}^N)\cap H^2(\mathbb{R}^N) \text{ as } \lambda \to 0,$$

where u_0 is the unique positive solution of $(1.1)_0$.

We shall organise this paper as follows. In Section 2, we give some notations and preliminary results. In Section 3, we assert that there exists $\lambda^* > 0$ such that $(1.1)_\lambda$ has a minimal solution for $\lambda \in [0, \lambda^*)$. In Section 4, we establish the existence of a second solution U_λ for $\lambda \in (0, \lambda^*)$ and some asymptotic behaviour of the solution of $(1.1)_\lambda$. In Section 5, we shall give some further properties, and bifurcation of solutions of $(1.1)_\lambda$.

2. Preliminaries

In this section, we shall give some notations and some known results. In order to get the existence of positive solutions of $(1.1)_\lambda$, we consider the energy functional $I_\lambda : H^1(\mathbb{R}^N) \to \mathbb{R}$ defined by

$$I_\lambda(u) = \int_{\mathbb{R}^N}\left[\frac{1}{2}(|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1}K(x)(u^+)^{p+1} - h(x)u\right]dx,$$

where $u^\pm(x) = \max\{\pm u(x), 0\}$. Then the critical points of I_λ are the positive solutions of $(1.1)_\lambda$. Consider the equation

$$(2.1)_\lambda \qquad \begin{cases} -\Delta u + u = \lambda K_\infty u^p \text{ in } \mathbb{R}^N, \\ u > 0 \text{ in } \mathbb{R}^N,\ u \in H^1(\mathbb{R}^N), \end{cases}$$

and its associated energy functional I_λ^∞ defined by

$$I_\lambda^\infty(u) = \int_{\mathbb{R}^N}\left[\frac{1}{2}(|\nabla u|^2 + |u|^2) - \frac{\lambda}{p+1}K_\infty(u^+)^{p+1}\right]dx,\ u \in H^1(\mathbb{R}^N).$$

It is well known that equation $(2.1)_\lambda$ has a unique ground state solution ω_λ and $I_\lambda^\infty(\omega_\lambda) = \sup_{t>0} I_\lambda^\infty(t\omega_\lambda)$ (see Bahri and Lions [**3**] and the references there).

Now, we given the following known propositions for later use.

Proposition 2.1. *Let $K(x)$ satisfy $(k1)$ and $\{u_k\}$ be a $(PS)_c$–sequence of I_λ in $H^1(\mathbb{R}^N)$:*

$$I_\lambda(u_k) = c + o(1) \text{ as } k \to \infty,$$
$$I'_\lambda(u_k) = o(1) \text{ strongly in } H^{-1}(\mathbb{R}^N).$$

Then there exist an integer $l \geqslant 0$, *sequence* $\{x_k^i\} \subseteq \mathbb{R}^N$, *functions* $\overline{u} \in H^1(\mathbb{R}^N)$, $\overline{u}_i \in H^1(\mathbb{R}^N)$, $1 \leqslant i \leqslant l$, *such that for some subsequence* $\{u_k\}$, *we have*

$$\begin{cases} u_k - \left(\overline{u} + \sum_{i=1}^{l} \overline{u}_i(\cdot - x_k^i)\right) \to 0, \text{ as } k \to \infty, \\ u_k \rightharpoonup \overline{u} \text{ weakly in } H^1(\mathbb{R}^N); \\ c = I_\lambda(\overline{u}) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{u}_i); \\ -\Delta\overline{u} + \overline{u} = \lambda K(x)\overline{u}^p + h(x) \text{ in } H^{-1}(\mathbb{R}^N); \\ -\Delta\overline{u}_i + \overline{u}_i = \lambda K_\infty \overline{u}_i^p \text{ in } H^{-1}(\mathbb{R}^N),\ 1 \leqslant i \leqslant l; \\ |x_k^i| \to \infty,\ \left|x_k^i - x_k^j\right| \to \infty,\ 1 \leqslant i \neq j \leqslant l. \end{cases}$$

where we agree that in the case $l = 0$ *the above holds without* $\overline{u}_i$, x_k^i.

PROOF: The proof can be obtained by using the arguments in Bahri and Lions [**3**] (also see [**13, 14**]). We omit it. ∎

3. Existence of minimal solution and decay

In this section, by the barrier method, we prove that the existence of minimal positive solution u_λ for all λ in some finite interval $[0, \lambda^*]$ (that is, for any positive solution u of $(1.1)_\lambda$, then $u \geqslant u_\lambda$). Furthermore, we establish a decay estimate for solutions of $(1.1)_\lambda$.

LEMMA 3.1. *Let* $K(x)$ *satisfy* $(k1)$. *Then* $(1.1)_\lambda$ *has a solution* u_λ *if* $0 \leqslant \lambda < \lambda_1$ *where* λ_1 *is given by* (1.2).

PROOF: For $\lambda = 0$, the existence question is equivalent to the existence of $u_0 \in H^1(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla\phi + u_0\phi = \int_{\mathbb{R}^N} h\phi \tag{3.1}$$

for all $\phi \in H^1(\mathbb{R}^N)$. Now, we have that

$$\left|\int_{\mathbb{R}^N} h\phi\right| \leqslant \|h\|_{H^{-1}}\|\phi\|.$$

According to the Lax–Milgram theorem, there exists a unique $u_0 \in H^1(\mathbb{R}^N)$ satisfies (3.1). Since $0 \not\equiv h \geqslant 0$ in $\mathbb{R}^N$, by strong maximum principle (see Gilbarg and Trudinger [**10**]), we conclude that $u_0 > 0$ in $\mathbb{R}^N$.

We consider next the case $\lambda > 0$. We show first that for sufficiently small λ, say $\lambda = \lambda_0$, there exists $t_0 = t(\lambda_0) > 0$ such that $I_{\lambda_0}(u) > 0$ for $\|u\| = t_0$. From the definitions of I_λ, we have

$$I_\lambda(u) \geqslant \frac{1}{2}\|u\|^2 - \frac{\lambda}{p+1}\|K\|_\infty M^{-(p+1)/2}\|u\|^{p+1} - \|h\|_{H^{-1}}\|u\|$$

Set

$$f(t) = \frac{1}{2}t - \lambda c_1 t^p - c_2,$$

where $c_1 = \|K\|_\infty/(p+1)M^{-(p+1)/2}$ and $c_2 = \|h\|_{H^{-1}}$.

It then follows that $f(t)$ achieves a maximum at $t_\lambda = (2p\lambda c_1)^{-(p-1)^{-1}}$. Set

$$B_\lambda = \{u \in H^1(\mathbb{R}^N) : \|u\| < t_\lambda\}.$$

Then for all $u \in \partial B_\lambda = \{u \in H^1(\mathbb{R}^N) : \|u\| = t_\lambda\}$,

$$I_\lambda(u) \geqslant t_\lambda f(t_\lambda) \geqslant t_\lambda\big[t_\lambda(p-1)/2p - c_2\big] > 0$$

provided that $\lambda < \lambda_1$ which λ_1 is given by (1.2). Fix such a value of λ, say λ_0, and set $t_0 = t(\lambda_0)$. Let $0 \not\equiv \phi \geqslant 0$, $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} h\phi\,dx > 0$. Then

$$I_{\lambda_0}(t\phi) = \frac{t^2}{2}\|\phi\|^2 - \frac{\lambda_0}{p+1}t^{p+1}\int_{\mathbb{R}^N} K\phi^{p+1} - t\int_{\mathbb{R}^N} h\phi < 0$$

for sufficiently small $t > 0$, and it is easy to see that I_{λ_0} is bounded below on B_{t_0}. Set $\alpha = \inf\{I_{\lambda_0}(u) \mid u \in B_{t_0}\}$. Then $\alpha < 0$, and since $I_{\lambda_0}(u) > 0$ on ∂B_{t_0}, the continuity of I_{λ_0} on $H^1(\mathbb{R}^N)$ implies that there exists $0 < t_1 < t_0$ such that $I_{\lambda_0}(u) > \alpha$ for all $u \in H^1(\mathbb{R}^N)$ and $t_1 \leqslant \|u\| \leqslant t_0$. By the Ekeland's variational principle [**9**], there exists a sequence $\{u_k\}_{k=1}^\infty \subset B_{t_1}$ such that $I_{\lambda_0}(u_k) = \alpha + o(1)$ and $I'_{\lambda_0}(u_k) = o(1)$ strongly in $H^{-1}(\mathbb{R}^N)$, as $k \to \infty$. By Proposition 2.1, we have that there exist a subsequence $\{u_k\}$, an integer $l \geqslant 0$, $\omega_i > 0$, $1 \leqslant i \leqslant l$ (if $l \geqslant 1$), $\overline{u} > 0$ in $\mathbb{R}^N$ and $\overline{u}$ in $\overline{B}_{t_1}$ such that

$$\begin{cases} u_k \rightharpoonup \overline{u} \text{ weakly in } H^1(\mathbb{R}^N), \\ -\Delta\overline{u} + \overline{u} = \lambda_0 K(x)\overline{u}^p + h(x) \text{ in } H^{-1}(\mathbb{R}^N), \\ -\Delta\omega_i + \omega_i = \lambda_0 K_\infty \omega_i^p \text{ in } H^{-1}(\mathbb{R}^N),\ 1 \leqslant i \leqslant l. \end{cases}$$

Moreover,

$$I_{\lambda_0}(u_k) = I_{\lambda_0}(\overline{u}) + \sum_{i=1}^{l} I_{\lambda_0}^\infty(\omega_i) + o(1) \text{ as } k \to \infty.$$

Note that $I_{\lambda_0}^\infty(\omega_i) = I_{\lambda_0}^\infty(\omega_{\lambda_0}) > 0$ for $i = 1, 2, \cdots, l$. Since $\overline{u} \in B_{t_0}$, we have $I_{\lambda_0}(\overline{u}) \geqslant \alpha$. We conclude that $l = 0$, $I_{\lambda_0}(\overline{u}) = \alpha$ and $I'_{\lambda_0}(\overline{u}) = 0$, that is, $\overline{u}$ is a weak positive solution of $(1.1)_{\lambda_0}$. ∎

Now, by the standard barrier method, we get the following Lemma.

LEMMA 3.2. *Let $K(x)$ satisfy $(k1)$. Then there exists $\lambda^* > 0$ such that for each $\lambda \in [0, \lambda^*)$, problem $(1.1)_\lambda$ has a minimal positive solution u_λ and u_λ is strictly increasing in λ.*

PROOF: Denoting

$$\lambda^* = \sup\{\lambda \geqslant 0 : (1.1)_\lambda \text{ has a positive solution }\}.$$

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By Lemma 3.1, we have $\lambda^* > 0$. Now, consider $\lambda \in [0, \lambda^*)$. By the definition of λ^*, we know that there exists $\lambda' > \lambda$ such that $\lambda' < \lambda^*$ and $(1.1)_{\lambda'}$ has a positive solution $u_{\lambda'} > 0$, that is,

$$\begin{aligned} -\Delta u_{\lambda'} + u_{\lambda'} &= \lambda' K(x) u_{\lambda'}^p + h(x) \\ &> \lambda K(x) u_{\lambda'}^p + h(x). \end{aligned}$$

Then $u_{\lambda'}$ is a supersolution of $(1.1)_\lambda$. From $h(x) \geqslant 0$ and $h(x) \not\equiv 0$, it is easily proved that 0 is a subsolution of $(1.1)_\lambda$. By the standard barrier method, there exists a solution u_λ of $(1.1)_\lambda$ such that $0 \leqslant u_\lambda \leqslant u_{\lambda'}$. Since 0 is not a solution of $(1.1)_\lambda$ and $\lambda' > \lambda$, the maximum principle implies that $0 < u_\lambda < u_{\lambda'}$. Again using a result of Amann [**1**], we can choose a minimum positive solution u_λ of $(1.1)_\lambda$. This completes the proof of Lemma 3.2. ∎

Now, we consider a solution u of $(1.1)_\lambda$. Let $\sigma_\lambda(u)$ be defined by

$$\sigma_\lambda(u) = \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla w|^2 + |w|^2\right) dx : w \in H^1(\mathbb{R}^N), \int_{\mathbb{R}^N} pKu^{p-1} w^2\, dx = 1 \right\} \tag{3.2}$$

By the standard direct minimisation procedure, we can show that $\sigma_\lambda(u)$ is attained by a function $\varphi_\lambda > 0$, $\varphi_\lambda \in H^1(\mathbb{R}^N)$, satisfying

$$-\Delta \varphi_\lambda + \varphi_\lambda = \sigma_\lambda(u) pKu^{p-1} \varphi_\lambda \text{ in } \mathbb{R}^N. \tag{3.3}$$

Lemma 3.3. *Let $K(x)$ satisfy $(k1)$. For $\lambda \in [0, \lambda^*)$, let u_λ be the minimal solution of $(1.1)_\lambda$ and $\sigma_\lambda(u_\lambda)$ be the corresponding number given by (3.2). Then*

(i) *$\sigma_\lambda(u_\lambda) > \lambda$ and is strictly decreasing in λ, $\lambda \in [0, \lambda^*)$;*

(ii) *$\lambda^* < \infty$, and $(1.1)_{\lambda^*}$ has a minimal solution u_{λ^*}.*

Proof: Consider $u_{\lambda'}$, u_λ, where $\lambda^* > \lambda' > \lambda \geqslant 0$. Let φ_λ be a minimiser of $\sigma_\lambda(u_\lambda)$, then by Lemma 3.2, we obtain that

$$\int_{\mathbb{R}^N} pKu_{\lambda'}^{p-1} \varphi_\lambda^2\, dx > \int_{\mathbb{R}^N} pKu_\lambda^{p-1} \varphi_\lambda^2\, dx = 1,$$

and there is t, $0 < t < 1$ such that

$$\int_{\mathbb{R}^N} pKu_{\lambda'}^{p-1} (t\varphi_\lambda)^2 = 1.$$

Therefore,

$$\sigma_{\lambda'}(u_{\lambda'}) \leqslant t^2 \|\varphi_\lambda\|^2 < \|\varphi_\lambda\|^2 = \sigma_\lambda(u_\lambda), \tag{3.4}$$

showing the monotonicity of $\sigma_\lambda(u_\lambda)$, $\lambda \in [0, \lambda^*)$.

Consider now $\lambda \in (0, \lambda^*)$. Let $\lambda < \lambda' < \lambda^*$. From (3.3) and the monotonicity of u_λ, we get

$$\begin{aligned}
&\sigma_\lambda(u_\lambda) p \int_{\mathbb{R}^N} (u_{\lambda'} - u_\lambda) K u_\lambda^{p-1} \varphi_\lambda \, dx \\
&\qquad = \int_{\mathbb{R}^N} \nabla(u_{\lambda'} - u_\lambda) \cdot \nabla \varphi_\lambda \, dx + \int_{\mathbb{R}^N} (u_{\lambda'} - u_\lambda) \varphi_\lambda \, dx \\
&\qquad = (\lambda' - \lambda) \int_{\mathbb{R}^N} K u_{\lambda'}^p \varphi_\lambda \, dx + \lambda \int_{\mathbb{R}^N} K (u_{\lambda'}^p - u_\lambda^p) \varphi_\lambda \, dx \\
&\qquad > \lambda p \int_{\mathbb{R}^N} K \varphi_\lambda \int_{u_\lambda}^{u_{\lambda'}} t^{p-1} \, dt \, dx \\
&\qquad \geqslant \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1} (u_{\lambda'} - u_\lambda) \varphi_\lambda \, dx,
\end{aligned} \tag{3.5}$$

which implies that $\sigma_\lambda(u_\lambda) > \lambda$, $\lambda \in (0, \lambda^*)$. This completes the proof of (i).

We show next that $\lambda^* < \infty$. Let $\lambda_0 \in (0, \lambda^*)$ be fixed. For any $\lambda \geqslant \lambda_0$, (3.4) and (3.5) imply

$$\sigma_{\lambda_0}(u_{\lambda_0}) \geqslant \sigma_\lambda(u_\lambda) > \lambda$$

for all $\lambda \in [\lambda_0, \lambda^*)$. Thus, $\lambda^* < \infty$.

By (3.2) and $\sigma_\lambda(u_\lambda) > \lambda$, we have

$$\int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx - \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p+1} \, dx > 0.$$

and

$$\int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx - \int_{\mathbb{R}^N} \lambda K u_\lambda^{p+1} \, dx - \int_{\mathbb{R}^N} h u_\lambda = 0.$$

Thus

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx &= \int_{\mathbb{R}^N} \lambda K u_\lambda^{p+1} \, dx + \int_{\mathbb{R}^N} h u_\lambda \, dx \\
&< \frac{1}{p} \int_{\mathbb{R}^N} \left(|\nabla u_\lambda|^2 + |u_\lambda|^2\right) dx + \|h\|_{L^2(\mathbb{R}^N)} \|u_\lambda\| \\
&< \left(\frac{1}{p} + \frac{\delta}{2}\right) \|u_\lambda\|^2 + \frac{1}{2\delta} \|h\|^2_{L^2(\mathbb{R}^N)},
\end{aligned}$$

for any $\delta > 0$. Since $p > 1$, we can obtain that $\|u_\lambda\| \leqslant c < +\infty$ for all $\lambda \in (0, \lambda^*)$ by taking δ small enough. By Lemma 3.2, the solution u_λ is strictly increasing with respect to λ; we may suppose that

$$u_\lambda \rightharpoonup u_{\lambda^*} \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } \lambda \to \lambda^*,$$

and hence u_{λ^*} is a minimal solution of $(1.1)_{\lambda^*}$. This completes the proof of Lemma 3.3. ∎

Lemma 3.4. *If $K(x)$ satisfies $(k1)$, then $\lambda_1 \leqslant \lambda^* \leqslant \lambda_2 \leqslant \lambda_3$, where λ_1, λ_2 and λ_3 are given by* (1.2).

Proof: By Lemma 3.1 and the definition of λ^*, we conclude that $\lambda^* \geqslant \lambda_1$.

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As in Lemma 3.3, we have $\sigma_\lambda(u_\lambda) > \lambda$ for all $\lambda \in (0, \lambda^*)$, so for any $w \in H^1(\mathbb{R}^N)\backslash\{0\}$, we have

$$\int_{\mathbb{R}^N} \left(|\nabla w|^2 + |w|^2\right) dx > \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1} w^2 \, dx. \tag{3.6}$$

Let u_0 be the unique solution of $(1.1)_0$, then by (3.6) and $u_\lambda > u_0$ for all $\lambda \in (0, \lambda^*]$, we obtain that

$$\int_{\mathbb{R}^N} \left(|\nabla w|^2 + |w|^2\right) dx > \lambda p \int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx,$$

that is,

$$\lambda \leqslant \inf_{w \in H^1(\mathbb{R}^N)\backslash\{0\}} \left(\frac{\|w\|^2}{p \int_{\mathbb{R}^N} K u_0^{p-1} w^2 \, dx} \right) = \lambda_2. \tag{3.7}$$

This implies that $\lambda^* \leqslant \lambda_2$.

For all $\lambda \in [0, \lambda^*]$, let u_λ is a minimal solution of $(1.1)_\lambda$ and take $w = u_\lambda$ in (3.6), then we have that

$$\begin{aligned} \|u_\lambda\|^2 = &\ \lambda \int_{\mathbb{R}^N} K u_\lambda^{p+1} \, dx + \int_{\mathbb{R}^N} h u_\lambda \, dx \\ < &\ \frac{1}{p} \|u_\lambda\|^2 + \|h\|_{H^{-1}} \|u_\lambda\|. \end{aligned}$$

This implies that

$$\|u_\lambda\| \leqslant \frac{p}{p-1} \|h\|_{H^{-1}}. \tag{3.8}$$

Take $w = u_\lambda$ in (3.7), and by (3.8) and the monotonicity of u_λ, we get that

$$\begin{aligned} \lambda_2 \leqslant &\ \frac{\|u_\lambda\|^2}{p \int_{\mathbb{R}^N} K u_0^{p-1} u_\lambda^2 \, dx} \\ \leqslant &\ \frac{p \|h\|_{H^{-1}}^2}{(p-1)^2 \displaystyle\int_{\mathbb{R}^N} K u_0^{p+1} \, dx} = \lambda_3. \end{aligned}$$

∎

Finally, we establish the decay estimate for solutions of $(1.1)_\lambda$ and this result will be used in Section 4 and Section 5. Now, we quote two Regularity Lemmas (see Hsu [**11**] for the proof).

Lemma 3.5. *Let $f : \mathbb{X} \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for almost every $x \in \mathbb{X}$, there holds*

$$\left|f(x, u)\right| \leqslant c\left(|u| + |u|^p\right) \text{ uniformly in } x \in \mathbb{X}, \tag{3.9}$$

where $\mathbb{X}$ is a $C^{1,1}$ domain in $\mathbb{R}^N$, $1 < p < (N+2)/(N-2)$ if $N \geqslant 3$, $1 < p < \infty$ if $N = 2$. Also, let $u \in H_0^1(\mathbb{X})$ be a weak solution of equation $-\Delta u = f(x, u) + h(x)$ in $\mathbb{X}$, where $h \in L^{N/2}(\mathbb{X}) \cap L^2(\mathbb{X})$. Then $u \in L^q(\mathbb{X})$ for $q \in [2, \infty)$.

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LEMMA 3.6. *Let $\mathbb{X}$ be a $C^{1,1}$ domain in $\mathbb{R}^N$, $g \in L^2(\mathbb{X}) \cap L^q(\mathbb{X})$ for some $q \in [2, \infty)$ and $u \in H_0^1(\mathbb{X})$ be a weak solution of the equation $-\Delta u + u = g$ in $\mathbb{X}$. Then $u \in W^{2,q}(\mathbb{X})$ satisfies*

$$\|u\|_{W^{2,q}(\mathbb{X})} \leqslant c\big(\|u\|_{L^q(\mathbb{X})} + \|g\|_{L^q(\mathbb{X})}\big),$$

where $c = c(N, q, \partial\mathbb{X})$.

LEMMA 3.7. *Let $h(x)$ satisfy $(h1)$ and u be a weak solution of $(1.1)_\lambda$, then*

(i) *$u(x) \to 0$ as $|x| \to \infty$.*

(ii) *there exists positive constant c_1 such that*

$$u(x) \geqslant c_1 \exp\big(-|x|\big)|x|^{-(N-1)/2} \text{ as } |x| \to \infty. \tag{3.10}$$

PROOF: (i) Let u satisfy

$$-\Delta u + u = \lambda K(x)u^p + h(x) \qquad \text{in } H^{-1}(\mathbb{R}^N).$$

Since K is bounded and $h \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$ for some $q > N/2$. Hence

$$h \in L^2(\mathbb{R}^N) \cap L^{N/2}(\mathbb{R}^N)$$

and by Lemma 3.5, we have $u \in L^q(\mathbb{R}^N)$ for $q \in [2, \infty)$. Hence

$$\lambda K(x)u^p + h(x) \in L^2(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$$

for some $q > N/2$. Then by Lemma 3.6, we have $u \in W^{2,q}(\mathbb{R}^N)$ for some $q > N/2$. By the Sobolev embedding theorem, $u \in C_b(\mathbb{R}^N)$ and there exists $c > 0$, such that for any $r > 1$,

$$\|u\|_{L^\infty(\overline{B}_r^c)} \leqslant c\|u\|_{W^{2,q}(\overline{B}_r^c)}$$

where

$$\overline{B}_r^c = \{x \in \mathbb{R}^N : |x| > r\}.$$

Hence $\lim_{|x|\to\infty} u(x) = 0$.

(ii) It is very easy to show that $\big(1 + 1/\sqrt{|x|}\big)e^{-|x|}/|x|^{(N-1)/2}$ is a subsolution of $(1.1)_\lambda$ for all $|x|$ large. Therefore (3.10) is proved by means of the maximum principle. $\square$

LEMMA 3.8. *Let u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in [0, \lambda^*]$ and $\sigma_\lambda(u_\lambda) > \lambda$. Then for any $g(x) \in H^{-1}(\mathbb{R}^N)$, problem*

$$-\Delta w + w = \lambda p K u_\lambda^{p-1} w + g(x), w \in H^1(\mathbb{R}^N) \tag{3.11$_\lambda$}$$

has a solution.

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PROOF: Consider the functional

$$\Phi(w) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla w|^2 + w^2\right) dx - \frac{1}{2}\lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1} w^2 \, dx - \int_{\mathbb{R}^N} g(x) w \, dx,$$

where $w \in H^1(\mathbb{R}^N)$. From Hölder inequality and Young's inequality, we have, for any $\varepsilon > 0$, that

$$\begin{aligned} \Phi(w) \geqslant\ & \frac{1}{2}(1 - \lambda\sigma_\lambda(u_\lambda)^{-1})\|w\|^2 - \frac{1}{2}\varepsilon\|w\|^2 - \frac{C_\varepsilon}{2}\|g\|^2_{H^{-1}(\mathbb{R}^N)} \\ \geqslant\ & -C\|g\|^2_{H^{-1}(\mathbb{R}^N)} \end{aligned} \tag{3.12}$$

if we choose ε small.

Now, let $\{w_n\} \subset H^1(\mathbb{R}^N)$ be the minimising sequence of variational problem

$$d = \inf\left\{\Phi(w) \mid w \in H^1(\mathbb{R}^N)\right\}.$$

From (3.12) and $\sigma_\lambda(u_\lambda) > \lambda$, we can also deduce that $\{w_n\}$ is bounded in $H^1(\mathbb{R}^N)$, if we choose ε small. So we may suppose that

$$\begin{aligned} &w_n \rightharpoonup w \text{ weakly in } H^1(\mathbb{R}^N) \text{ as } n \to \infty, \\ &w_n \to w \text{ almost everywhere in } \mathbb{R}^N \text{ as } n \to \infty. \end{aligned}$$

By Fatou's Lemma,

$$\|w\|^2 \leqslant \liminf \|w_n\|^2.$$

By Lemma 3.7, we have that $u_\lambda(x) \to 0$ as $|x| \to \infty$ and the weak convergence imply

$$\int_{\mathbb{R}^N} g w_n \, dx \to \int_{\mathbb{R}^N} g w \, dx, \int_{\mathbb{R}^N} K u_\lambda^{p-1} w_n^2 \, dx \to \int_{\mathbb{R}^N} K u_\lambda^{p-1} w^2 \, dx \text{ as } n \to \infty.$$

Therefore

$$\Phi(w) \leqslant \lim_{n\to\infty} \Phi(w_n) = d,$$

and hence $\Phi(w) = d$ which gives that w is a solution of $(3.11)_\lambda$. ∎

REMARK 3.9. From Lemma 3.8, we know that $(3.11)_\lambda$ has a solution $w \in H^1(\mathbb{R}^N)$. Now, we also assume that $K(x), h(x)$, and $g(x)$ are in $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then by the elliptic regular theory ([10]), we can deduce that $w \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

LEMMA 3.10. *Suppose u_{λ^*} is a solution of $(1.1)_{\lambda^*}$, then $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$ and the solution u_{λ^*} is unique.*

PROOF: Define $F : \mathbb{R} \times H^1(\mathbb{R}^N) \longrightarrow H^{-1}(\mathbb{R}^N)$ by

$$F(\lambda, u) = \Delta u - u + \lambda K(u^+)^p + h(x).$$

Since $\sigma_\lambda(u_\lambda) \geqslant \lambda$ for $\lambda \in (0, \lambda^*)$, so $\sigma_{\lambda^*}(u_{\lambda^*}) \geqslant \lambda^*$. If $\sigma_{\lambda^*}(u_{\lambda^*}) > \lambda^*$, the equation $F_u(\lambda^*, u_{\lambda^*})\phi = 0$ has no nontrivial solution. From Lemma 3.8, F_u maps $\mathbb{R} \times H^1(\mathbb{R}^N)$ onto

$H^{-1}(\mathbb{R}^N)$. Applying the implicit function theorem to F, we can find a neighbourhood $(\lambda^*-\delta, \lambda^*+\delta)$ of λ^* such that $(1.1)_\lambda$ possesses a solution u_λ if $\lambda \in (\lambda^*-\delta, \lambda^*+\delta)$. This is contradictory to the definition of λ^*. Hence, we obtain that $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$.

Next, we are going to prove that u_{λ^*} is unique. In fact, suppose $(1.1)_{\lambda^*}$ has another solution $U_{\lambda^*} \geqslant u_{\lambda^*}$. Set $w = U_{\lambda^*} - u_{\lambda^*}$; we have

$$-\Delta w + w = \lambda^* K\big[(w+u_{\lambda^*})^p - u_{\lambda^*}^p\big],\ w > 0 \text{ in } \mathbb{R}^N. \tag{3.13}$$

By $\sigma_{\lambda^*}(u_{\lambda^*}) = \lambda^*$, we have that the problem

$$-\Delta\phi + \phi = \lambda^* p K u_{\lambda^*}^{p-1}\phi, \qquad \phi \in H^1(\mathbb{R}^N) \tag{3.14}$$

possesses a positive solution ϕ_1.

Multiplying (3.13) by ϕ_1 and (3.14) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \lambda^* K\big[(w+u_{\lambda^*})^p - u_{\lambda^*}^p - p u_{\lambda^*}^{p-1} w\big]\phi_1\,dx \\ &= \frac{1}{2}p(p-1)\int_{\mathbb{R}^N} \lambda^* K \xi_{\lambda^*}^{p-2} w^2 \phi_1\,dx, \end{aligned}$$

where $\xi_{\lambda^*} \in (u_{\lambda^*}, u_{\lambda^*}+w)$. Thus $w \equiv 0$. ∎

4. Existence of second solution

The existence of a second solution of $(1.1)_\lambda$, $\lambda \in (0, \lambda^*)$, will be established via the mountain pass theorem. When $0 < \lambda < \lambda^*$, we have known that $(1.1)_\lambda$ has a minimal positive solution u_λ by Lemma 3.2, then we need only to prove that $(1.1)_\lambda$ has another positive solution in the form of $U_\lambda = u_\lambda + v_\lambda$, where v_λ is a solution of the following problem:

$$\begin{cases} -\Delta v + v = \lambda K\big[(v+u_\lambda)^p - u_\lambda^p\big] \text{ in } \mathbb{R}^N, \\ v \in H^1(\mathbb{R}^N), v > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{4.1$_\lambda$}$$

The corresponding variational functional of $(4.1)_\lambda$ is

$$J_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla v|^2 + v^2) - \lambda\int_{\mathbb{R}^N}\int_0^{v^+} K\big[(s+u_\lambda)^p - u_\lambda^p\big]\,ds\,dx,\ v \in H^1(\mathbb{R}^N).$$

The following lemma comes from the fact

$$\lim_{s\to 0}\frac{(u_\lambda+s)^p - u_\lambda^p - p u_\lambda^{p-1}s}{s} = 0$$

and

$$\lim_{s\to\infty}\frac{(u_\lambda+s)^p - u_\lambda^p - p u_\lambda^{p-1}s}{s^p} = 1$$

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Lemma 4.1. *For any $\varepsilon > 0$, there is a positive constant c_ε such that*

$$(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s \leqslant \varepsilon u_\lambda^{p-1}s + c_\varepsilon s^p$$

for all $s \geqslant 0$.

Lemma 4.2. *Under condition $(k1)$, then there exist positive constants ρ and α, such that*

$$J_\lambda(v) \geqslant \alpha > 0, \ v \in H^1(\mathbb{R}^N), \ \|v\| = \rho.$$

Proof: By Lemma 4.1, we have

$$\begin{aligned} J_\lambda(v) &= \frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx - \frac{1}{2}\lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \, dx \\ &\qquad - \lambda \int_{\mathbb{R}^N} \int_0^{v^+} K\left[(u_\lambda + s)^p - u_\lambda^p - pu_\lambda^{p-1}s\right] ds\, dx \\ &\geqslant \frac{1}{2}\left[\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx - \lambda p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \, dx\right] \\ &\qquad - \lambda \int_{\mathbb{R}^N} K\left[\frac{\varepsilon}{2} u_\lambda^{p-1}(v^+)^2 + c_\varepsilon \frac{(v^+)^{p+1}}{p+1}\right] dx. \end{aligned} \tag{4.2}$$

Furthermore, from the definition $\sigma_\lambda(u_\lambda)$ in (3.2), we have

$$\int_{\mathbb{R}^N} \left(|\nabla v|^2 + v^2\right) dx \geqslant \sigma_\lambda(u_\lambda) p \int_{\mathbb{R}^N} K u_\lambda^{p-1}(v^+)^2 \, dx,$$

and, therefore, by (4.2) we obtain

$$J_\lambda(v) \geqslant \frac{1}{2}\sigma_\lambda(u_\lambda)^{-1}\left(\sigma_\lambda(u_\lambda) - \lambda - \frac{\varepsilon}{2}\lambda\right)\|v\|^2 - \lambda c_\varepsilon (p+1)^{-1} \int_{\mathbb{R}^N} K(v^+)^{p+1} \, dx.$$

Since $\sigma_\lambda(u_\lambda) > \lambda$, by property (ii) in Lemma 3.2, the boundedness of K, and the Sobolev inequality imply that for small $\varepsilon > 0$,

$$J_\lambda(v) \geqslant \frac{1}{4}\sigma_\lambda(u_\lambda)^{-1}(\sigma_\lambda(u_\lambda) - \lambda)\|v\|^2 - \lambda c\|v\|^{p+1},$$

and the conclusion in Lemma 4.2 follows. ∎

We need the following concentration compactness principle to prove our result:

Lemma 4.3. *Assume condition $(k1)$ holds. Let $\{v_k\}$ be a $(PS)_c$ sequence of J_λ in $H^1(\mathbb{R}^N)$:*

$$\begin{aligned} J_\lambda(v_k) &= c + o(1) \text{ as } k \to \infty, \\ J'_\lambda(v_k) &= o(1) \text{ strong in } H^{-1}(\mathbb{R}^N). \end{aligned}$$

Then there exists a subsequence (still denoted by) $\{v_k\}$ for which the following holds: there exist an integer $l \geqslant 0$, sequence $\{x_k^i\} \subset \mathbb{R}^N$, a solution v_λ of $(4.1)_\lambda$ and solutions

$\overline{v}_\lambda^i$ of $(2.1)_\lambda$, for $1 \leqslant i \leqslant l$, such that

$$\begin{cases} v_k \rightharpoonup v_\lambda \text{ weakly in } H^1(\mathbb{R}^N); \\ v_k - \left[v_\lambda + \sum_{i=1}^{l} \overline{v}_\lambda^i(\cdot - x_k^i) \right] \to 0 \text{ strongly in } H^1(\mathbb{R}^N); \\ J_\lambda(v_k) = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i) + o(1). \end{cases}$$

where we agree that in the case $l = 0$ the above holds without $\overline{v}_\lambda^i$, x_k^i.

PROOF: Lemma 4.3 can be derived directly from the arguments in Bahri and Lions [**3**] (or [**4, 14, 17**]). We omit it. ∎

LEMMA 4.4. *Assume condition $(k1)$ holds, then*

(i) *there exists $t_0 > 0$, such that*

$$J_\lambda(t\omega_\lambda) < 0 \text{ for all } t \geqslant t_0.$$

(ii) *the following inequality holds*

$$0 < \sup_{t \geqslant 0} J_\lambda(t\omega_\lambda) < I_\lambda^\infty(\omega_\lambda) = M_\lambda^\infty.$$

PROOF: By ω_λ is the ground state solution of $(2.1)_\lambda$ and condition $(k1)$, then we have

$$\begin{aligned} J_\lambda(t\omega_\lambda) = {} & \frac{1}{2}t^2 \int_{\mathbb{R}^N} (|\nabla\omega_\lambda|^2 + |\omega_\lambda|^2)dx - \frac{1}{p+1}t^{p+1} \int_{\mathbb{R}^N} \lambda K(x)\omega_\lambda^{p+1}\,dx \\ & - \int_{\mathbb{R}^N} \int_0^{t\omega_\lambda} \lambda K(x)\left[(s+u_\lambda)^p - u_\lambda^p - s^p\right] ds\,dx \\ \leqslant {} & \frac{1}{2}t^2\|\omega_\lambda\|^2 - \frac{1}{p+1}t^{p+1} \int_{\mathbb{R}^N} \lambda K_\infty \omega_\lambda^{p+1}(x)\,dx \\ \leqslant {} & c_1 t^2 - c_2 t^{p+1} \end{aligned} \tag{4.3}$$

where $c_1 = 1/2\|\omega_\lambda\|^2$, $c_2 = 1/(p+1)\int_{\mathbb{R}^N} \lambda K_\infty \omega_\lambda^{p+1}(x)\,dx$ are independent of t. From (4.3), we conclude the result (i).

From (i), we easily see that the left hand of (ii) holds and we need only to show that the right hand of (ii) holds. By (i), we have that there exists $t_2 > 0$ such that

$$\sup_{t \geqslant 0} J_\lambda(t\omega_\lambda) = \sup_{0 \leqslant t \leqslant t_2} J_\lambda(t\omega_\lambda).$$

Since J is continuous in $H^1(\mathbb{R}^N)$, there exists $t_1 > 0$ such that

$$J_\lambda(t\omega_\lambda) < M_\lambda^\infty, \text{ for } 0 \leqslant t < t_1.$$

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Then, to prove (ii) we now only to prove the following inequality:

$$\sup_{t_1 \leqslant t \leqslant t_2} J_\lambda(t\omega_\lambda) < I_\lambda^\infty(\omega_\lambda) = M_\lambda^\infty.$$

By the definition of J_λ, we get

$$\begin{aligned} J_\lambda(t\omega_\lambda) = \frac{t^2}{2}\int_{\mathbb{R}^N}\left(|\nabla\omega_\lambda|^2 + \omega_\lambda^2\right)dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\lambda K_\infty \omega_\lambda^{p+1}\,dx \\ + \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^N}\lambda\big(K_\infty - K(x)\big)\omega_\lambda^{p+1}\,dx \\ - \int_{\mathbb{R}^N}\int_0^{t\omega_\lambda}\lambda K(x)\big[(s+u_\lambda)^p - u_\lambda^p - s^p\big]\,ds\,dx. \end{aligned}$$

Since ω_λ is the ground state solution of $(2.1)_\lambda$ and $\sup\limits_{t \geqslant 0} I_\lambda^\infty(t\omega_\lambda) = I_\lambda^\infty(\omega_\lambda)$, then we have

$$\begin{aligned} J_\lambda(t\omega_\lambda) \leqslant I_\lambda^\infty(\omega_\lambda) - \frac{t_2^{p+1}}{p+1}\int_{\mathbb{R}^N}\lambda(K(x) - K_\infty)\omega_\lambda^{p+1}\,dx \\ - \int_{\mathbb{R}^N}\int_0^{t\omega_\lambda}\lambda K(x)\big[(s+u_\lambda)^p - u_\lambda^p - s^p\big]\,ds\,dx. \end{aligned}$$

By condition $(k1)$ and $(t_1+t_2)^p \geqslant (\not\equiv) t_1^p + t_2^p$ for all $t_1 \geqslant 0, t_2 \geqslant 0, p > 1$. Therefore, we obtain that

$$\begin{aligned} J_\lambda(t\omega_\lambda) &\leqslant I_\lambda^\infty(\omega_\lambda) - \inf_{t_1 \leqslant t \leqslant t_2}\int_{\mathbb{R}^N}\int_0^{t\omega_\lambda}\lambda K(x)\big[(s+u_\lambda)^p - u_\lambda^p - s^p\big]\,ds\,dx \\ &< M_\lambda^\infty. \end{aligned}$$

Therefore (ii) holds. ∎

PROPOSITION 4.5. *Suppose condition $(k1)$ holds. Then problem $(4.1)_\lambda$ has at least one solution for $\lambda \in (0, \lambda^*)$.*

PROOF: By Lemma 4.4 (i), we know that there is $t_0 > 0$ such that $J_\lambda(t_0\omega_\lambda) < 0$. We set

$$\Gamma = \left\{\gamma \in C\big([0,1], H^1(\mathbb{R}^N)\big) : \gamma(0) = 0,\ \gamma(1) = t_0\omega_\lambda\right\},$$

then, by Lemma 4.2 and Lemma 4.4 (ii) we get

$$0 < c = \inf_{\gamma\in\Gamma}\max_{0 \leqslant s \leqslant 1} J_\lambda\big(\gamma(s)\big) < M_\lambda^\infty. \tag{4.4}$$

Applying the mountain pass lemma of Ambrosetti and Rabinowitz [2], there exists a $(PS)_c$-sequence $\{v_k\}$ such that

$$J_\lambda(v_k) \to c \text{ and } J'_\lambda(v_k) \to 0 \text{ in } H^{-1}(\mathbb{R}^N).$$

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By Lemma 4.3, there exist a subsequence, still denoted by $\{v_k\}$, an integer $l \geqslant 0$, a solution v_λ of $(4.1)_\lambda$ and solutions $\overline{v}_\lambda^i$ of $(2.1)_\lambda$, for $1 \leqslant i \leqslant l$, such that

$$c = J_\lambda(v_\lambda) + \sum_{i=1}^{l} I_\lambda^\infty(\overline{v}_\lambda^i). \tag{4.5}$$

By the strong maximum principle, to complete the proof, we only need to prove $v_\lambda \not\equiv 0$ in $\mathbb{R}^N$. In fact, by (4.4) and (4.5), we have

$$c = J_\lambda(v_\lambda) \geqslant \alpha > 0 \text{ if } l = 0, M_\lambda^\infty > c \geqslant J_\lambda(v_\lambda) + M_\lambda^\infty \text{ if } l \geqslant 1.$$

This implies $v_\lambda \not\equiv 0$ in $\mathbb{R}^N$. ∎

5. Properties and bifurcation of solutions

In this section, we shall give some further properties and bifurcation of solutions for problem $(1.1)_\lambda$. Now, we set

$$A = \big\{(\lambda, u) : u \text{ satisfies } (1.1)_\lambda, \lambda \in [0, \lambda^*]\big\}.$$

For each $(\lambda, u) \in A$, let $\sigma_\lambda(u)$ denote the number defined by (3.2), which is the smallest eigenvalue of the problem (3.3).

We always assume that condition $(h1)$ and $(k1)$ hold. By Lemma 3.6, we have $A \subset L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$. Moreover, if we assume that,

$$h(x),\ K(x) \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N),$$

then by elliptic regular theory ([**10**]), we can deduce that $A \subset C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$.

Lemma 5.1. *Let u be a solution and u_λ be the minimal solution of $(1.1)_\lambda$ for $\lambda \in (0, \lambda^*)$. Then*

(i) *$\sigma_\lambda(u) > \lambda$ if and only if $u = u_\lambda$;*

(ii) *$\sigma_\lambda(U_\lambda) < \lambda$, where U_λ is the second solution of $(1.1)_\lambda$ constructed in Section 4.*

Proof: Now, let $\psi \geqslant 0$ and $\psi \in H^1(\mathbb{R}^N)$. Since u and u_λ are the solution of $(1.1)_\lambda$, then

$$\begin{aligned}\int_{\mathbb{R}^N} \nabla\psi \cdot \nabla(u_\lambda - u)\,dx + \int_{\mathbb{R}^N} \psi(u_\lambda - u)\,dx &= \lambda \int_{\mathbb{R}^N} K(u_\lambda^p - u^p)\psi\,dx \\ &= \lambda \int_{\mathbb{R}^N} \Big(\int_u^{u_\lambda} t^{p-1}\,dt\Big) pK\psi\,dx \\ &\geqslant \lambda \int_{\mathbb{R}^N} pKu^{p-1}(u_\lambda - u)\psi\,dx.\end{aligned} \tag{5.1}$$

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Let $\psi = (u - u_\lambda)^+ \geqslant 0$ and $\psi \in H^1(\mathbb{R}^N)$. If $\psi \not\equiv 0$, then (5.1) implies

$$-\int_{\mathbb{R}^N} \left(|\nabla \psi|^2 + \psi^2\right) dx \geqslant -\lambda \int_{\mathbb{R}^N} pKu^{p-1}\psi^2\, dx$$

and, therefore, the definition of $\sigma_\lambda(u)$ implies

$$\begin{aligned}\int_{\mathbb{R}^N} \left(|\nabla \psi|^2 + \psi^2\right) dx &\leqslant \lambda \int_{\mathbb{R}^N} pKu^{p-1}\psi^2\, dx \\ &< \sigma_\lambda(u) \int_{\mathbb{R}^N} pKu^{p-1}\psi^2\, dx \\ &\leqslant \int_{\mathbb{R}^N} \left(|\nabla \psi|^2 + \psi^2\right) dx,\end{aligned}$$

which is impossible. Hence $\psi \equiv 0$, and $u = u_\lambda$ in $\mathbb{R}^N$. On the other hand, by Lemma 3.3, we also have that $\sigma_\lambda(u_\lambda) > \lambda$. This completes the proof of (i).

By (i), we get that $\sigma_\lambda(U_\lambda) \leqslant \lambda$ for $\lambda \in (0, \lambda^*)$. We claim that $\sigma_\lambda(U_\lambda) = \lambda$ can not occur. We proceed by contradiction. Set $w = U_\lambda - u_\lambda$; we have

$$-\Delta w + w = \lambda K\left[U_\lambda^p - (U_\lambda - w)^p\right],\ w > 0 \text{ in } \mathbb{R}^N. \tag{5.2}$$

By $\sigma_\lambda(U_\lambda) = \lambda$, we have that the problem

$$-\Delta \phi + \phi = \lambda pKU_\lambda^{p-1}\phi, \qquad \phi \in H^1(\mathbb{R}^N) \tag{5.3}$$

possesses a positive solution ϕ_1.

Multiplying (5.2) by ϕ_1 and (5.3) by w, integrating and subtracting we deduce that

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} \lambda K\left[U_\lambda^p - (U_\lambda - w)^p - pU_\lambda^{p-1}w\right]\phi_1\, dx \\ &= -\frac{1}{2}p(p-1)\int_{\mathbb{R}^N} \lambda K \xi_\lambda^{p-2} w^2 \phi_1\, dx,\end{aligned}$$

where $\xi_\lambda \in (u_\lambda, U_\lambda)$. Thus $w \equiv 0$, that is $U_\lambda = u_\lambda$ for $\lambda \in (0, \lambda^*)$. This is a contradiction. Hence, we have that $\sigma_\lambda(U_\lambda) < \lambda$ for $\lambda \in (0, \lambda^*)$. ∎

REMARK 5.2. Since $\sigma_\lambda(U_\lambda) < \lambda$, one may employ a similar argument to the used for u_λ to show that U_λ is strictly decreasing in λ, $\lambda \in (0, \lambda^*)$.

PROPOSITION 5.3. *Let u_λ be the minimal solution of* $(1.1)_\lambda$. *Then u_λ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for all $\lambda \in [0, \lambda^*]$ and*

$$u_\lambda \to u_0 \text{ in } L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N) \text{ as } \lambda \to 0.$$

where u_0 is the unique positive solution of $(1.1)_0$.

PROOF: By Lemma 3.2, 3.3, and 3.7, we can deduce $\|u_\lambda\|_\infty \leqslant \|u_{\lambda^*}\|_\infty \leqslant c$, for $\lambda \in [0, \lambda^*]$. By (3.8), we have that

$$\|u_\lambda\| \leqslant \frac{p}{p-1}\|h\|_{H^{-1}}.$$

Hence, u_λ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ for $\lambda \in [0, \lambda^*]$.

Now, let $w_\lambda = u_\lambda - u_0$, then w_λ satisfies the following equation

$$(5.4)_\lambda \qquad -\Delta w_\lambda + w_\lambda = \lambda K u_\lambda^p \text{ in } \mathbb{R}^N,$$

and by u_λ is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, we have that

$$\begin{aligned} \|w_\lambda\|^2 &= \int_{\mathbb{R}^N} \lambda K u_\lambda^p w_\lambda \, dx \\ &\leqslant \lambda \|K\|_\infty \|u_\lambda\|_\infty^{p-1} \|u_\lambda\|_2 \|w_\lambda\|_2 \\ &\leqslant c\lambda, \end{aligned}$$

where c is independent of λ. Hence, we obtain that $u_\lambda \to u_0$ in $H^1(\mathbb{R}^N)$ as $\lambda \to 0$.

Now, let $q_0 = N/2 + 2 > \max\{N/2, 2\}$ and by u_λ is uniformly bounded in

$$L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N),$$

then we have that $\lambda K u_\lambda^p \in L^{q_0}(\mathbb{R}^N)$. By Lemma 3.6 and using $(5.4)_\lambda$, we have

$$w_\lambda \in W^{2,2}(\mathbb{R}^N) \cap W^{2,q_0}(\mathbb{R}^N).$$

By the Sobolev embedding theorem, Lemma 3.6 and $u_{\lambda^*} \geqslant u_\lambda > 0$ for $\lambda \in [0, \lambda^*]$, we have that

$$\begin{aligned} \|w_\lambda\|_\infty &\leqslant c_1 \|w_\lambda\|_{W^{2,q_0}(\mathbb{R}^N)} \\ &\leqslant c_2 \left(\|\lambda K u_\lambda^p\|_{q_0} + \|w_\lambda\|_{q_0} \right) \\ &\leqslant c_3 \left(\lambda \|u_{\lambda^*}^p\|_{q_0} + \|w_\lambda\|_\infty^{(q_0-2)/(q_0)} \|w_\lambda\|_2^{2/(q_0)} \right) \\ &\leqslant c(\lambda + \lambda^{1/(q_0)}) \end{aligned}$$

where c is independent of λ. Hence, we obtain that $u_\lambda \to u_0$ in $L^\infty(\mathbb{R}^N)$ as $\lambda \to 0$. ∎

Proposition 5.4. *For $\lambda \in (0, \lambda^*)$, let U_λ be the positive solution of $(1.1)_\lambda$ with $U_\lambda > u_\lambda$, then U_λ is unbounded in $L^\infty(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$, that is*

$$\lim_{\lambda \to 0} \|U_\lambda\| = \lim_{\lambda \to 0} \|U_\lambda\|_\infty = \infty.$$

Proof: Firstly, we show that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H^1(\mathbb{R}^N)$. Since $U_\lambda = u_\lambda + v_\lambda$, we only need to show that $\{v_\lambda : \lambda > 0\}$ is unbounded in $H^1(\mathbb{R}^N)$. If not, then

$$(5.5) \qquad \|v_\lambda\| \leqslant M$$

for all $\lambda \in (0, \lambda^*)$. It is easily to see that for any $\delta > 0$, $\{U_\lambda\}_{\lambda \geqslant \delta}$ is bounded in $H^1(\mathbb{R}^N)$, we may assume $\lambda \in (0, \delta]$.

Choose $\lambda_n \downarrow 0$ and let v_{λ_n} be the corresponding solutions constructed by Proposition 4.5. By the Hölder inequality and the Sobolev embedding theorem, we obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla v_{\lambda_n}|^2 + |v_{\lambda_n}|^2)\, dx &= \int_{\mathbb{R}^N} \lambda_n K[U_{\lambda_n}^p - u_{\lambda_n}^p] v_{\lambda_n}\, dx \\ &\leqslant c\lambda_n \|U_{\lambda_n}\|_{p+1}^p \|v_{\lambda_n}\|_{p+1} \\ &\leqslant c\lambda_n \|U_{\lambda_n}\|^p \|v_{\lambda_n}\| \\ &\leqslant c_1\lambda_n\end{aligned}$$

for some constant c_1, independent of v_{λ_n}, where we have used (5.5) and the boundedness of $\{u_{\lambda_n}\}$ in $H^1(\mathbb{R}^N)$. Hence, we have $\lim\limits_{n\to\infty} \|v_{\lambda_n}\|^2 = 0$. It implies that

$$\lim_{n\to\infty} \|v_{\lambda_n}\|_2 = 0. \tag{5.6}$$

On the other hand, we notice that $U_\lambda = u_\lambda + v_\lambda$ is decreasing and u_λ is increasing in λ. Therefore, v_λ is decreasing in λ, which implies

$$v_{\lambda_n} \geqslant v_\delta \text{ for all } n,$$

then we obtain that

$$\|v_{\lambda_n}\|_2 \geqslant \|v_\delta\|_2 > 0 \text{ for all } n.$$

which contradicts (5.6). This implies that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H^1(\mathbb{R}^N)$.

Now, we show that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $L^\infty(\mathbb{R}^N)$. We proceed by contradiction. Assume to the contrary that there exists $c_0 > 0$ such that

$$\|U_\lambda\|_\infty \leqslant c_0 < \infty \text{ for all } \lambda \in (0, \lambda^*).$$

Since U_λ is a solution of $(1.1)_\lambda$, we have that

$$\begin{aligned}\|U_\lambda\|^2 &= \int_{\mathbb{R}^N} \lambda K U_\lambda^{p+1}\, dx + \int_{\mathbb{R}^N} hU_\lambda\, dx \\ &\leqslant \lambda c_0^{p-1} \|K\|_\infty \|U_\lambda\|_2^2 + \|h\|_2 \|U_\lambda\|_2 \\ &\leqslant c_1\lambda \|U_\lambda\|^2 + c_2 \|U_\lambda\|,\end{aligned}$$

where c_1 and c_2 are independent of λ. If we choose

$$\lambda_0 = \min\left\{\lambda^*, \frac{1}{2c_1}\right\},$$

then there exists $c > 0$, independent of λ, such that $\|U_\lambda\| \leqslant c$ for all $\lambda \leqslant \lambda_0$. This is a contradiction to that $\{U_\lambda : \lambda \in (0, \lambda^*)\}$ is unbounded in $H^1(\mathbb{R}^N)$. This completes the proof of Proposition 5.4. ∎

In order to get bifurcation results we need the following Bifurcation Theorem which can be found in Crandall and Rabinowitz [7].

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Theorem A. *Let X, Y be Banach space. Let $(\overline{\lambda}, \overline{x}) \in \mathbb{R} \times X$ and let F be a continuously differentiable mapping of an open neighbourhood of $(\overline{\lambda}, \overline{x})$ into Y. Let the null-space*

$$N\big(F_x(\overline{\lambda}, \overline{x})\big) = span\{x_0\}$$

be one-dimensional and codim $R\big(F_x(\overline{\lambda}, \overline{x})\big) = 1$. Let $F_\lambda(\overline{\lambda}, \overline{x}) \notin R\big(F_x(\overline{\lambda}, \overline{x})\big)$. If Z is the complement of span$\{x_0\}$ in X, then the solutions of $F(\lambda, x) = F(\overline{\lambda}, \overline{x})$ near $(\overline{\lambda}, \overline{x})$ form a curve

$$\big(\lambda(s), x(s)\big) = \big(\overline{\lambda} + \tau(s), \overline{x} + sx_0 + z(s)\big),$$

where

$$s \to \big(\tau(s), z(s)\big) \in \mathbb{R} \times Z$$

is continuously differentiable function near $s = 0$ and $\tau(0) = \tau'(0) = 0$, $z(0) = z'(0) = 0$.

Proof of Theorem 1.1 and Theorem 1.2: Theorem 1.1 now follows from Lemma 3.2, 3.3, 3.4, 3.10, 5.1 and Proposition 4.5. The conclusion (i) and (ii) of Theorem 1.2 follow immediately from Lemma 3.2, Remark 5.2 and Proposition 5.3, 5.4. Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point in $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ by using an idea in [**12**]. We also assume that $K(x)$ and $h(x)$ are in $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ and define

$$F : \mathbb{R}^1 \times C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \to C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$$

by

$$F(\lambda, u) = \Delta u - u + \lambda K (u^+)^p + h(x).$$

where $C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ and $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$ are endowed with the natural norm; then they become Banach spaces. It can be proved easily that $F(\lambda, u)$ is differentiable. From Lemma 3.8 and Remark 3.9, we know that

$$F_u(\lambda, u)w = \Delta w - w + \lambda p K u_\lambda^{p-1} w$$

is an isomorphism of $\mathbb{R}^1 \times C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ onto $C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. It follows from Implicit Function Theorem that the solutions of $F(\lambda, u) = 0$ near (λ, u_λ) are given by a continuous curve.

Now we are going to prove that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point of F. We show first that at the critical point $(\lambda^*, u_{\lambda^*})$, Theorem A applies. Indeed, from Lemma 3.10, problem (3.14) has a solution $\phi_1 > 0$ in $\mathbb{R}^N$. By the standard elliptic regular theory, we have that $\phi_1 \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$ if $h \in C^\alpha(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Thus

$$F_u(\lambda^*, u_{\lambda^*})\phi = 0, \ \phi \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N)$$

has a solution $\phi_1 > 0$. This implies that $N\big(F_u(\lambda^*, u_{\lambda^*})\big) = span\{\phi_1\} = 1$ is one dimensional and codim $R\big(F_u(\lambda^*, u_{\lambda^*})\big) = 1$ by the Fredholm alternative. It remains to check that $F_\lambda(\lambda^*, u_{\lambda^*}) \notin R\big(F_u(\lambda^*, u_{\lambda^*})\big)$.

Assuming the contrary would imply existence of $v \not\equiv 0$ such that

$$\Delta v - v + \lambda^* pKu_{\lambda^*}^{p-1} v = Ku_{\lambda^*}^p, \qquad v \in H^1(\mathbb{R}^N).$$

From $F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0$, we conclude that $\int_{\mathbb{R}^N} Ku_{\lambda^*}^p \phi_1 \, dx = 0$. This is impossible because $K(x) \geqslant 0, K(x) \not\equiv 0, u_{\lambda^*}(x) > 0$ and $\phi_1(x) > 0$ in $\mathbb{R}^N$.

Applying Theorem A, we conclude that $(\lambda^*, u_{\lambda^*})$ is a bifurcation point near which, the solution of $(1.1)_\lambda$ form a curve $\big(\lambda^* + \tau(s), u_{\lambda^*} + s\phi_1 + z(s)\big)$ with s near $s = 0$ and $\tau(0) = \tau'(0) = 0, z(0) = z'(0) = 0$. We claim that $\tau''(0) < 0$ which implies that the bifurcation curve turns strictly to the left in (λ, u) plane.

Since $u_{\lambda^*}(x) \to 0$ as $|x| \to \infty$, we have, for $|x|$ large,

$$0 = \Delta\phi_1 - \phi_1 + \lambda^* pKu_{\lambda^*}^{p-1}\phi_1 \leqslant \Delta\phi_1 - \frac{1}{4}\phi_1.$$

It is well-known that the equation $\Delta w - w/4 = -w^p$ in $\mathbb{R}^N$ has a unique positive radial symmetric solution, denoted by $\overline{w}$ (see Bahri and Lions [**3**] and the references there), and there exists $c_1 > 0$ such that

$$\overline{w}\big(|x|\big) e^{|x|/2} |x|^{(N-1)/2} \to c_1.$$

Since $\Delta\overline{w} - \overline{w}/4 = -\overline{w}^p \leqslant 0$ in $\mathbb{R}^N$, hence we obtain by the maximum principle that

$$\phi_1(x) \leqslant c_2 e^{-|x|/2} |x|^{-(N-1)/2} \qquad \text{for } |x| \text{ large}, \tag{5.7}$$

for some $c_2 > 0$.

From (3.10) and (5.7) and the Holder's inequality, we derive that

$$\begin{aligned} \int_{\mathbb{R}^N} Ku_{\lambda^*}^{p-2}\phi_1^3 \, dx \leqslant\ & c\int_{\mathbb{R}^N} Ku_{\lambda^*}^{p-1}\phi_1 \, dx \\ \leqslant\ & c\Big(\int_{\mathbb{R}^N} u_{\lambda^*}^{p+1} \, dx\Big)^{(p-1)/(p+1)} \Big(\int_{\mathbb{R}^N} e^{-(p+1)/4|x|} \, dx\Big)^{2/(p+1)} < \infty. \end{aligned} \tag{5.8}$$

Since $\lambda = \lambda^* + \tau(s)$, $u = u_{\lambda^*} + s\phi_1 + z(s)$ in

$$-\Delta u + u - \lambda K u^p - h = 0,\ u > 0,\ u \in C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N). \tag{5.9}$$

Differentiating (5.9) in s twice, we have

$$-\Delta u_{ss} + u_{ss} - \lambda pKu^{p-1}u_{ss} - 2\lambda_s pKu^{p-1}u_s - \lambda p(p-1)Ku^{p-2}u_s^2 - \lambda_{ss}Ku^p = 0.$$

Setting here $s = 0$ and using the facts that $\tau'(0) = 0$, $u_s = \phi_1(x)$ and $u = u_{\lambda^*}$ as $s = 0$, we obtain

$$-\Delta u_{ss} + u_{ss} - \lambda^* pKu_{\lambda^*}^{p-1}u_{ss} - \lambda^* p(p-1)Ku_{\lambda^*}^{p-2}\phi_1^2 - \tau''(0)Ku_{\lambda^*}^p = 0 \tag{5.10}$$

Multiplying $F_u(\lambda^*, u_{\lambda^*})\phi_1 = 0$ by u_{ss}, and (5.10) by ϕ_1, integrating and subtracting the result, and by (5.8) we obtain

$$\int_{\mathbb{R}^N} \lambda^* p(p-1) K u_{\lambda^*}^{p-2} \phi_1^3 \, dx + \tau''(0) \int_{\mathbb{R}^N} K u_{\lambda^*}^p \phi_1 \, dx = 0,$$

which immediately gives $\tau''(0) < 0$. Thus

$$\begin{array}{llll} u_\lambda \to u_{\lambda^*} & \text{in} \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) & \text{as} & \lambda \to \lambda^*, \\ U_\lambda \to u_{\lambda^*} & \text{in} \quad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) & \text{as} & \lambda \to \lambda^*. \end{array}$$

Using Lemma 3.8, Remark 3.9, the Implicit Function Theorem and the uniqueness of the positive ground-state solution of $(1.1)_0$, we can easily prove that

$$u_\lambda \to u_0 \qquad in \qquad C^{2,\alpha}(\mathbb{R}^N) \cap H^2(\mathbb{R}^N) \qquad as \qquad \lambda \to 0,$$

which proves Theorem 1.2. □

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