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Analytic continuation of holonomy germs of Riccati foliations along Brownian paths

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Abstract. Consider a Riccati foliation whose monodromy representation is non-elementary and parabolic and consider a non-invariant section of the fibration whose associated developing map is onto. We prove that any holonomy germ from any non-invariant fibre to the section can be analytically continued along a generic Brownian path. To prove this theorem, we prove a dual result about complex projective structures. Let Σ be a hyperbolic Riemann surface of finite type endowed with a branched complex projective structure: such a structure gives rise to a non-constant holomorphic map $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$, from the universal cover of Σ to the Riemann sphere $\mathbb{CP}^1$, which is ρ-equivariant for a morphism $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$. The dual result is the following. If the monodromy representation ρ is parabolic and non-elementary and if $\mathcal{D}$ is onto, then, for almost every Brownian path ω in $\tilde{\Sigma}$, $\mathcal{D}(\omega(t))$ does not have limit when t goes to ∞. If, moreover, the projective structure is of parabolic type, we also prove that, although $\mathcal{D}(\omega(t))$ does not converge, it converges in the Cesàro sense.

1. *Introduction*

Given a complex algebraic foliation, the study of the holonomy maps is crucial since they encode the dynamics of the leaves. This paper is devoted to the problem of analytic continuation of these holonomy maps. This problem, which goes back to the times of Painlevé, regained interest recently with the works of Loray [**L**], Il'yashenko [**Il**] and Calsamiglia *et al* [**CDFG**].

Let us explain the context. Consider the following differential equation in $\mathbb{C}^2$:

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)} \tag{1}$$

where P and Q are polynomials in $\mathbb{C}[X, Y]$ without common factors.

The solutions of (1) define a singular holomorphic foliation of complex dimension one in $\mathbb{C}^2$ which can be extended to a singular holomorphic foliation $\mathcal{F}$ of $\mathbb{CP}^2$.

Let C_0, C_1 be two complex curves in $\mathbb{CP}^2$ and L be a leaf of $\mathcal{F}$ which intersects C_0 in p_0 and C_1 in p_1. Assume that p_0 and p_1 are not singularities of the foliation and that the curve C_0 (respectively, C_1) is transverse to $\mathcal{F}$ in p_0 (respectively, p_1). Let $\gamma : [0, 1] \to L$ be a continuous path such that $\gamma(0) = p_0$ and $\gamma(1) = p_1$. Then one can find a continuous family $\gamma_p : [0, 1] \to \mathbb{CP}^2$ of paths parametrized by $p \in C_0$ close enough to p_0 such that:

(1) $\gamma_p(0) = p$;

(2) $\gamma_p(1) \in C_1$;

(3) $\gamma_{p_0} = \gamma$; and

(4) for all p, γ_p belongs to the leaf through p.

The germ of the holomorphic map $p \mapsto \gamma_p(1)$ in p_0 is uniquely determined by the relative (i.e. with fixed endpoints) homotopy class of γ under the above conditions and is called the *holonomy germ* associated to γ.

A rather general question is to define the domain of definition of such a germ.

In [**L**], Loray makes the following conjecture.

CONJECTURE 1.1. (Loray) *Let $\mathcal{F}$ be a singular holomorphic foliation in $\mathbb{CP}^2$. Let L_1 and L_2 be two non-invariant projective lines and $h : (L_1, p_1) \to (L_2, p_2)$ be a holonomy germ. Then h can be analytically continued along any continuous path which avoids a countable set of points called singularities of h.*

This was motivated by the following result which can be found in [**CDFG**, Theorem 1.1] and which is a consequence of Theorem 1 of Painlevé (see [**L**]). If the polynomials P and Q of equation (1) are such that $w = P\,dx - Q\,dy$ is a closed one-form, then Loray's conjecture is true.

In the same vein, Il'yashenko asks the following [**Il**] questions.

Question 1.2. (Ilyashenko) Consider the foliation in $\mathbb{C}^2$ associated to equation (1) and let $h : (L_1, p_1) \to (L_2, p_2)$ be a holonomy germ between two lines. Can h be analytically continued along a generic ray emerging from p_1?

In [**CDFG**], the authors prove that Loray's conjecture fails to be true. More precisely, they prove the following results.

- For an algebraic foliation of $\mathbb{CP}^2$ with hyperbolic singularities and without invariant curves (these are generic properties), there is a holonomy germ between a projective line and a curve whose set of singularities contains a Cantor set.
- There exist algebraic foliations of $\mathbb{CP}^2$ admitting a holonomy germ $h : (L_1, p_1) \to (L_2, p_2)$ between complex lines whose set of singularities is the whole L_1.

Our main result is particularly linked to the second assertion. To see this, let us explain briefly how they built such a foliation. They consider a parabolic projective structure on the punctured Riemann sphere whose monodromy group is dense in $PSL(2, \mathbb{C})$. Suspending the monodromy representation, one obtains a $\mathbb{CP}^1$-fibre bundle over the punctured Riemann sphere endowed with a non-singular foliation transverse to every fibre and a section Δ (given by the developing map). There exist local models (introduced by Brunella in [**B**]) over the cusps which allow to one compactify the $\mathbb{CP}^1$-fibre bundle, the foliation and the section. After the compactification, one gets a singular holomophic foliation on a $\mathbb{CP}^1$-fibre bundle over $\mathbb{CP}^1$ whose fibres are transverse to the foliation,

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with the exception the ones over the punctures which are invariant lines containing the singularities of the foliation. Now consider a holonomy germ h between a transverse fibre and the section given by a developing map of the projective structure. Then h is a local inverse of the developing map. If the monodromy group of the projective structure is dense in $PSL(2, \mathbb{C})$, the authors prove that h has full singular set. The $\mathbb{CP}^1$-bundles over $\mathbb{CP}^1$ are parametrized by an integer $n \geq 0$. Choosing, conveniently, the local models around the cusps, this number is $n = 1$, so that the ambient space is the first Hirzebruch surface $\mathbb{F}_1$ which has a unique exceptional curve disjoint from $\overline{\Delta}$. Blow-down gives $\mathbb{CP}^2$ with the desired property.

This paper is based on the following observation. With the same hypothesis, even if the germ h has full singular set, h can be analytically continued along a generic Brownian path, that is, the Brownian motion does not see this full set of singularities.

The foliations previously defined on Hirzebruch surfaces are examples of Riccati foliations. More generally, a *Riccati foliation* is the data of $(\Pi, M, X, \mathcal{F})$, where M is a compact complex surface, X is a compact Riemann surface, $\Pi : M \to X$ is a $\mathbb{CP}^1$-fibre bundle and $\mathcal{F}$ is a singular holomorphic foliation transverse to all the fibres except a finite number of them which are invariant lines for the foliation and contain the singularities. The main theorem of this paper now follows.

THEOREM A. *Let $\mathcal{F}$ be a Riccati foliation with a parabolic and non-elementary monodromy representation. Let F be a non-invariant fibre and s_0, s_1 be two sections of the bundle. Denote by $\overline{S_0}$ and $\overline{S_1}$ the images of X by s_0 and s_1. Endow F, $\overline{S_0}$ and $\overline{S_1}$ with complete metrics in their conformal class. Assume, moreover, that the developing map associated to $\overline{S_0}$ is onto.*

(1) *If $h : (F, p) \to (\overline{S_0}, p_0)$ is a holonomy germ, then h can be analytically continued along almost every Brownian path in F starting at p.*

(2) *If $h : (\overline{S_1}, p_1) \to (\overline{S_0}, p_0)$ is a holonomy germ, then h can be analytically continued along almost every Brownian path in $\overline{S_1}$ starting at p_1.*

Remark 1.3. A holomorphic $\mathbb{CP}^1$-fibre bundle always admits a holomorphic section (see [**BPV**, p. 139]).

Theorem A is a consequence of a theorem concerning complex projective structures, which we now explain.

1.1. *Complex projective structures.* Let Σ be a Riemann surface. A *branched complex projective structure* in Σ is a $(PSL(2, \mathbb{C}), \mathbb{CP}^1)$-structure where $\mathbb{CP}^1$ is the Riemann sphere and $PSL(2, \mathbb{C})$ is the group of Möbius transformations acting on $\mathbb{CP}^1$. Such a structure gives rise to a non-constant holomorphic map $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$ from the universal cover of Σ to the Riemann sphere $\mathbb{CP}^1$ and to a morphism $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ satisfying the equivariance relation

$$\text{for all } x \in \tilde{\Sigma} \text{ and for all } \alpha \in \pi_1(\Sigma), \quad \mathcal{D}(\alpha \cdot x) = \rho(\alpha) \cdot \mathcal{D}(x).$$

The map $\mathcal{D}$ (well defined up to a post-composition by a Möbius transformation) is called a developing map, the morphism ρ is called a monodromy representation and the group $\rho(\pi_1(\Sigma))$ is called a monodromy group (see §2 for more details on projective structures).

If Σ is not compact, we will need a parabolicity hypothesis around the cusps: a representation is said to be *parabolic* if the holonomy around each cusp is parabolic (i.e. it is conjugated to the group generated by the transformation $z \mapsto z + 1$). A complex projective structure is said to be *parabolic* if, in some coordinate z around each puncture, some developing map writes $\mathcal{D}(z) = (1/2i\pi) \log z$. Our Theorem B is proved under the hypothesis of parabolicity of the monodromy representation, while Theorem C is proved under the stronger hypothesis of parabolicity of the projective structure.

1.2. *The image of a generic Brownian path by the developing map.* In [**CDFG**], the authors prove that if the monodromy group is a dense subgroup of $PSL(2, \mathbb{C})$ and if h is a germ of $\mathcal{D}^{-1}$ in z_0, then the set of singularities for the analytic continuation of h is all the Riemann sphere $\mathbb{CP}^1$ (see Proposition 3.2). In other words, for any point z in $\mathbb{CP}^1$, there is a continuous path c from z_0 to z such that h cannot be analytically continued along c (we will give a proof of this fact in §3). As it has been explained earlier, with the same hypothesis, h can be analytically continued along a generic Brownian path (i.e. the Brownian motion does not see this full set of singularities). This is stated more precisely in the following theorem.

THEOREM B. *Let Σ be a Riemann surface of finite type endowed with a branched projective structure. Let $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$ be a developing map and $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ be the monodromy representation associated to $\mathcal{D}$. Let (x_0, z_0) be a couple of points in $\tilde{\Sigma} \times \mathbb{CP}^1$ such that $\mathcal{D}(x_0) = z_0$ and let h be the germ of $\mathcal{D}^{-1}$ such that $h(z_0) = x_0$. We also define the Brownian motion in $\tilde{\Sigma}$ as the one associated to the hyperbolic metric with constant curvature -1 and the Brownian motion in $\mathbb{CP}^1$ as the one associated to any complete metric in its conformal class.*

Assume that $\mathcal{D}$ is onto and that the monodromy representation ρ is parabolic and non-elementary.

Then the two following equivalent assertions are satisfied.

(1) *For almost every Brownian path ω starting from x_0, $\mathcal{D}(\omega(t))$ does not have any limit when t goes to ∞.*
(2) *For almost every Brownian path ω starting from z_0, the map h can be analytically continued along $\omega([0, \infty[)$.*

The equivalence of the two assertions is a direct consequence of the conformal invariance of the Brownian motion. In order to prove the first assertion, we will use the discretization procedure of Furstensberg–Lyons–Sullivan. In our context, this procedure associates a sequence $X_n(\omega)$ of elements of $\pi_1(\Sigma)$ to every Brownian path ω in $\tilde{\Sigma}$, which corresponds, more or less, to the sequence of fundamental domains visited by ω. The sequence $X_n(\omega)$ turns out to be the realization of a right random walk, that is, $X_{n+1}(\omega) = X_n(\omega) \cdot \gamma_{n+1}(\omega)$, the $\gamma_n(\omega)$ being independent and identically distributed. Pushing $X_n(\omega)$ forward by ρ gives a right random walk $Y_n(\omega)$ in $\rho(\pi_1(\Sigma)) < PSL(2, \mathbb{C})$. Random walks in such matrix groups have been widely studied. A classical result of the theory is the following. If the support of the measure μ defining the random walk Y_n is non-elementary and if ν is a μ-stationary measure on $\mathbb{CP}^1$, then, for almost every ω, there exists $z(\omega) \in \mathbb{CP}^1$ such that

$$Y_n(\omega) \cdot \nu \underset{n\to\infty}{\longrightarrow} \delta_{z(\omega)}. \tag{2}$$

In view of this property, Theorem B is surprising because one could think at first glance that this contraction property would imply that $\mathcal{D}(\omega(t)) \underset{t\to\infty}{\longrightarrow} z(\omega)$.

In §6, we will give a new statement of the last theorem, including the case where $\mathcal{D}$ is not onto. In this case, the opposite conclusion holds. For almost every Brownian path ω starting from x_0, there is a point $z(\omega)$ such that $\lim_{t\to\infty} \mathcal{D}(\omega(t)) = z(\omega)$, which is equivalent to the following: for almost every Brownian path ω starting from z_0, the map h cannot be analytically continued along $\omega([0, \infty[)$.

1.3. *The family of harmonic measures.* At the beginning of this study, we did not think that Theorem B was realistic. On the contrary, we expected to prove that, in both cases ($\mathcal{D}$ onto and $\mathcal{D}$ not onto), the following would hold: for all x in $\tilde{\Sigma}$, for almost every Brownian path ω starting at x, there is a point $z(\omega)$ such that $\lim_{t\to\infty} \mathcal{D}(\omega(t)) = z(\omega)$. The existence of such a point $z(\omega)$ would allow us to associate to any projective structure on Σ a family of measures $(\nu_x)_{x\in\tilde{\Sigma}}$ in $\mathbb{P}^1$ in the following way: if $\mathbb{P}_x$ is the Wiener measure on the set Ω_x of continuous paths starting at x, for any Borel set A in $\mathbb{P}^1$, we would have defined

$$\nu_x(A) = \mathbb{P}_x(w \in \Omega_x \text{ such that } z(\omega) \in A).$$

Although $\mathcal{D}(\omega(t))$ does not converge when t goes to ∞ (in the case where $\mathcal{D}$ is onto) in the classical sense, $\mathcal{D}(\omega(t))$ converges almost surely in the Cesàro sense. This gives the following theorem.

THEOREM C. *Let $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$ and $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ be a couple developing map-monodromy representation associated with a branched complex projective structure of parabolic type on a hyperbolic Riemann surface Σ of finite type. Then, for every x in the universal cover $\tilde{\Sigma}$ and for almost every Brownian path ω starting from x, there exists $z(\omega) \in \mathbb{CP}^1$ such that*

$$\frac{1}{t} \cdot \int_0^t \delta_{\mathcal{D}(\omega(s))} \cdot ds \underset{t\to\infty}{\longrightarrow} \delta_{z(\omega)}.$$

Remark 1.4.

- The limit point $z(\omega)$ in Theorem C is nothing but the attractive point $z(\omega)$ of the random walk $\rho(X_n(\omega))$ defined above in (2). Equivalently, $z(\omega)$ is the projectivization of Oseledets' contracting direction of the cocycle $\rho(X_n(\omega))^{-1}$. Hence, the limit point $z(\omega)$ does not depend on the developing map $\mathcal{D}$ but it only depends on the monodromy representation ρ.
- The monodromy representation of a projective structure of parabolic type is always non-elementary (see [**DD**, p. 3] and the references therein).

Then, to any complex projective structure on Σ satisfying the hypothesis of the previous theorem, one can associate a family $(\nu_x)_{x\in\tilde{\Sigma}}$ of harmonic measures on $\mathbb{CP}^1$: it is the distribution law of the point $z(\omega)$ (given by the previous theorem) for a Brownian path starting at x. This family of measures gives interesting information about the projective structure. It has been recently studied by Deroin and Dujardin in [**DD**]. In a recent work in collaboration with Alvarez [**AH**], we prove the following. For all $x \in \tilde{\Sigma}$, the image of a generic geodesic ray starting at x by the developing map has a limit in $\mathbb{CP}^1$. The distribution law of this limit point (with respect to the angular measure at x for the Poincaré metric) is a measure μ_x which is proved to be equal to ν_x.

1.4. *Organization of the paper.* First, §2 is devoted to the basic definitions and examples concerning branched projective structures. Then §3 deals with generalities about analytic continuation of holomorphic maps. Section 4, where we prove a contraction property for random walks in $PSL(2, \mathbb{C})$, and §5, where we explain the discretization procedure of Furstenberg–Lyons–Sullivan, provide the necessary background for the proof of Theorem B in §6. In §7, we prove Theorem A and, finally, in §8, we prove Theorem C.

2. *Projective structures*

This section gives basic concepts about complex projective structures which will be useful in the subsequent work. For further insight into this notion, we refer the reader to the survey of Dumas [**Du**].

Definition 2.1. Let Σ be a Riemann surface. A *branched projective structure* on Σ is a maximal atlas $(\phi_i : U_i \to \mathbb{CP}^1)$, where the U_i are open sets in Σ and the ϕ_i are non-constant holomorphic maps on U_i such that, on the intersection of two domains $U_i \cap U_j$, the relation $\phi_i = \gamma_{ij} \circ \phi_j$ holds for some Möbius transformation γ_{ij} (i.e. for some element of $PSL(2, \mathbb{C})$).

Let $\phi_i : U_i \to V_i$ be a chart of such an atlas. If U_j is another chart such that $U_i \cap U_j \neq 0$, then the map $\gamma_{ij} \circ \phi_j : U_j \to \mathbb{CP}^1$ is equal to ϕ_i on $U_i \cap U_j$ and allows us to continue ϕ_i to U_j. Continuing in this way, we obtain a globally defined holomorphic map whose domain of definition is the universal covering space $\tilde{\Sigma}$. This map, denoted by $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$ is called a *developing map*. $\mathcal{D}$ is defined up to a post-composition by a Möbius transformation.

Associated with this, we can define a morphism $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ called a *monodromy representation* which satisfies the equivariance relation

$$\text{for all } x \in \tilde{\Sigma} \text{ for all } \alpha \in \pi_1(\Sigma), \quad \mathcal{D}(\alpha \cdot x) = \rho(\alpha) \cdot \mathcal{D}(x).$$

The group $\Gamma := \rho(\pi_1(\Sigma))$ is called a *monodromy group* of the branched projective structure. As the developing map $\mathcal{D}$ is defined up to a post-composition by a Möbius transformation, Γ is defined up to a conjugacy by this transformation.

In this paper, we will consider Riemann surfaces of finite type: that is, compact Riemann surfaces with a finite number of points deleted. Our Theorems B and C, concerning projective structures, both assume that the monodromy representation is non-elementary. Theorem B assumes that the monodromy representation is parabolic and Theorem C assumes that the projective structure is parabolic (which is a stronger condition). We will now recall the definitions of these notions.

Definition 2.2.

- A representation $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ is said to be parabolic if the monodomy is parabolic around each puncture (i.e. it is conjugated to the group generated by the transformation $z \mapsto z + 1$).
- A branched projective structure on a Riemann surface of finite type is said to be parabolic if, for any puncture p, there is a neighbourhood V_p of p and a biholomorphism $\phi : D(0, e^{-2\pi}) - \{0\} \to V_p$ such that some developing map satisfies $\mathcal{D} \circ \phi(z) = (1/(2i\pi)) \log z$ (in this definition, the developing must be seen as a multivalued holomorphic map from Σ to $PSL(2, \mathbb{C})$).
- A subgroup Γ of $PSL(2, \mathbb{C})$ is said to be elementary if there exists a finite set in $\mathbb{CP}^1$ which is globally invariant by the action of Γ or if it is conjugate to a subgroup of the projective special unitary group $PSU(2, \mathbb{C})$. A representation $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ is said to be elementary if $\rho(\pi_1(\Sigma))$ is elementary.

Remark 2.3.

- If the monodromy representation is non-elementary, then the Riemann surface is necessarily hyperbolic. Indeed, The monodromy group of a branched projective structure on the sphere is trivial (since $\pi_1(\mathbb{CP}^1)$ is trivial) and the monodromy group of a branched projective structure on a parabolic Riemann surface Σ is abelian (since $\pi_1(\Sigma)$ is).
- At the universal covering level, the parabolicity of a projective structure at a puncture p implies that the developing map in any connected component of the preimage of V_p is holomorphically conjugated to the inclusion map. More precisely, consider the universal covering map $q : \mathbb{H}_{\geq 1} = \{\mathrm{Im}\, z \geq 1\} \to D(0, e^{-2\pi}) - \{0\}$, $\tau \mapsto e^{2i\pi\tau}$. Let $\mathcal{H}_p$ be a connected component of $\mathrm{proj}^{-1}(V_p)$, where $\mathrm{proj} : \tilde{\Sigma} \to \Sigma$ is the universal covering map. If $\phi : D(0, e^{-2\pi}) - \{0\} \to V_p$ satisfies $\mathcal{D} \circ \phi(z) = (1/2i\pi) \log z$, then, lifting ϕ by q and proj, one gets a biholomorphism $\tilde{\phi} : \mathbb{H}_{\geq 1} \to \mathcal{H}_p$ satisfying $\mathcal{D} \circ \tilde{\phi}(\tau) = \tau$. Moreover, in [**AH**] the following lemma is proved.

  LEMMA 2.4. *$\tilde{\phi}$ is bi-Lipschitz for the distances associated to the hyperbolic metrics in $\mathbb{H}_{\geq 1}$ and in $\mathcal{H}_p$.*

- If $\mathcal{D}(z) = (1/2i\pi) \log z$ in a coordinate z around a puncture, then $\mathcal{D}(e^{2i\pi} z) = \mathcal{D}(z) + 1$. So the parabolicity of the projective structure implies the parabolicity of the monodromy representation. But, in general, the converse is false. Indeed, on the puncture disc, the projective structure given by $\mathcal{D}_n(z) = (1/2i\pi) \log z + 1/z^n$ has a parabolic monodromy representation ($\mathcal{D}_n(e^{2i\pi} z) = \mathcal{D}_n(z) + 1$) but it is not parabolic for $n \in \mathbb{N}^*$ (to see this, one can check, for example, that $\mathcal{D}_n(z)$ does not have limit when z goes to zero).

Examples 2.5.

(1) Let Σ be a hyperbolic Riemann surface. The universal covering space of Σ is the upper half-plane $\mathbb{H}$ and $\Sigma = \mathbb{H}/\Gamma$, where Γ is a subgroup of $PSL(2, \mathbb{R})$ whose action on $\mathbb{H}$ is free and properly discontinuous. The couple $(\mathcal{D}, \rho) = (i : \mathbb{H} \hookrightarrow \mathbb{CP}^1,\ i : \Gamma \hookrightarrow PSl(2, \mathbb{C}))$ (where i is the inclusion map) defines a projective structure on Σ called the *uniformizing projective structure* of Σ.

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(2) Let Γ be a Kleinian group (i.e. a discrete subgroup of $PSL(2, \mathbb{C})$) such that the set of discontinuity $\Omega(\Gamma) \in \mathbb{CP}^1$ is not vacuous. The quotient $\Omega(\Gamma)/\Gamma$ is a Riemann surface which can be endowed with a projective structure in the following way. We cover $\Omega(\Gamma)/\Gamma$ by open sets U_i small enough and we choose local inverses s_i of the projection $p : \Omega(\Gamma) \to \Omega(\Gamma)/\Gamma$ defined on U_i. The $s_i : U_i \to \Omega(\Gamma) \subset \mathbb{CP}^1$ define an atlas of Σ whose transition functions are elements of Γ (i.e. Möbius transformations). Note that, by Ahlfors' finiteness theorem [**Ah**], the Riemann surface $\Omega(\Gamma)/\Gamma$ is of finite type and the projective structure is parabolic.

(3) In the two previous examples, the developing map is not onto. Starting with the uniformizing projective structure of Σ, as in example (1), there is a surgery operation introduced by Heijal [**He**], called *grafting*, that produces new projective structures having the same monodromy representation but such that the new developing map is onto.

3. *Analytic continuation*

Recall that one of the goals of this paper is to show that, with some good assumptions on the projective structure, any local inverse h of the developing map can be analytically continued along a generic Brownian path. In this part, following the paper [**CDFG**], we show that, however, there are many paths along which h cannot be analytically continued. Let us start with some basic definitions about analytic continuation of holomorphic maps.

Let C_0 and C_1 be two Riemann surfaces and a germ of holomorphic map $h : (C_0, p_0) \to (C_1, p_1)$. Let $\tau : [0, t] \to C_0$ be a continuous path such that $\tau(0) = p_0$. We say that τ is covered by the sequence of open discs $D_1, \ldots, D_n$ if there is a sequence of times $0 = t_0 < t_1 < \cdots < t_n = t$ such that $\tau([t_k, t_{k+1}]) \subset D_{k+1}$. We say that h can be *analytically continued* along $\tau([0, t])$ if there is a sequence of discs $D_1, \ldots, D_n$ covering τ, and holomorphic maps $f_k : D_k \to C_1$, such that the germ of f_1 in p_0 is h and such that for all $k \in \{1, \ldots, n-1\}$, we have $f_k = f_{k+1}$ on $D_k \cap D_{k+1}$.

Definition 3.1. A point $q \in C_0$ is called a singularity for h if there is a continuous path $\tau : [0, 1] \to C_0$ such that:

(1) $\tau(0) = p_0$ and $\tau(1) = q$;

(2) for all $\epsilon > 0$, h can be analytically continued along $\tau([0, 1-\epsilon])$; and

(3) h cannot be analytically continued along $\tau([0, 1])$.

The set of singularities could be, in principle, any subset of C_0. If it is the whole C_0, we say that h has *full singular set*.

There may also exist an open set $D \subset C_0$ containing p_0 such that, for any path $\tau : [0, 1] \to C_0$ with $\tau(0) = p_0$, $\tau(1) \in \partial D$ and $\tau([0, 1[) \subset D$, h can be analytically continued along $\tau([0, 1-\epsilon])$ but not along $\tau([0, 1])$. In the case where ∂D is a topological disc, we say that h has a *natural boundary* for analytic continuation.

PROPOSITION 3.2. [**CDFG**] *Let Σ be a hyperbolic Riemann surface endowed with a branched projective structure. Let $\mathcal{D}$ be a developing map and h be a germ of $\mathcal{D}^{-1}$.*

(1) *If the projective structure is the one given by uniformization, then h has a natural boundary for analytic continuation.*

(2) *If the monodromy group is dense in $PSL(2, \mathbb{C})$, then h has full singular set.*

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Proof. For a complete proof, see [**CDFG**]. We will now state some of the ideas contained in this proof because we think it could be helpful for the comprehension of the proof of Theorem B.

(1) In this case, the developing map is the inclusion $i : \mathbb{H} \hookrightarrow \mathbb{CP}^1$. Then $\partial\mathbb{H} \subset \mathbb{CP}^1$ is a natural boundary for analytic continuation of h.

(2) Let h be a germ of $\mathcal{D}^{-1}$ at $z_0 \in \mathbb{CP}^1$ and $p_0 = h(z_0)$. The proof is based on the following lemma.

LEMMA 3.3. [**CDFG**] *For all $z \in \mathbb{CP}^1$, there is a finite set $\mathcal{A} \subset \pi_1(\Sigma)$ and an infinite sequence $(\alpha_n)_{n\in\mathbb{N}^*}$ of elements of $\mathcal{A}$ which have the following properties. Denoting $A_n = \alpha_1\alpha_2 \cdots \alpha_n$ and $A_0 = \mathrm{id}$:*

(a) *the diameter of the ball*

$$B_n = \left\{ w \in \mathbb{CP}^1 \text{ such that } |(\rho(A_n))'(w)| \geq \frac{1}{2^n} \right\}$$

converges to zero exponentially fast when n tends to infinity;

(b) *for all $n \in \mathbb{N}$, $\rho(A_n)(\mathbb{CP}^1 - B_n) \subset D(z, \mathrm{cst}/2^n)$; and*

(c) *for all $n \in \mathbb{N}$, neither z_0 nor $\rho(\alpha_n)(z_0)$ belong to B_{n-1}.*

In this lemma (the proof of which can be found in [**CDFG**]), $\mathbb{CP}^1$ is endowed with the standard spherical metric. In any of the two charts, this metric is written as $|ds| = |dz|/(1 + |z|^2)$. If γ is a Möbius transformation, γ' is the derivative of γ and $|\gamma'(z)|$ is the spherical norm in z. If $z \in \mathbb{CP}^1$ and $\alpha \in \mathbb{R}$, then $D(z, \alpha)$ is the spherical disc of radius α centred at z. We now prove that the previous lemma implies the following proposition. With properties (a) and (c) of the previous lemma, one can construct for all $n \in \mathbb{N}$, a C^∞ path $c_n : [0, 1] \to \tilde{\Sigma}$ from p_0 to $\alpha_n(p_0)$, whose length is bounded by a constant that is independent of n and such that, for n big enough, $\mathcal{D} \circ c_n$ does not meet B_{n-1}. Then we define the path $c : [0, \infty[\to \tilde{\Sigma}$ as the infinite concatenation of paths $a_n := A_{n-1}c_n$ (from $A_{n-1}(p_0)$ to $A_n(p_0)$). The ρ-equivariance gives

$$\mathcal{D} \circ a_n = \rho(A_{n-1}) \circ \mathcal{D} \circ c_n.$$

As $\mathcal{D} \circ c_n$ does not meet B_{n-1}, we deduce, from property (a) of the previous lemma, that the length of the path $\mathcal{D} \circ a_n$ converges exponentially fast to zero and so $\mathcal{D} \circ c(t)$ converges, when t goes to infinity, toward a point in $\mathbb{CP}^1$. Using property (b) of the previous lemma, this point is necessarily z (because $\mathcal{D} \circ a_n \subset D(z, \mathrm{cst}/2^{n-1})$). So z is a singularity for analytic continuation of h. □

4. *Random walks*

In this section, after explaining some basic facts about random walks and stationary measures, we prove Proposition 4.4, which is the key of the proof of Theorems B and C.

In this part, Γ is a subgroup of $PSL(2, \mathbb{C})$, finitely generated, and μ is a probability measure on Γ. Also, $\mathrm{supp}(\mu)$ is the support of μ and $\langle \mathrm{supp}(\mu) \rangle$ is the group generated by $\mathrm{supp}(\mu)$. Define $\Omega = \Gamma^{\mathbb{N}^*}$ and $\mathbb{P} = \mu^{\mathbb{N}^*}$. The coordinate maps $h_i : \Omega \to \Gamma$ are $\mathbb{P}$-independent and identically distributed with law μ. This part deals with the statistical behaviour of the action on $\mathbb{CP}^1$ of the right random walk in Γ with law μ: $X_n(\omega) = h_1(\omega) \cdots h_n(\omega)$.

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The action of Γ on $\mathbb{CP}^1$ gives an action of Γ on the set $\mathcal{P}(\mathbb{CP}^1)$ of Borel probability measures on $\mathbb{CP}^1$. If $\gamma \in \Gamma$, $\nu \in \mathcal{P}(\mathbb{CP}^1)$ and A is a Borel set in $\mathbb{CP}^1$, this action is defined by $\gamma \cdot \nu(A) = \nu(\gamma^{-1}(A))$.

We also define $\mu^{*n} := \mu * \mu * \cdots * \mu$. The measure μ^{*n} on Γ is the push-forward of the product measure $\mu^{\otimes n}$ on Γ^n by the map $\Gamma \times \cdots \times \Gamma \to \Gamma$, $(\gamma_1, \ldots, \gamma_n) \mapsto \gamma_1 \cdots \gamma_n$. The law of X_n is μ^{*n}. If $\nu \in \mathcal{P}(\mathbb{CP}^1)$, we also define the measure $\mu * \nu$ as the push-forward on $\mathbb{CP}^1$ of the product measure on $\Gamma \times \mathbb{CP}^1$ by the map $\Gamma \times \mathbb{CP}^1 \to \mathbb{CP}^1$, $(\gamma, x) \mapsto \gamma \cdot x$. So, if A is a Borel set in $\mathbb{CP}^1$,

$$\mu * \nu(A) = \sum_{\gamma \in \Gamma} \mu(\gamma)\nu(\gamma^{-1}(A)).$$

Definition 4.1. The measure $\nu \in \mathcal{P}(\mathbb{CP}^1)$ is said to be μ-stationary if $\mu * \nu = \nu$, which means that for any Borel set A in $\mathbb{CP}^1$,

$$\sum_{\gamma \in \Gamma} \mu(\gamma)\nu(\gamma^{-1}(A)) = \nu(A).$$

The following results are classical.

THEOREM 4.2. (Furstenberg)
(1) *There always exists a μ-stationary measure on $\mathbb{CP}^1$* [**Fur**].
(2) *Let ν be a μ-stationary measure on $\mathbb{CP}^1$. Then, for almost every $\omega \in \Omega$, there is a measure $\lambda(\omega) \in \mathcal{P}(\mathbb{CP}^1)$ such that the sequence of probability measures $X_n(\omega) \cdot \nu$ converges weakly towards $\lambda(\omega)$* [**Fur2**].
(3) *If $\langle \mathrm{supp}(\mu) \rangle$ is not an elementary group. Then, for almost every $\omega \in \Omega$, there is $z(\omega) \in \mathbb{CP}^1$ such that $\lambda(\omega) = \delta_{z(\omega)}$ (Dirac in $z(\omega)$)* [**Fur2**].
(4) *If $\langle \mathrm{supp}(\mu) \rangle$ is not an elementary group, then any μ-stationary measure on $\mathbb{CP}^1$ is non-atomic* [**Wo**].

4.1. *The Lyapunov exponent.* The positivity of the Lyapunov exponent is a central result in the theory of random walks and is one of the key points of the proofs of Theorems B and C.

THEOREM 4.3. (Furstenberg) *If*
(1) $\int_\Gamma \log \|\gamma\| \, d\mu(\gamma) < +\infty$ *and*
(2) $\langle \mathrm{supp}(\mu) \rangle$ *is not an elementary group,*

then there exists $\lambda > 0$ such that $\mathbb{P}$-almost surely and

$$\frac{1}{n} \log \|X_n\| \longrightarrow \lambda,$$

where λ is called the Lyapunov exponent of the random walk. The fact that $(1/n) \log \|X_n\|$ converges almost surely to $\lambda \in [0, \infty[$ is a direct consequence of Kingman's subadditive ergodic theorem and requires the first hypothesis of the theorem ($\int_\Gamma \log \|\gamma\| \, d\mu(\gamma) < +\infty$). The fact that $\lambda > 0$ requires the second hypothesis and was first proved by Furstenberg [**Fur**, Theorem 8.6] (see also [**BLa**]).

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4.2. *A corollary of the positivity of the Lyapunov exponent.* We work with the distance induced by the Fubini–Study metric on $\mathbb{CP}^1$,

$$ds^2 = \frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2},$$

and we denote the closed disc centred in x with radius α by $D(x, \alpha)$ and its complementary set by $(D(x, \alpha))^c$. As a rather direct consequence of Theorem 4.3, we get the following proposition (see [**Hu**] for a proof).

PROPOSITION 4.4. *If*

(1) $\int_\Gamma \log \|\gamma\| \, d\mu(\gamma) < +\infty$ *and*

(2) $\langle \mathrm{supp}(\mu) \rangle$ *is not an elementary group,*

then there are constants $0 < \lambda' < \lambda''$ *such that, for* $\mathbb{P}$*-almost every* $\omega \in \Omega$*, there is* $N(\omega)$ *such that, for all* $n > N(\omega)$*, there are* $y_n(\omega), z_n(\omega) \in \mathbb{CP}^1$ *such that*

(1) $X_n(\omega)((D(y_n(\omega), e^{-\lambda' n}))^c) \subset D(z_n(\omega), e^{-\lambda' n})$ *and*

(2) $X_n(\omega)(D(y_n(\omega), e^{-2\lambda'' n})) \subset (D(z_n(\omega), \frac{1}{2}))^c$.

Remark 4.5.

- Almost surely, the sequence $(z_n(\omega))$, defined in the previous Proposition 4.4, converges to the point $z(\omega)$, defined in Theorem 4.2(3). Indeed, let α be an accumulation point of the sequence (z_n) that is different from z. Let $(n_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} z_{n_i} = \alpha$. Theorem 4.2 gives

$$X_{n_i} \cdot \nu(D(\alpha, (d(z, \alpha))/2)) \to \delta_z(D(\alpha, (d(z, \alpha))/2)) = 0.$$

  We deduce, from Proposition 4.4, that $\nu(D(y_{n_i}, e^{-\lambda' n_i})) \to 1$. Extracting a new time, one can suppose that $y_{n_i} \to y \in \mathbb{CP}^1$. Then $\nu(\{y\}) = 1$, which contradicts the fact that ν is a non-atomic measure.
- The limit $z(\omega)$ of the sequence $(z_n(\omega))$ has also a dynamical interpretation: it is the projectivization of Oseledets' contracting direction of $X_n(\omega)^{-1}$. More precisely, when applying Oseledets' theorem to our situation (see [**Ar**]), we get, for almost every ω, a one-dimensional vector space $F(\omega)$ in $\mathbb{C}^2$ (which depends measurably of ω) such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|X_n(\omega)^{-1} \cdot v\| = \begin{cases} \lambda & \text{if } v \in \mathbb{C}^2 - F(\omega), \\ -\lambda & \text{if } v \in F(\omega) - \{0, 0\}. \end{cases}$$

  The point $z(\omega) \in \mathbb{CP}^1$ is simply the projectivization of the vectorial space $F(\omega) \subset \mathbb{C}^2$.

5. *Brownian motion and discretization*

This part deals with the Brownian motion. Firstly, we recall the classical conformal invariance property of the Brownian motion. Secondly, we include a detailed treatment of the discretization procedure of Furstenberg–Lyons–Sullivan which is close, but not identical, to that of Lyons and Sullivan (see [**LS**, **BL**]).

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5.1. *Brownian motion and conformal invariance.* Let (M, g) be a connected Riemannian manifold with bounded geometry. The Brownian motion on (M, g) is the diffusion process associated to the Laplace–Beltrami operator Δ. It is defined on a probability space $(\Omega, \mathbb{P})$ and denoted by $(B_t)_{t\geq 0}$. We will make use of the following classical result of Lévy [**Le**] which states that conformal maps are Brownian paths preserving up to a change of timescale.

THEOREM 5.1. (Lévy) *Let (S_1, g_1) and (S_2, g_2) be two connected, Riemannian surfaces and $f : (S_1, g_1) \to (S_2, g_2)$ be a conformal map. Let $(B_t)_{t\in[0,T[}$ be a Brownian motion starting from a point $b_0 \in S_1$, running up to a stopping time T. Then the process $(f(B_t))_{t\in[0,T[}$ is a changed time Brownian motion. In other words, there exists a family of strictly increasing functions $\sigma_\omega : [0, T(\omega)[\longrightarrow [0, +\infty[$ and a Brownian motion $(B'_s)_{s\geq 0}$ starting from $f(b_0)$ such that*

$$f \circ B = B' \circ \sigma.$$

Remarks 5.2.

(1) $f \circ B = B' \circ \sigma$ means that, for all $\omega \in \Omega$ and for all $t \in [0, T(\omega)[$, we have $f \circ B_t(\omega) = B'_{\sigma_\omega(t)}(\omega)$.

(2) If $|f'(z)|$ denotes the modulus of the derivative of f in z relative to the metrics g_1 and g_2, then the timescale change is explicitly given by

$$\sigma_\omega(t) = \int_0^t |f'(B_u(\omega))|^2\, du.$$

5.2. *Discretization of the Brownian motion.* In the most general context, this procedure associates a Markov chain in a discrete $*$-recurrent set $X \subset M$, with time homogeneous transition probabilities, to the Brownian motion in a Riemannian manifold (M, g). Here we explain the discretization in the case where $M = \tilde{\Sigma} = \mathbb{D}$ is the universal covering space of a hyperbolic Riemann surface Σ of finite type, and $X = \pi_1(\Sigma) \cdot 0$. We follow the presentation of [**KL**].

Let Σ be a hyperbolic Riemann surface of finite type. The fundamental group $\pi_1(\Sigma)$ of Σ acts on $\tilde{\Sigma}$ $(= \mathbb{D})$, the universal covering space of Σ, by isometry for the Poincaré metric of the disc. For all $X \in \pi_1(\Sigma)$, we define $F_X = X.\overline{D(0, \delta)}$ and $V_X = X.D(0, \delta')$, with $\delta < \delta'$. We also require that δ and δ' are small enough so that $F_X \cap V_{X'} = \varnothing$ for $X \neq X'$. Let $(\Omega_x, \mathbb{P}_x)$ be the set of Brownian paths starting from x in $\mathbb{D}$ with the Wiener measure associated to the Poincaré metric in the hyperbolic disc. Also $\bigcup_{X\in\pi_1(\Sigma)} F_X$ is a recurrent set for the Brownian motion (because Σ is of finite type). Let $X \in \pi_1(\Sigma)$. For $x \in F_X$, consider $\epsilon_x^{\partial V_X}$ which is the exit measure of V_X for a Brownian motion starting from x. The Harnack constant C_X of the couple (F_X, V_X) is defined by

$$C_X = \sup\left\{\frac{d\epsilon_x^{\partial V_X}}{d\epsilon_y^{\partial V_X}}(z);\ x, y \in F_X, z \in \partial V_X\right\},$$

where $(d\epsilon_x^{\partial V_X})/(d\epsilon_y^{\partial V_X})$ is the Radon–Nikodym derivative. Notice that, as elements of $\pi_1(\Sigma)$ act isometrically on $\mathbb{D}$, the Harnack constant of (F_X, V_X) does not depend on $X \in \pi_1(\Sigma)$ (i.e. there is a constant C such that for all $X \in \pi_1(\Sigma)$, $C_X = C$). Hence, the family of couples $(F_X, V_X)_{X\in\pi_1(\Sigma)}$ defines a system of Lyons–Sullivan (L–S) data in the sense of Ballmann and Ledrappier [**BL**, p. 4].

If $x \in V_{\mathrm{Id}}$ and $\omega \in \Omega_x$, we define, recursively,

$$S_0(\omega) = \inf\{t \geq 0;\ \omega(t) \notin V_{\mathrm{Id}}\}$$

and, for $n \geq 1$,

$$R_n(\omega) = \inf\left\{t \geq S_{n-1}(\omega);\ \omega(t) \in \bigcup F_X\right\},$$

$$S_n(\omega) = \inf\left\{t \geq R_n(\omega);\ \omega(t) \notin \bigcup V_X\right\}.$$

We also define $X_n(\omega)$ by

$$X_0(\omega) = \mathrm{Id} \quad \text{and} \quad w(R_n(\omega)) \in F_{X_n(\omega)} \quad \text{for } n \geq 1,$$

$$\kappa_n(\omega) = \frac{1}{C}\left(\frac{d\epsilon_{X_n(\omega)\cdot 0}^{\partial V_{X_n(\omega)}}}{d\epsilon_{\omega(R_n(\omega))}^{\partial V_{X_n(\omega)}}}(\omega(S_n(\omega)))\right).$$

By definition of C and κ_n, note that $1/C^2 \leq \kappa_n \leq 1$.

Now we define $(\Omega_0 \times [0, 1]^{\mathbb{N}}, \mathbb{P}_0 \otimes \mathrm{leb}^{\otimes\mathbb{N}}) = (\tilde{\Omega}, \tilde{\mathbb{P}})$. Let

$$\begin{array}{rccc} N_k : & \tilde{\Omega} & \longrightarrow & \mathbb{N} \\ & (\omega, \alpha) = (\omega, (\alpha_n)_{n\in\mathbb{N}}) = \tilde{\omega} & \longmapsto & N_k(\tilde{\omega}) \end{array}$$

be the random variable defined, recursively, by

$$N_0(\tilde{\omega}) = 0,$$

$$N_k(\omega, \alpha) = \inf\{n > N_{k-1}(\omega, \alpha);\ \alpha_n < \kappa_n(\omega)\}.$$

The following theorem is stated in [**LS**] in the cocompact case but it is observed in [**K**, Proposition 4] that it is also valid in the general set-up.

THEOREM 5.3. [**LS**, Theorem 6] *The distribution law of X_{N_1} defines a probability measure μ on $\pi_1(\Sigma)$ which satisfies, for any Borel set A in $\mathbb{D}$,*

$$\tilde{\mathbb{P}}(X_{N_1} = x_1; \ldots; X_{N_k} = x_k, \omega(S_{N_k}) \in A) = \mu(x_1)\mu(x_1^{-1}x_2)\cdots\mu(x_{k-1}^{-1}x_k)\epsilon_{x_k.0}^{\partial V_{x_k}}(A).$$

COROLLARY 5.4. [**LS**] *$(X_{N_k})_{k\in\mathbb{N}}$ is the realization of a right random walk in $\pi_1(\Sigma)$ with law μ: in other words, $(\gamma_{N_k} := X_{N_{k-1}}^{-1} X_{N_k})_{k\in\mathbb{N}^*}$ is a sequence of independent, identically distributed random variables with law μ.*

The following two propositions will be useful later.

PROPOSITION 5.5. [**KL**, Corollaire 3.4] *There is a constant $T > 0$ such that almost surely S_{N_k}/k converges to T when k goes to infinity.*

Note that there is a constant D such that, for all $X \in \pi_1(\Sigma)$ and for all $z \in \partial F_X$, the Green's function $G_{V_X}(X \cdot 0, z) = D$. This is because the Green's function of a hyperbolic disc centred at zero is radial. Hence the L–S data $(F_X, V_X)_{X\in\pi_1(\Sigma)}$ are balanced (see definition in [**BL**, p. 9]). This gives the next proposition.

PROPOSITION 5.6. [**BL**, Theorem 3.2(b)] *The measure μ has full support and has a finite first moment with respect to the distance d associated to the Poincaré metric in $\mathbb{D}$: in other words, $\int_{\gamma\in\pi_1(\Sigma)} d(\gamma \cdot 0, 0)\, d\mu(\gamma) < +\infty$.*

6. *Proof of Theorem B*

Actually, we are going to prove the following theorem which is a reformulation of Theorem B including the case where $\mathcal{D}$ is not onto. In this theorem, the Brownian motion in $\mathbb{D}$ (respectively, $\mathbb{CP}^1$) is the one associated to the hyperbolic metric (respectively, any complete metric in its conformal class).

THEOREM 6.1. *Let Σ be a Riemann surface of finite type endowed with a branched projective structure. Let $\mathcal{D} : \tilde{\Sigma} \to \mathbb{CP}^1$ be a developing map and $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ be the monodromy representation associated to $\mathcal{D}$. Assume that ρ is parabolic and non-elementary. Let (x_0, z_0) be a couple of points in $\tilde{\Sigma} \times \mathbb{CP}^1$ such that $\mathcal{D}(x_0) = z_0$ and let h be the germ of $\mathcal{D}^{-1}$ such that $h(z_0) = x_0$.*

First case: $\mathcal{D}$ is onto. Then the two following equivalent assertions are satisfied.

(1) *For almost every Brownian path ω starting from x_0, $\mathcal{D}(\omega(t))$ does not have limit when t goes to ∞.*
(2) *For almost every Brownian path ω starting from z_0, h can be analytically continued along $\omega([0, \infty[)$.*

Second case: $\mathcal{D}$ is not onto. Then the two following equivalent assertions are satisfied.

(1) *For almost every Brownian path ω starting from x_0, there is a point $z(\omega)$ such that $\lim_{t\to\infty} \mathcal{D}(\omega(t)) = z(\omega)$.*
(2) *For almost every Brownian path ω starting from z_0, h cannot be analytically continued along $\omega([0, \infty[)$.*

Firstly, notice that, according to Remark 2.3, as the monodromy group $\Gamma := \rho(\pi_1(\Sigma))$ is non-elementary, Σ is a hyperbolic Riemann surface. Notice also that in either of the two cases ($\mathcal{D}$ onto and $\mathcal{D}$ not onto), the two conclusions are equivalent because of the conformal invariance of the Brownian motion. More precisely, in the first case, if $(B_t)_{t\in[0,\infty[}$ is a Brownian motion in $\tilde{\Sigma}$, then $(\mathcal{D} \circ B_{\sigma^{-1}(s)})_{0\le s\le T}$ is a Brownian motion in $\mathbb{CP}^1$ stopped at time $T = \lim_{t\to\infty} \sigma(t)$. If $\mathcal{D} \circ B_t$ does not have limit when t goes to ∞, then almost surely $T = \infty$. Thus, almost surely, the germ h of a local inverse of $\mathcal{D}$ can be analytically continued along the Brownian motion (defined for every positive time) $(\mathcal{D} \circ B_{\sigma^{-1}(s)})_{0\le s\le\infty}$. Conversely, if h can be analytically continued along a generic Brownian path in $\mathbb{CP}^1$, then, almost surely, $\mathcal{D} \circ B_t$ does not have limit when t goes to ∞. Otherwise, we would have $T(\omega) < \infty$ for ω belonging to a set A with strictly positive Wiener measure. Hence, for all $\omega \in A$, the germ h could not be analytically continued along the Brownian path $(\mathcal{D} \circ B_{\sigma_\omega^{-1}(s)})_{0\le s\le T(\omega)}$. The proof of the equivalence of the two assertions in the second case (i.e. in the case where $\mathcal{D}$ is not onto) is similar.

6.1. *Proof in the case where $\mathcal{D}$ is onto.*

The discretization. In order to prove the theorem, we are going to use the discretization procedure explained in the previous part and the contraction property 4.4 proved in §4. To simplify the notation, we take $x_0 = 0$ and $\omega \in \Omega_0$. If $\tilde{\omega} = (\omega, \alpha) \in \tilde{\Omega}$, then the path ω can be written as an infinite concatenation of paths

$$\omega = \beta_0 * \omega_0 * \beta_1 * \omega_1 * \cdots ,$$

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where $\beta_0 = \omega_{|[0,S_{N_0}]}$, for $k \geq 0$, $\omega_k = \omega_{|[S_{N_k},R_{N_{k+1}}]}$ and for $k \geq 1$, $\beta_k = \omega_{|[R_{N_k},S_{N_k}]}$. Let $c_k(t) = X_{N_k}^{-1} \cdot \omega_k(t - S_{N_k})$. The $(c_k)_{k\in\mathbb{N}}$ form a family of portions of Brownian paths independent and identically distributed: the distribution law of their starting point is the exit measure of $V_{\mathrm{Id}} = D(0, \delta')$ for a Brownian motion starting at zero and they are stopped at time $R_{N_{k+1}} - S_{N_k}$. So

$$\omega = \beta_0 * X_{N_0}c_0 * \beta_1 * X_{N_1}c_1 * \cdots .$$

Because of the ρ-equivariance,

$$\mathcal{D}(\omega) = \mathcal{D}(\beta_0) * \rho(X_{N_0})\mathcal{D}(c_0) * \mathcal{D}(\beta_1) * \rho(X_{N_1})\mathcal{D}(c_1) * \cdots .$$

Now we are going to push forward the right random walk X_{N_k} by ρ in order to obtain a right random walk in the monodromy group Γ and then apply Proposition 4.4. For this, we write $\tilde{\mu} = \rho_*\mu$ (where μ is the probability measure in $\pi_1(\Sigma)$ defined by the discretization procedure of the previous part) and $Y_{N_k} = \rho(X_{N_k})$. The process $(Y_{N_k})_{k\geq 0}$ is a realization of a right random walk in Γ with law $\tilde{\mu}$. The parabolicity of the monodromy representation implies the following (see [**A**, Theorem 3.4.2]) for a proof). There is a constant a such that, for all $\alpha \in \pi_1(\Sigma)$, we have $\log(\|\rho(\alpha)\|) \leq a \cdot d(0, \alpha \cdot 0)$. We deduce, using Proposition 5.6, that $\int_{\alpha\in\pi_1(\Sigma)} \log(\|\rho(\alpha)\|)\, d\mu(\alpha) < +\infty$ and so $\int_{\gamma\in\Gamma} \log(\|\gamma\|)\, d\tilde{\mu}(\gamma) < +\infty$. Then the hypotheses of Proposition 4.4 are satisfied. Consequently, there are $0 < \lambda' < \lambda''$ such that, for $\tilde{\mathbb{P}}$-almost every $\tilde{\omega} \in \tilde{\Omega}$, there is $N(\tilde{\omega})$ such that, for all $k > N(\tilde{\omega})$, there is $y_k(\tilde{\omega}), z_k(\tilde{\omega}) \in \mathbb{CP}^1$ such that:

(1) $Y_{N_k}((D(y_k, e^{-\lambda' k}))^c) \subset D(z_k, e^{-\lambda' k})$; and

(2) $d(Y_{N_k}(D(y_k, e^{-2\lambda'' k})), z_k) \geq \frac{1}{2}$.

Then the theorem follows from the next proposition.

PROPOSITION 6.2. *For almost every $\tilde{\omega}$, there is a sequence $(k_n)_{n\in\mathbb{N}}$ converging to infinity such that*

$$\mathcal{D}(c_{k_n}) \cap D(y_{k_n}, e^{-2\lambda'' k_n}) \neq \varnothing \quad \textit{and} \quad \mathcal{D}(c_{k_n}) \cap (D(y_{k_n}, e^{-\lambda' k_n}))^c \neq \varnothing.$$

Proposition 6.2 implies Theorem 6.1. Indeed, by Proposition 4.4, the previous proposition implies that, for an infinite number of values of k, the portion $\rho(X_{N_k})\mathcal{D}(c_k)$ of the path $\mathcal{D}(\omega)$ visits $D(z_k, e^{-\lambda' k})$ and $D(z_k, \frac{1}{2})^c$, which proves that $\mathcal{D}(\omega(t))$ does not have limit when t goes to infinity.

The technical lemma. We still have to prove Proposition 6.2. For that purpose, let us define

$$E_k = \{\mathcal{D}(c_k) \cap D(y_k, e^{-2\lambda'' k}) \neq \varnothing\} \cap \{\mathcal{D}(c_k) \cap (D(y_k, e^{-\lambda' k}))^c \neq \varnothing\}.$$

We need to prove that

$$\tilde{\mathbb{P}}\bigg(\bigcap_{n\geq 0} \bigcup_{k\geq n} E_k\bigg) = 1. \tag{3}$$

It turns out that there is a constant c such that, for all $k \in \mathbb{N}^*$, $\tilde{\mathbb{P}}(E_k) \geq c/k$, which implies that $\sum_{k\geq 1} \tilde{\mathbb{P}}(E_k) = \infty$. So, if the sequence $(E_k)_{k\in\mathbb{N}}$ were a sequence of independent events, one could conclude that (3) is true using the Borel–Cantelli lemma. Unfortunately, one can be convinced easily that the E_k are not independent: this is due to the fact that the y_k are not mutually independent. This observation makes the proof of (3) more technical: instead of proving that $\tilde{\mathbb{P}}(E_k) \geq c/k$, we are going to prove the following lemma.

LEMMA 6.3. *There exist constants $c > 0$ and $N_0 \in \mathbb{N}^*$ such that, for all $N \geq N_0$ and $k > N$,*

$$\tilde{\mathbb{P}}(E_k \mid E_{k-1}^c, \ldots, E_N^c) \geq \frac{c}{k}.$$

Lemma 6.3 implies Proposition 6.2. Let us assume that Lemma 6.3 is proved. To prove Proposition 6.2, it is enough to prove (3). So it is enough to prove that, for all $N \in \mathbb{N}$, $\tilde{\mathbb{P}}(\bigcap_{n=N}^{\infty} E_n^c) = 0$. Let $N \geq N_0$. Then

$$\tilde{\mathbb{P}}\bigg(\bigcap_{n=N}^{\infty} E_n^c\bigg) = \lim_{k\to\infty} \tilde{\mathbb{P}}\bigg(\bigcap_{n=N}^{k} E_n^c\bigg).$$

Let $k > N$, $u_k = \tilde{\mathbb{P}}(\bigcap_{n=N}^{k} E_n^c)$ and $\alpha_k = \tilde{\mathbb{P}}(E_k^c | E_{k-1}^c, \ldots, E_N^c)$. Then

$$\begin{aligned}
u_k &= \alpha_k \cdot u_{k-1} \\
&= \alpha_k \alpha_{k-1} \cdots \alpha_{N+1} \cdot u_N \\
&\leq \bigg(1 - \frac{c}{k}\bigg)\bigg(1 - \frac{c}{k-1}\bigg) \cdots \bigg(1 - \frac{c}{N+1}\bigg) \cdot u_N \\
&= \prod_{n=N+1}^{k} \bigg(1 - \frac{c}{n}\bigg) \cdot u_N \\
&\leq \prod_{n=N+1}^{k} e^{-c/n} \cdot u_N \\
&= \exp\bigg(-\sum_{n=N+1}^{k} \frac{c}{n}\bigg) \cdot u_N \underset{k\to\infty}{\longrightarrow} 0.
\end{aligned}$$

So, for all $N > N_0$, $\tilde{\mathbb{P}}(\bigcap_{n=N}^{\infty} E_n^c) = 0$. And, if $N < N_0$, then $\bigcap_{n=N}^{\infty} E_n^c \subset \bigcap_{n=N_0}^{\infty} E_n^c$. So $\tilde{\mathbb{P}}(\bigcap_{n=N}^{\infty} E_n^c) = 0$, which finishes the proof of (3).

Proof of Lemma 6.3. The proof of this lemma will occupy the rest of this section. To do it, we will need the following lemma.

LEMMA 6.4. *There exists $\beta > 0$, $r > 0$, and $N_0 \in \mathbb{N}$ such that, for all $y \in \mathbb{CP}^1$, there exists $x \in D(0, r)$ such that, for all $k \geq N_0$,*

$$D(x, \beta e^{-2\lambda'' k}) \subset \mathcal{D}^{-1}(D(y, e^{-2\lambda'' k})).$$

Proof. $\mathcal{D}$ is onto, so there exists $r > 0$ such that $\mathcal{D}(D(0, r)) = \mathbb{CP}^1$. Let $1/\beta = \sup_{D(0,2r)} |\mathcal{D}'|$. Let $N_0 \in \mathbb{N}$ such that $\beta e^{-2\lambda'' N_0} < r$. Let $y \in \mathbb{CP}^1$ and let $x \in D(0, r)$ such that $\mathcal{D}(x) = y$. Let $k \geq N_0$ and $x_1 \in D(x, \beta e^{-2\lambda'' k})$. Then $d(\mathcal{D}(x), \mathcal{D}(x_1)) \leq$

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$\sup_{D(x,\beta e^{-2\lambda'' k})} |\mathcal{D}'| \cdot d(x, x_1)$. As $D(x, \beta e^{-2\lambda'' k}) \subset D(0, 2r)$, we deduce that $d(\mathcal{D}(x), \mathcal{D}(x_1)) \leq 1/\beta \cdot \beta \cdot e^{-2\lambda'' k} = e^{-2\lambda'' k}$, which finishes the proof. □

Let us notice that, for big enough k,

$$E_k = \{\mathcal{D}(c_k) \cap D(y_k, e^{-2\lambda'' k}) \neq \varnothing\}.$$

Indeed, for big enough k, the event $\mathcal{D}(c_k) \cap (D(y_k, e^{-\lambda' k}))^c \neq \varnothing$ is certain. To see this, note that

$$\{\mathcal{D}(c_k) \cap (D(y_k, e^{-\lambda' k}))^c \neq \varnothing\} = \{c_k \cap \mathcal{D}^{-1}(D(y_k, e^{-\lambda' k}))^c \neq \varnothing\} = \varnothing.$$

Moreover, if D is a compact disc in $\mathbb{D}$, then $\mathcal{D}^{-1}(D(y_k, e^{-\lambda' k})) \cap D$ is a finite union of topological discs whose diameters converge to zero when k goes to infinity, and the number of these discs is bounded by the degree of $\mathcal{D}_{|D}$. So the sequence of continuous paths c_k from ∂V_{Id} to $\bigcup F_\gamma$ cannot, for big enough k, be included in $\mathcal{D}^{-1}(D(y_k, e^{-\lambda' k}))$.

Let $N \in \mathbb{N}$ big enough and $k > N$. Write $D_k(\tilde{\omega}) = \mathcal{D}^{-1}(D(y_k(\tilde{\omega}), e^{-2\lambda'' k}))$. We are going to prove the following lemma.

LEMMA 6.5.

$$\tilde{\mathbb{P}}(E_k \mid E_{k-1}^c, \ldots, E_N^c) \geq \inf_{x \in D(0,r)} \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\}),$$

where r and β are given in Lemma 6.4.

Proof. From the proof of Proposition 4.4, we see that, by construction, y_k depends only on the set $X_{N_1}, \ldots, X_{N_k}$ (i.e. it depends on the set $\gamma_{N_1}, \ldots, \gamma_{N_k}$) and c_k depends only on $X_{N_k}^{-1} X_{N_{k+1}} = \gamma_{N_{k+1}}$. As the γ_{N_i} are mutually independent, we deduce that y_k and c_k are independent. Thus

$$\begin{aligned}
&\tilde{\mathbb{P}}(E_k \mid E_{k-1}^c, \ldots, E_N^c) \\
&\quad \geq \inf_{y \in \mathbb{CP}^1} \tilde{\mathbb{P}}(\{c_k \cap \mathcal{D}^{-1}(D(y, e^{-2\lambda'' k})) \neq \varnothing\} \mid c_{k-1} \cap D_{k-1} = \varnothing, \ldots, c_N \cap D_N = \varnothing\}) \\
&\quad = \inf_{y \in \mathbb{CP}^1} \tilde{\mathbb{P}}(\{c_k \cap \mathcal{D}^{-1}(D(y, e^{-2\lambda'' k})) \neq \varnothing\})
\end{aligned}$$

(this is because the event $\{c_{k-1} \cap D_{k-1} = \varnothing, \ldots, c_N \cap D_N = \varnothing\}$ and the event $\{c_k \cap \mathcal{D}^{-1}(D(y, e^{-2\lambda'' k})) \neq \varnothing\}$ are independent)

$$\geq \inf_{x \in D(0,r)} \tilde{\mathbb{P}}(\{c_k \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\})$$

(this last inequality comes from Lemma 6.4)

$$= \inf_{x \in D(0,r)} \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\})$$

(because the paths c_k are independent and identically distributed). □

So we still have to prove the following lemma.

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LEMMA 6.6. *There is a constant c such that, for big enough k,*

$$\inf_{x\in D(0,r)} \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\}) \geq \frac{c}{k}.$$

Proof. The proof of this fact is a little bit technical. So we start with the general idea. We will prove that the value of $\inf_{x\in D(0,r)} \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\})$ is almost the same as the probability that a Brownian path in $\mathbb{C}$ (with Euclidean metric) starting from $z = \frac{1}{2}$ would reach $D(0, e^{-k})$ before reaching $\partial D(0, 1)$. Using Brownian invariance by the exponential map, this probability is equal to the probability that a plane Brownian motion starting from $z = -\log 2$ would reach the line $x = -k$ before reaching the line $x = 0$. As the two canonical coordinates of a plane Brownian motion are one-dimensional Brownian motions, the previous probability is equal to $\mathbb{P}_{-\log 2}(T_{-k} \leq T_0)$ (the probability that a Brownian motion in $\mathbb{R}$ starting from $-\log 2$ would reach the point $-k$ before reaching the point zero). For all $x \in [-k; 0]$, the map $f(x) = \mathbb{P}_x(T_{-k} \leq T_0)$ is harmonic and satisfies $f(-k) = 1,\ f(0) = 0$. We deduce that $f(x) = -x/k$. Hence the desired probability is $f(-\log 2) = \log 2/k$.

Let us give a precise proof. Recall that $\mathbb{P}_y$ is the Wiener measure of the Brownian motion starting from y (Brownian motion associated to the Poincaré metric of the disc if y belongs to the Poincaré disc and associated to the Euclidean metric if y belongs to $\mathbb{C}$). Define $\mathbb{P}_m := \int \mathbb{P}_y\, dm(y)$, where m is the exit measure of $V_{\mathrm{Id}} = D(0, \delta')$ for a Brownian path starting from zero. For a closed set A, and a Brownian path ω, denote the reaching time of the set A by $T_A(\omega)$. Let $\epsilon > 0$ and $x \in D(0, r)$. Choose $\gamma \in \pi_1(\Sigma)$ such that $F_\gamma \cap D(x, \epsilon) = \varnothing$. Then

$$\begin{aligned}\inf_{x\in D(0,r)} &\tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\})\\ &\geq \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\} \cap \{\gamma_{N_1} = \gamma\}).\end{aligned}$$

As the event $\{\gamma_{N_1} = \gamma\} = \{X_{N_1} = \gamma\}$ contains the event $\{N_1 = 1\} \cap \{X_1 = \gamma\}$, we deduce that the previous probability is greater than

$$\begin{aligned}&\tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\} \cap \{X_1 = \gamma\} \cap \{N_1 = 1\})\\ &\quad\geq \tilde{\mathbb{P}}\bigg(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\} \cap \{X_1 = \gamma\} \cap \bigg\{\alpha_1 \leq \frac{1}{C^2}\bigg\}\bigg)\\ &\quad= \frac{1}{C^2} \cdot \mathbb{P}_m(\{T_{D(x,\beta e^{-2\lambda'' k})} \leq T_{\cup F_\alpha}\} \cap \{T_{F_\gamma} \leq T_{\cup F_\alpha}\}).\end{aligned}$$

If k is big enough so that $\beta e^{-2\lambda'' k} < \epsilon/2$, then, by the strong Markov property, the last quantity is

$$\begin{aligned}&\geq \frac{1}{C^2} \cdot \mathbb{P}_m(T_{D(x,\epsilon/2)} \leq T_{\cup F_\alpha}) \cdot \inf_{y\in\partial D(x,\epsilon/2)} \mathbb{P}_y(T_{D(x,\beta e^{-2\lambda'' k})} \leq T_{\partial D(x,\epsilon)})\\ &\quad\cdot \inf_{z\in\partial D(x,\epsilon)} \mathbb{P}_z(T_{F_\gamma} \leq T_{\cup F_\alpha}).\end{aligned}$$

As $x \in D(0, r)$, there exists $a > 0$ (which does not depend on x) such that

$$\mathbb{P}_m(T_{D(x,\epsilon/2)} \leq T_{\cup F_\alpha}) \cdot \inf_{z\in\partial D(x,\epsilon)} \mathbb{P}_z(T_{F_\gamma} \leq T_{\cup F_\alpha}) \geq a. \qquad \square$$

LEMMA 6.7. *There exists $b > 0$ (which does not depend on x) such that*

$$\text{for all } y \in \partial D\left(x, \frac{\epsilon}{2}\right), \quad \mathbb{P}_y(T_{D(x,\beta e^{-2\lambda'' k})} \leq T_{\partial D(x,\epsilon)}) \geq \frac{b}{k}.$$

Proof. For $p \in \mathbb{C}$, denote the disc with centre p and radius α in $\mathbb{C}$ for the Euclidean metric by $D_{\text{eucl}}(p, \alpha)$. Let $y \in \partial D(x, \epsilon/2)$. There are constants $c_1 > 0$, $0 < c_2 < 1$ such that, for k big enough, there is a biholomorphism Ψ_k which identifies:

- $D(x, \beta e^{-2\lambda'' k})$ and $D_{\text{eucl}}(0, c_1 e^{-2\lambda'' k}) := D_1(k)$;
- $D(x, \epsilon/2)$ and $D_{\text{eucl}}(0, c_2) := D_2$;
- $D(x, \epsilon)$ and $D_{\text{eucl}}(0, 1) := D_3$; and
- y and c_2.

By the conformal invariance of the Brownian motion

$$\mathbb{P}_y(T_{D(x,\beta e^{-2\lambda'' k})} \leq T_{\partial D(x,\epsilon)}) = \mathbb{P}_{c_2}(T_{D_1(k)} \leq T_{\partial D_3}).$$

The exponential map sends:

- the line $\Delta_1(k) := \{x = \log(c_1 e^{-2\lambda'' k})\}$ onto $\partial D_1(k)$;
- the line $\Delta_2 := \{x = \log(c_2)\}$ onto ∂D_2; and
- the line $\Delta_3 := \{x = 0\}$ onto ∂D_3.

So, by the conformal invariance of the Brownian motion, there is a constant b such that, for big enough k,

$$\mathbb{P}_{c_2}(T_{D_1(k)} \leq T_{\partial D_3}) = \mathbb{P}_{\log(c_2)}(T_{\Delta_1(k)} \leq T_{\Delta_3}) = \frac{-\log(c_2)}{2\lambda'' k - \log(c_1)} \geq \frac{b}{k}. \qquad \square$$

So we found a constant $c = ab/C^2$ such that, for big enough k, $\inf_{x \in D(0,r)} \tilde{\mathbb{P}}(\{c_0 \cap D(x, \beta e^{-2\lambda'' k}) \neq \varnothing\}) \geq c/k$. This ends the proof of Lemma 6.3. $\square$

This also completes the proof of the theorem in the case 'D is onto'.

Remark 6.8. In Theorem B, we made the assumption that Γ is non-elementary (this assumption was necessary to get the positivity of the Lyapounov exponent). Note that the conclusion holds if Γ is conjugate to a subgroup of $PSU(2, \mathbb{C})$. Indeed, in this case, there exists $k_1 > 0$ such that, for almost every $\tilde{\omega} \in \Omega$ and for all $n \in \mathbb{N}$, the path $\mathcal{D}(c_n(\tilde{\omega}))$ contains two points at a spherical distance greater than k_1. As a group conjugated to a subgroup of $PSU(2, \mathbb{C})$ quasi-preserves the spherical metric, and

$$\text{there exists } k_2 > 0 \text{ such that, for all } \gamma \in \Gamma$$
$$\text{and for all } z, z' \in \mathbb{CP}^1,\ d(\gamma \cdot z, \gamma \cdot z') > k_2 \cdot d(z'z').$$

So, almost surely, for all $n \in \mathbb{N}$, the path $\mathcal{D}(\omega_n) = Y_n \cdot \mathcal{D}(c_n)$ contains two points at a distance greater than $k_1 \cdot k_2$. So, almost surely, $\mathcal{D}(\omega(t))$ does not have limit when t goes to infinity.

6.2. *Proof in the case where $\mathcal{D}$ is not onto.* Let (x_0, z_0) be a couple of points in $\tilde{\Sigma} \times \mathbb{CP}^1$ such that $\mathcal{D}(x_0) = z_0$ and let h be the germ of $\mathcal{D}^{-1}$ satisfying $h(z_0) = x_0$. We are going to prove that, for almost every Brownian path ω starting from z_0, the germ h cannot be analytically continued along $\omega([0, \infty[)$.

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Let U be the open set in $\mathbb{CP}^1$ defined by $U := \mathcal{D}(\tilde{\Sigma})$. Its complementary U^c is a closed Γ-invariant set (infinite because Γ is not elementary). As Γ is non-elementary, we are in one of the following situations (see [**S**, Paragraph 1] for a proof):

(1) either Γ is dense in $PSL(2, \mathbb{C})$;
(2) Γ is discrete; or
(3) replacing Γ by a subgroup of index two, if necessary, Γ is conjugate to a dense subgroup of $PSL(2, \mathbb{R})$.

Case (1) is impossible because Γ leaves invariant the closed set $U^c \neq \mathbb{CP}^1$. In case (2), Γ is Kleinian. As the limit set $\Lambda(\Gamma)$ is the smallest closed Γ-invariant set of $\mathbb{CP}^1$, $\Lambda(\Gamma) \subset U^c$. As Γ is non-elementary, a theorem of Myrberg [**My**] (see also [**Do**]) asserts that the logarithmic capacity of $\Lambda(\Gamma)$ is strictly positive. Hence $\Lambda(\Gamma)$ (and so U^c) is visited by the Brownian motion in finite time, which implies that h cannot be analytically continued along a generic Brownian path. In case (3), U^c contains a Jordan curve. So U^c is also visited by the Brownian motion.

7. *Analytic continuation of holonomy germs of algebraic foliations*

7.1. *Riccati foliations and branched projective structures.* Let $(\Pi, M, X, \mathcal{F})$ be a Riccati foliation (see the definition in the introduction). Using the transversality of a generic fibre with $\mathcal{F}$, we can define a monodromy representation associated to such foliations. Define $\{x_1, \ldots, x_n\}$ as the points in X such that the fibre over x_i is an invariant line. Denote $\Sigma = X - \{x_1, \ldots, x_n\}$. Fix $x_0 \in X$. Let $\alpha : [0, 1] \to \Sigma$ be a closed curve in Σ based in x_0. Let $z \in \Pi^{-1}(x_0) := F_{x_0}$. There is a unique path $\tilde{\alpha} : [0, 1] \to M$ lifting α, belonging to the leaf through z and satisfying $\tilde{\alpha}(0) = z$. The map $z \mapsto \phi_\alpha(z) = \tilde{\alpha}(1)$ is a biholomorphism of F_{x_0} that only depends on the homotopy class of α. Then a local trivialization of the fibre bundle around x_0 gives an identification $F_{x_0} \cong \mathbb{CP}^1$ and we obtain a representation

$$\rho : \pi_1(\Sigma, x_0) \longrightarrow PSL(2, \mathbb{C})$$

called a *monodromy representation of the foliation*. Take any holomorphic section $s : X \to M$ not invariant by the foliation (recall that such a section always exists; see Remark 1.3). We can transport, by the foliation, the unique complex projective structure on F_{x_0} (or on any other non-invariant fibre). We obtain a branched complex projective structure on $S := s(\Sigma) \cong \Sigma$, whose monodromy representation is the monodromy representation of the foliation (the branched points are the points of S where the foliation is tangential to S). By definition, if $p \in S$ is not a branched point and if $h : (F_{x_0}, p_0) \to (S, p)$ is a holonomy germ of the foliation, then the analytic continuation of h^{-1} defines a developing map of the complex projective structure on S.

We have just explained how to pass from a Riccati foliation to a complex projective structure. Conversely, starting from a parabolic branched complex projective structure on a Riemann surface Σ of finite type, we can obtain a Riccati foliation after suspending the representation and compactifying with local models, as explained briefly in the introduction (see also [**DD**] or [**CDFG**]).

7.2. *Proof of Theorem A.* Item (1) is a direct application of Theorem B. Proof of item (2) proceeds as follows. Let $(s_i)_{i=0,1}$ be two sections of Π, $S_i = s_i(\Sigma)$ and $\overline{S_i} = s_i(X)$. Let g_1 be a complete metric on $\overline{S_1}$ in its conformal class. Let $h : (\overline{S_1}, p_1) \to (\overline{S_0}, p_0)$ be a holonomy germ. We want to prove that h can be analytically continued along a Brownian motion $(B_t)_{t\geq 0}$ (with respect to the metric g_1), starting at p_1. Firstly, using the strong Markov property, one can assume that $p_1 \in S_1$. Moreover, (B_t) does not visit the points $\{x_1, \ldots, x_n\}$. Secondly, h can be written as $h = \mathcal{D}_0^{-1} \circ \mathcal{D}_1$, where, for $i \in \{0, 1\}$, $\mathcal{D}_i$ is a developing map associated to the branched projective structure on S_i. By the conformal invariance of the Brownian motion, after time reparametrization, $\mathcal{D}_1 \circ B_t$ is a Brownian motion in $\mathbb{CP}^1$ along which $\mathcal{D}_2^{-1}$ can be analytically continued, by Item (1). This concludes the proof.

8. *Proof of Theorem C*

Let Σ be a hyperbolic Riemann surface of finite type. Let $\mathcal{D} : \tilde{\Sigma} = \mathbb{D} \to \mathbb{CP}^1$ and $\rho : \pi_1(\Sigma) \to PSL(2, \mathbb{C})$ be a couple developing map-monodromy representation associated with a branched complex projective structure on Σ. Assume that this structure is parabolic and recall that, with this hypothesis, ρ is necessarily non-elementary. We need to prove that, for almost every Brownian path ω starting at $0 \in \mathbb{D}$, there exists $z(\omega) \in \mathbb{CP}^1$ such that

$$\frac{1}{t} \cdot \int_0^t \delta_{\mathcal{D}(\omega(s))} \cdot ds \underset{t\to\infty}{\longrightarrow} \delta_{z(\omega)}.$$

As in the proof of the previous theorem, we are going to use the discretization procedure of Furstenberg, Lyons and Sullivan. Nevertheless, the notation is slightly modified. If $\tilde{\omega} = (\omega, \alpha) \in \tilde{\Omega}$, then the infinite path ω can be written as an infinite concatenation of paths

$$\omega = \beta_0 * \omega_0 * \omega_1 * \cdots ,$$

where $\beta_0 = \omega_{|[0,S_{N_0}]}$ and, for $k \geq 0$, $\omega_k = \omega_{|[S_{N_k}, S_{N_{k+1}}]}$. For $k \geq 0$, we define $c_k := X_{N_k}^{-1} \cdot \omega_k$. Then

$$\omega = \beta_0 * X_{N_0} c_0 * X_{N_1} c_1 * \cdots .$$

Using ρ-equivariance of $\mathcal{D}$,

$$\mathcal{D}(\omega) = \mathcal{D}(\beta_0) * \rho(X_{N_0})\mathcal{D}(c_0) * \rho(X_{N_1})\mathcal{D}(c_1) * \cdots .$$

The sequence of random variables X_{N_k} is a realization of a right random walk in $\pi_1(\Sigma)$ with law μ and the sequence $Y_{N_k} = \rho(X_{N_k})$ is a realization of a right random walk in $\rho(\pi_1(\Sigma))$ with law $\tilde{\mu} = \rho_* \mu$. Let $y_k(\tilde{\omega})$ and $z_k(\tilde{\omega})$ be the two sequences of random points in $\mathbb{CP}^1$ defined in Proposition 4.4. According to Remark 4.5, almost surely $z_k(\tilde{\omega}) \to z(\tilde{\omega})$.

In order to prove the theorem, it is enough to prove that, for $\tilde{\mathbb{P}}$-almost every $\tilde{\omega} = (\omega, \alpha) \in \tilde{\Omega}$ and for all $\epsilon > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \cdot \mathrm{leb}\{u \in [0, t] \text{ such that } \mathcal{D}(\omega(u)) \in D(z(\tilde{\omega}), \epsilon)\} = 1. \tag{4}$$

For $k \geq 0$, define

$$T_k(\tilde{\omega}) = \mathrm{leb}\{t \in [S_{N_k}, S_{N_{k+1}}] \text{ such that } \mathcal{D}(c_k(\tilde{\omega})(t)) \in D(y_k(\tilde{\omega}), e^{-\lambda' k})\}.$$

We get the following proposition.

PROPOSITION 8.1. *Almost surely* $\lim_{k\to\infty} T_k = 0$.

Before proving this proposition, let us show why this implies the theorem. First, if we assume that almost surely $\lim_{k\to\infty} T_k = 0$, then, almost surely,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} T_k = 0. \tag{5}$$

Now the fact that equality (5) implies (4) is a direct consequence of the following three facts.

(1) According to Proposition 5.5, there is a constant T such that, almost surely,

$$\lim_{n\to\infty} \frac{S_{N_n}}{n} = T.$$

(2) According to Remark 4.5, almost surely,

$$\lim_{k\to\infty} z_k = z.$$

(3) According to Proposition 4.4, almost surely,

$$\rho(X_{N_k})((D(y_k, e^{-\lambda' k}))^c) \subset D(z_k, e^{-\lambda' k}).$$

8.1. *Beginning of the proof of Proposition 8.1.* Using Borel–Cantelli, it is enough to prove that, for all $\epsilon > 0$, $\tilde{\mathbb{P}}(T_k \geq \epsilon) \leq 2/k^2$. Let K be a positive constant (to be determined later) and define

$$A_k = \{\tilde{\omega} \text{ s.t. } c_0(\tilde{\omega}) \cap D(0, K \log(k))^c \neq \varnothing\}.$$

We are going to prove the following lemma.

LEMMA 8.2.

$$\tilde{\mathbb{P}}(T_k \geq \epsilon) \leq \tilde{\mathbb{P}}(A_k) + \sup_{y\in\mathbb{CP}^1} \mathbb{P}_0(\tau_{y,k} \geq \epsilon),$$

where $\tau_{y,k} = \mathrm{leb}\{t \in [0, T_{\partial D(0, K\log k)}] \text{ s.t. } \mathcal{D}(\omega(t)) \in D(y, e^{-\lambda' k})\}$.

Proof. Let us define

$$U_{k,y} = \mathrm{leb}\{t \in [S_{N_k}, S_{N_{k+1}}] \text{ s.t. } \mathcal{D}(c_k(\tilde{\omega})(t)) \in D(y, e^{-\lambda' k})\},$$
$$V_{k,y} = \mathrm{leb}\{t \in [S_{N_0}, S_{N_1}] \text{ s.t. } \mathcal{D}(c_0(\tilde{\omega})(t)) \in D(y, e^{-\lambda' k})\}.$$

As explained in the proof of Theorem 6.1, c_k and y_k are independent. So

$$\begin{aligned}
\tilde{\mathbb{P}}(T_k \geq \epsilon) &\leq \sup_{y\in\mathbb{CP}^1} \tilde{\mathbb{P}}(U_{k,y} \geq \epsilon) \\
&= \sup_{y\in\mathbb{CP}^1} \tilde{\mathbb{P}}(V_{k,y} \geq \epsilon) \\
&= \sup_{y\in\mathbb{CP}^1} (\tilde{\mathbb{P}}(\{V_{k,y} \geq \epsilon\} \cap A_k) + \tilde{\mathbb{P}}(\{V_{k,y} \geq \epsilon\} \cap A_k^c)) \\
&\leq \tilde{\mathbb{P}}(A_k) + \sup_{y\in\mathbb{CP}^1} \mathbb{P}_0(\tau_{y,k} \geq \epsilon).
\end{aligned}$$

The last inequality is due to the fact that, for all $y \in \mathbb{CP}^1$, $\{V_{k,y} \geq \epsilon\} \cap A_k^c \subset \{\tau_{y,k} \geq \epsilon\} \times [0; 1]^{\mathbb{N}}$, which implies that $\tilde{\mathbb{P}}(\{V_{k,y} \geq \epsilon\} \cap A_k^c) \leq \mathbb{P}_0(\tau_{y,k} \geq \epsilon)$. □

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Now we are going to bound the two terms of the right side of the inequality in Lemma 8.2 by $1/k^2$. For the term $\tilde{\mathbb{P}}(A_k)$, we have the following proposition.

PROPOSITION 8.3. *There exists K such that for k big enough, $\tilde{\mathbb{P}}(A_k) \leq 1/k^2$.*

Proof. In [**DD2**, Proposition 2.15], the authors prove that there is $\alpha > 0$ such that

$$\mathbb{E}[e^{\alpha S_{N_1}}] = M < \infty.$$

Using Markov inequality, one deduces that

$$\tilde{\mathbb{P}}[S_{N_1} \geq t] = \tilde{\mathbb{P}}[e^{\alpha S_{N_1}} \geq e^{\alpha t}] \leq e^{-\alpha t}\mathbb{E}[e^{\alpha S_{N_1}}] = Me^{-\alpha t}.$$

If ω is a Brownian path, define

$$\xi_t(\omega) = \sup_{0 \leq u \leq t} d(\omega(0), \omega(t)).$$

Let $C_1 > 0$ satisfying $\alpha C_1 > 2$. Then

$$\begin{aligned}\tilde{\mathbb{P}}(A_k) &\leq \tilde{\mathbb{P}}(A_k \cap \{S_{N_1} \geq C_1 \log(k)\}) + \tilde{\mathbb{P}}(A_k \cap \{S_{N_1} \leq C_1 \log(k)\}) \\ &\leq \tilde{\mathbb{P}}(S_{N_1} \geq C_1 \log(k)) + \mathbb{P}_0(\xi_{C_1 \log(k)} \geq K \log(k)).\end{aligned}$$

The first term of the right hand side satisfies

$$\tilde{\mathbb{P}}(S_{N_1} \geq C_1 \log(k)) \leq Me^{-\alpha C_1 \log k} \leq \frac{1}{2k^2},$$

for big enough k. In order to bound the second term, we will use the following estimate (see [**P**, Paragraph 6] for a proof). There is $c > 0$ such that, for all $y \in \mathbb{D}$ and for all $r \geq 2$, $\mathbb{P}_y(\xi_1 \geq r) \leq e^{-cr^2}$. Hence

$$\begin{aligned}\mathbb{E}_y[e^{\xi_1}] &= 1 + \int_{u>0} e^u \mathbb{P}_y(\xi_1 \geq u)\, du \\ &\leq 1 + \int_{u>0} e^{u-cu^2}\, du.\end{aligned}$$

The last integral converges. Let a_4 be the constant satisfying $e^{a_4} = 1 + \int_{u>0} e^{u-cu^2}\, du$. Denote the integral part of t by $\lfloor t \rfloor$. Using successively the Markov inequality and the strong Markov property of Brownian motion, gives

$$\begin{aligned}\mathbb{P}_0(\xi_t \geq r) &\leq e^{-r}\mathbb{E}[e^{\xi_t}] \\ &\leq e^{-r}\mathbb{E}\bigg[\exp\bigg(\sum_{k=0}^{\lfloor t \rfloor - 1} \sup_{k \leq s \leq k+1} d(\omega(k), \omega(s))\bigg)\bigg] \\ &\leq e^{-r} \cdot \Big(\sup_{y \in \mathbb{D}} \mathbb{E}_y[e^{\xi_1}]\Big)^t.\end{aligned}$$

For $t = C_1 \log(k)$ and $r = K \log(k)$, one gets

$$\mathbb{P}_0(\xi_{C_1 \log(k)} \geq K \log(k)) \leq k^{-K} \cdot k^{a_4 C_1}.$$

Consequently

$$\tilde{\mathbb{P}}(A_k) \leq \frac{1}{2k^2} + k^{-K + a_4 C_1}.$$

Choose K big enough so that $-K + a_4 C_1 < -2$. We get that, for k big enough, $\tilde{\mathbb{P}}(A_k) \leq 1/k^2$. □

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Up to now, we have fixed K to satisfy the previous proposition. In order to bound the second term $\sup_{y\in\mathbb{CP}^1}\mathbb{P}_0(\tau_{y,k}\geq\epsilon)$ of the inequality in Lemma 8.2, we will need the following proposition.

PROPOSITION 8.4. *There exist two positive constants α and β such that, for all $y\in\mathbb{CP}^1$ and for k big enough, the intersection of $\mathcal{D}^{-1}(D(y, e^{\lambda' k}))$ with $D(0, K\log k)$ is included in a union of at most k^α discs with radius $e^{-\beta k}$.*

Proof. Let us fix $y\in\mathbb{CP}^1$. Let F be the Dirichlet fundamental domain associated to the base point $0\in\mathbb{D}$ given by

$$F=\{x\in\mathbb{D}\text{ s.t. }\forall\gamma\in\pi_1(\Sigma),\ d(0,x)\leq d(\gamma\cdot 0,x)\}.$$

Let $D=F\cap D(0, K\log k)$. First, note that

$$D(0, K\log k)\subset\bigcup_{d(0,\gamma\cdot 0)\leq 2K\log k}\gamma\cdot D. \tag{6}$$

To see this, take $z\in D(0, K\log k)$. There exists $\gamma\in\pi_1(\Sigma)$ such that $z\in\gamma\cdot F$. We have $d(0,\gamma^{-1}\cdot z)=d(\gamma\cdot 0,z)\leq d(0,z)\leq K\log k$. So $z\in\gamma\cdot D$. Moreover, $d(0,\gamma\cdot 0)\leq d(0,z)+d(z,\gamma\cdot 0)\leq 2\cdot d(0,z)\leq 2K\log k$.

Second, the ρ-equivariance of $\mathcal{D}$ gives, for every $\gamma\in\pi_1(\Sigma)$,

$$\mathcal{D}^{-1}(D(y, e^{-\lambda' k}))\cap\gamma D=\gamma\cdot(\mathcal{D}^{-1}(\rho(\gamma^{-1})D(y, e^{-\lambda' k}))\cap D).$$

A direct calculation gives

$$\|\rho(\gamma)\|^2=\sup_{z\in\mathbb{CP}^1}|\rho(\gamma^{-1})'(z)|. \tag{7}$$

Indeed, if $\rho(\gamma^{-1})=\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, then

$$\begin{aligned}
\|\rho(\gamma)\|^2 &= \sup_{V\in\mathbb{C}^2-\{0;0\}}\frac{\|\rho(\gamma)\cdot V\|^2}{\|V\|^2}\\
&= \sup_{W\in\mathbb{C}^2-\{0;0\}}\frac{\|W\|^2}{\|\rho(\gamma^{-1})\cdot W\|^2}\\
&= \sup_{(z,w)\in\mathbb{C}^2-\{0;0\}}\frac{|z|^2+|w|^2}{|az+bw|^2+|cz+dw|^2}\\
&= \sup_{z\in\mathbb{C}}\frac{|z|^2+1}{|az+b|^2+|cz+d|^2}\\
&= \sup_{z\in\mathbb{C}}\frac{1}{|cz+d|^2}\cdot\frac{1+|z|^2}{1+|\rho(\gamma^{-1})(z)|^2}\\
&= \sup_{z\in\mathbb{CP}^1}|\rho(\gamma^{-1})'(z)|.
\end{aligned}$$

Moreover, as the monodromy representation is parabolic, we have already seen, at the beginning of the proof of Theorem B, that there exists a constant a such that

$$\log\|\rho(\gamma)\|\leq a\cdot d(0,\gamma\cdot 0). \tag{8}$$

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From equations (7) and (8), we deduce that if $\gamma \in \pi_1(\Sigma)$ is such that $d(0, \gamma \cdot 0) \leq 2K \log k$, then

$$\sup_{z \in \mathbb{CP}^1} |\rho(\gamma^{-1})'(z)| \leq k^{4aK},$$

which implies that

$$\rho(\gamma^{-1})D(y, e^{-\lambda' k}) \subset D(\rho(\gamma^{-1})y, k^{4aK} \cdot e^{-\lambda' k}).$$

If α_1 is a constant such that $0 < \alpha_1 < \lambda'$, then, for big enough k, $k^{4aK} \cdot e^{-\lambda' k} \leq e^{-\alpha_1 k}$. This implies that, for big enough k and γ satisfying $d(0, \gamma \cdot 0) \leq 2K \log k$,

$$\mathcal{D}^{-1}(D(y, e^{-\lambda' k})) \cap \gamma D \subset \gamma \cdot (\mathcal{D}^{-1}(D(\tilde{y}, e^{-\alpha_1 k})) \cap D) \tag{9}$$

with $\tilde{y} = \rho(\gamma^{-1})y$. To conclude, we will need the following lemma.

LEMMA 8.5. *There exist constants $N \in \mathbb{N}$ and $\beta > 0$ such that, for big enough k and for every $\tilde{y} \in \mathbb{CP}^1$, the set $\mathcal{D}^{-1}(D(\tilde{y}, e^{-\alpha_1 k})) \cap D$ is included in an union of at most N discs with radius at most $e^{-\beta k}$.*

Before proving the lemma, let us finish the proof of Proposition 8.4. Using (9) and the previous lemma, we get that, for all γ satisfying $d(0, \gamma \cdot 0) \leq 2K \log k$, the set $\mathcal{D}^{-1}(D(y, e^{-\lambda' k})) \cap \gamma D$ is included in an union of at most N discs with radius at most $e^{-\beta k}$. Now, noting that there is $\alpha > 0$ such that $\mathrm{Card}\{\gamma \in \pi_1(\Sigma) \text{ s.t. } d(0, \gamma \cdot 0) \leq 2K \log k\} \leq k^{\alpha}$ and using equation (6), we get the desired result. □

Proof of Lemma 8.5. Recall that as the projective structure is parabolic, for any puncture p in Σ, there is a neighbourhood V of p that satisfies the following. If $\mathcal{H}$ is the connected component of $\mathrm{proj}^{-1}(V)$ which meets the fundamental domain F, then there is a bi-Lipschitz biholomorphism $\tilde{\phi} : \mathbb{H}_{\geq 1} \to \mathcal{H}$ such that some developing map satisfies $\mathcal{D} \circ \tilde{\phi}(\tau) = \tau$. Denote the set of all such components for each puncture by $\mathcal{H}_1, \ldots, \mathcal{H}_r$. Recall that $D = F \cap D(0, K \log k)$ and define $F_0 = D \cap (\bigcup_j \mathcal{H}_j)^c$. We are going to analyse the intersection of $\mathcal{D}^{-1}(D(\tilde{y}, e^{-\lambda' n}))$ with the compact part F_0 and with $D - F_0$ separately.

The compact part F_0. Let us start with a heuristic argument. $\mathcal{D}'$ has a finite number of zeros a_i in F_0. Let V_i be a small neighbourhood of a_i. $|\mathcal{D}'|$ is bounded away from zero on $F_0 - \bigcup V_i$. So, if $\tilde{y} \in \mathbb{CP}^1$ and α is small, $\mathcal{D}^{-1}(D(\tilde{y}, \alpha))(\bigcap F_0 - \bigcup V_i)$ is a finite union of discs with radius of the order of α. For each i, in local coordinates (for V_i and $\mathcal{D}(V_i)$) the map $\mathcal{D}$ writes: $\mathcal{D}(z) = z^{n_i}$. This implies that $\mathcal{D}^{-1}(D(\tilde{y}, \alpha)) \cap V_i$ is the union of at most n_i discs with radius at most α^{1/n_i}.

Now we will give a rigorous proof. Denote the ϵ-neighbourhood of F_0 by $N_\epsilon(F_0) = \{\tau \in \mathbb{D} \text{ s.t. } d(\tau, F_0) \leq \epsilon\}$. As $\mathcal{D}$ is a non-constant holomorphic map, there is a constant N such that any $\tilde{y}$ has at most N preimages by $\mathcal{D}$ in $N_\epsilon(F_0)$. Moreover, in [**AH**, Lemma 5.1], we proved that there exists $C_0 > 0$ such that, for any $\tilde{y} \in \mathcal{D}(N_\epsilon(F_0))$ and any $z \in F_0$,

$$d(\mathcal{D}(z), \tilde{y}) \geq C_0 \prod_{\mathcal{D}(w) = \tilde{y}, w \in N_\epsilon(F_0)} d(z, w).$$

Let $\tilde{y} \in \mathbb{CP}^1$. If $\tilde{y} \notin \mathcal{D}(N_\epsilon(F_0))$, then, for big enough k, $\mathcal{D}^{-1}(D(\tilde{y}, e^{-\lambda' k})) \cap F_0 = \varnothing$. Otherwise, $\tilde{y} \in \mathcal{D}(N_\epsilon(F_0))$. Then if $N(\tilde{y})$ denotes the number of preimages of $\tilde{y}$ in $N_\epsilon(F_0)$, and if one takes $z \in \mathcal{D}^{-1}(D(\tilde{y}, e^{-\lambda' k})) \cap F_0$, one gets

$$e^{-\lambda' k} \geq d(\mathcal{D}(z), \tilde{y}) \tag{10}$$

$$\geq C_0 \prod_{\mathcal{D}(w)=\tilde{y}, w \in F_0+\epsilon} d(z, w) \tag{11}$$

$$\geq C_0 \Big(\inf_{\mathcal{D}(w)=\tilde{y}, w \in N_\epsilon(F_0)} d(z, w) \Big)^{N(\tilde{y})}. \tag{12}$$

This implies that

$$\inf_{\mathcal{D}(w)=\tilde{y}, w \in N_\epsilon(F_0)} d(z, w) \leq \left(\frac{e^{-\lambda' k}}{C_0} \right)^{1/N}.$$

As there exists β such that, for big enough k, $(e^{-\lambda' k}/C_0)^{1/N} \leq e^{-\beta k}$, we get that $z \in D(w, e^{-\beta k})$ for w a preimage of $\tilde{y}$ by $\mathcal{D}$ in $F_0 + \epsilon$.

The non-compact part. We are going to analyse the intersection of the set $\mathcal{D}^{-1}(D(\tilde{y}, e^{-\lambda' k}))$ with each portion of horodisc $D \cap \mathcal{H}_j$. Recall that, for each j, there is a bi-Lipschitz biholomorphism $\tilde{\phi}_j : \{\mathrm{Im}(z) \geq 1\} \to \mathcal{H}_j$ such that some developing map satisfies $\mathcal{D} \circ \tilde{\phi}_j(z) = z$ (see Remark 2.3). As $\tilde{\phi}_j$ is bi-Lipschitz (see Lemma 2.4), it preserves the lengths modulo multiplications by constants. Hence, we can assume that $\mathcal{H}_j = \{\mathrm{Im}(z) \geq 1\}$, the developing map is the inclusion $\iota : \{\mathrm{Im}(z) \geq 1\} \to \mathbb{CP}^1$ and $D \cap \mathcal{H}_j = D_{\mathrm{hyp}}(i, K \log k) \cap [-\frac{1}{2}, \frac{1}{2}] \times [1, +\infty[$. To evaluate the size of the preimage of the intersection of a disc with spherical radius $e^{-\alpha_1 k}$ with $D \cap \mathcal{H}_j$, we just have to compare the spherical metric ds_{sph} and the hyperbolic one ds_{hyp} inside $D_{\mathrm{hyp}}(i, K \log k) \cap [-\frac{1}{2}, \frac{1}{2}] \times [1, +\infty[$. Thus

$$ds_{\mathrm{hyp}} = \frac{1 + x^2 + y^2}{y} \cdot ds_{\mathrm{sph}}.$$

Furthermore, there is $\alpha > 0$ such that $D_{\mathrm{hyp}}(i, K \log k) \cap [-\frac{1}{2}, \frac{1}{2}] \times [1, +\infty[\subset [-\frac{1}{2}, \frac{1}{2}] \times [1; k^\alpha]$, so that $ds_{\mathrm{hyp}} \leq (\frac{5}{4} + k^{2\alpha}) \cdot ds_{\mathrm{sph}}$. This implies that a disc with spherical radius $e^{-\alpha_1 k}$ is included in a disc with hyperbolic radius $e^{-\alpha_1 k} \cdot (\frac{5}{4} + k^{2\alpha})$. There is β such that, for k big enough, $e^{-\alpha_1 k} \cdot (\frac{5}{4} + k^{2\alpha}) \leq e^{-\beta k}$. This ends the proof of Lemma 8.5. □

8.2. *End of the proof of Proposition 8.1.* Using the previous proposition, we are going to give a bound for the second term of the right side of the inequality in Lemma 8.2: namely, we are going to prove that, for all $\epsilon > 0$ and for k big enough,

$$\sup_{y \in \mathbb{CP}^1} \mathbb{P}_0(\tau_{y,k} \geq \epsilon) \leq \frac{1}{k^2}. \tag{13}$$

Then a combination of Lemma 8.2, inequality (13) and Proposition 8.3 implies that $\tilde{\mathbb{P}}(T_k \geq \epsilon) \leq 2/k^2$, which ends the proof of Proposition 8.1 and of the theorem.

The proof of equation (13) proceeds as follows. Fix $y \in \mathbb{P}_1$ and $\epsilon > 0$. We recall that

$$\tau_{y,k} = \mathrm{leb}\{t \in [0, T_{\partial D(0, K \log k)}] \text{ s.t. } \mathcal{D}(\omega(t)) \in D(y, e^{-\lambda' k})\}$$

and that, according to Proposition 8.4,

$$\mathcal{D}^{-1}(D(y, e^{\lambda' k})) \bigcap D(0, K \log k) \subset \bigcup_{i \in I} D_i,$$

where D_i are hyperbolic discs with radius $e^{-\beta k}$ and $\mathrm{Card}(I) \leq k^\alpha$. So

$$\begin{aligned}\mathbb{P}_0(\tau_{y,k} \geq \epsilon) &\leq \frac{1}{\epsilon} \cdot \mathbb{E}_0[\tau_{y,k}] \\ &\leq \frac{1}{\epsilon} \cdot \sum_{i \in I} \mathbb{E}_0[\tau_{D_i}],\end{aligned}$$

where $\tau_{D_i} = \mathrm{leb}\{t \in [0, \infty[\text{ s.t. } \omega(t) \in D_i\}$. Let D_i be such an hyperbolic disc with hyperbolic radius $e^{-\beta k}$. Then

$$\mathbb{E}_0[\tau_{D_i}] = \int_{D_i} G_{\mathbb{D}}(0, z)\, d\,\mathrm{hyp}(z),$$

where $G_{\mathbb{D}}(0, z) = -(1/\pi) \log |z|$ is the Green's function. In order to give an upper bound of this integral, we distinguish two cases.

- Either $D_i \subset D_{\mathrm{eucl}}(0, \frac{1}{4})^c$ and $-\log|z| \leq \log 4$ for every $z \in D_i$. Hence $\int_{D_i} G_{\mathbb{D}}(0, z)\, d\,\mathrm{hyp}(z) \leq \mathrm{cst} \cdot \mathrm{vol}_{\mathrm{hyp}}(D_i) \underset{k \to 0}{\sim} \mathrm{cst} \cdot e^{-2\beta k}$.
- Or $D \subset D_{\mathrm{eucl}}(0, \frac{1}{2})$ and, for every $z \in D_i$, $d\,\mathrm{hyp}(z) = |dz|^2/(1 - |z|^2)^2 \leq \frac{16}{9} \cdot |dz|^2$. In polar coordinates, $z = re^{i\theta}$ and $|dz|^2 = r \cdot dr \cdot d\theta$. Hence

  $$\int_{D_i} G_{\mathbb{D}}(0, z)\, d\,\mathrm{hyp}(z) \leq \mathrm{cst} \cdot \int_{D_i} -\log(r) r\, dr\, d\theta.$$

  As $-r \log r \leq e^{-1}$ on $[0, 1]$, we get that

  $$\begin{aligned}\int_{D_i} G_{\mathbb{D}}(0, z)\, d\,\mathrm{hyp}(z) &\leq \mathrm{cst} \cdot \int_{D_i} dr\, d\theta \\ &\leq \mathrm{cst} \cdot e^{-\beta k}.\end{aligned}$$

  The last line is due to the fact that the hyperbolic disc D_i with radius $e^{-\beta k}$ is also a Euclidean disc with radius less than $e^{-\beta k}$.

So, for k big enough, any hyperbolic disc D_i with radius $e^{-\beta k}$, satisfies $\mathbb{E}_0[\tau_{D_i}] \leq \mathrm{cst} \cdot e^{-\beta k}$. So

$$\frac{1}{\epsilon} \cdot \sum_{i \in I} \mathbb{E}_0[\tau_{D_i}] \leq \mathrm{cst} \cdot \mathrm{Card}(I) \cdot\ e^{-\beta k} \leq \mathrm{cst} \cdot k^\alpha \cdot e^{-\beta k} \leq \frac{1}{k^2}$$

for big enough k.

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