



PAPER

# Extended symmetry analysis of remarkable (1+2)-dimensional Fokker–Planck equation

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## Abstract

We carry out the extended symmetry analysis of an ultraparabolic Fokker–Planck equation with three independent variables, which is also called the Kolmogorov equation and is singled out within the class of such Fokker–Planck equations by its remarkable symmetry properties. In particular, its essential Lie invariance algebra is eight-dimensional, which is the maximum dimension within the above class. We compute the complete point symmetry pseudogroup of the Fokker–Planck equation using the direct method, analyse its structure and single out its essential subgroup. After listing inequivalent one- and two-dimensional subalgebras of the essential and maximal Lie invariance algebras of this equation, we exhaustively classify its Lie reductions, carry out its peculiar generalised reductions and relate the latter reductions to generating solutions with iterative action of Lie-symmetry operators. As a result, we construct wide families of exact solutions of the Fokker–Planck equation, in particular, those parameterised by an arbitrary finite number of arbitrary solutions of the (1+1)-dimensional linear heat equation. We also establish the point similarity of the Fokker–Planck equation to the (1+2)-dimensional Kramers equations whose essential Lie invariance algebras are eight-dimensional, which allows us to find wide families of exact solutions of these Kramers equations in an easy way.

## 1. Introduction

The Fokker–Planck and Kolmogorov equations provide powerful tools for adequately modelling a wide range of natural processes, which involve considering fluctuations of a quantity under the action of a random perturbation. Since the presence of a random noise is a common characteristic of many physical fields, the Fokker–Planck equations have acquired high popularity in applied sciences. However, theoretical studies of these equations have aroused lively interest as well, in particular, in the field of group analysis of differential equations.

For the Fokker–Planck and Kolmogorov equations, their Lie symmetries and other related objects are of course the most studied in dimension  $1 + 1$ . This study was initiated in the seminal paper [21] of Sophus Lie himself in the course of the group analysis of the wider class of second-order linear partial differential equations with two independent variables, including  $(1 + 1)$ -dimensional second-order linear evolution equations. A number of papers that restate, specify or develop the above Lie's result have been published over the past decades, see, for example, [34] for a review of these papers and a modern treatment of the problem. The equivalence of  $(1 + 1)$ -dimensional second-order linear evolution equations with respect to point transformations was considered in [7, 13, 21] and further in

[16, 18, 23]. Darboux transformations between such equations were studied, for example, in [8, 11, 22, 34]. The group classification problems for the classes of (1 + 1)-dimensional Fokker–Planck and Kolmogorov equations and their subclasses were solved in [34] up to the general point equivalence and in [27] with respect to the corresponding equivalence groups using the mapping method of group classification.

The most general form of (1 + 2)-dimensional ultraparabolic Fokker–Planck equations is

$$u_t + B(t, x, y)u_y = A^2(t, x, y)u_{xx} + A^1(t, x, y)u_x + A^0(t, x, y)u + C(t, x, y) \tag{1}$$

with  $A^2 \neq 0, B_x \neq 0$ .

We denote the entire class of these equations by  $\bar{\mathcal{F}}$ . Thus, the tuple  $\bar{\theta} := (B, A^2, A^1, A^0, C)$  of arbitrary elements of the class  $\bar{\mathcal{F}}$  runs through the solution set of the system of the inequalities  $A^2 \neq 0$  and  $B_x \neq 0$  with no restrictions on  $A^0, A^1$  and  $C$ . A partial preliminary group classification of the class  $\bar{\mathcal{F}}$  was carried out in [14]. Some subclasses of the class  $\bar{\mathcal{F}}$  were considered within the Lie-symmetry framework in [17, 20, 35–37, 41]. Despite the number of papers on this subject, there are still many open problems in the symmetry analysis of the entire class  $\bar{\mathcal{F}}$ , its subclasses and even particular equations from this class.

In the present paper, we carry out extended symmetry analysis of the equation

$$u_t + xu_y = u_{xx}, \tag{2}$$

which is of the simplest form within the class  $\bar{\mathcal{F}}$  and corresponds to the values  $B = x, A^2 = 1$  and  $A^1 = A^0 = C = 0$  of the arbitrary elements. This equation is distinguished within the class  $\bar{\mathcal{F}}$  by its remarkable symmetry properties. In particular, its essential Lie invariance algebra  $\mathfrak{g}^{\text{ess}}$  is eight-dimensional, which is the maximum dimension for equations from the class  $\bar{\mathcal{F}}$ . Moreover, it is, up to the point equivalence, a unique equation in  $\bar{\mathcal{F}}$  whose essential Lie invariance algebra is of this dimension. That is why we refer to (2) as the remarkable Fokker–Planck equation. The study of the equation (2) was initiated by Kolmogorov in 1934 [19], and hence, it is often called the Kolmogorov equation as well. In particular, he constructed its fundamental solution,<sup>1</sup>

$$F(t, x, y, t', x', y') = \frac{\sqrt{3}H(t-t')}{2\pi(t-t')^2} \exp\left(-\frac{(x-x')^2}{4(t-t')} - \frac{3(y-y' - \frac{1}{2}(x+x')(t-t'))^2}{(t-t')^3}\right), \tag{3}$$

where  $H$  denotes the Heaviside step function. A preliminary study of symmetry properties of (2) was carried out in [17, 20].

The algebra  $\mathfrak{g}^{\text{ess}}$  is nonsolvable, and its structure is complicated and specific, which makes the classification of subalgebras of  $\mathfrak{g}^{\text{ess}}$  nontrivial. More specifically, this algebra is isomorphic to a semidirect sum  $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{h}(2, \mathbb{R})$  of the real order-two special linear Lie algebra  $\mathfrak{sl}(2, \mathbb{R})$  and the real rank-two Heisenberg algebra  $\mathfrak{h}(2, \mathbb{R})$ , where the action of the former algebra on the latter is given by the direct sum of the one- and four-dimensional irreducible representations of  $\mathfrak{sl}(2, \mathbb{R})$ . Such structure has not been studied before in the literature on symmetry analysis of differential equations with regard to classifying subalgebras. This has been an obstacle to the complete classification of Lie reductions of the equation (2), which we successfully overcome in the present paper.

Further, using the direct method, we construct the complete point symmetry pseudogroup  $G$  of this equation and derive a nice representation for transformations from  $G$ . This essentially simplifies the subsequent classification of the one- and two-dimensional subalgebras of the maximal Lie invariance algebra  $\mathfrak{g}$  of the equation (2) up to the  $G$ -equivalence, which is required for optimally accomplishing Lie reductions of (2). We carefully analyse the structure of the pseudogroup  $G$  and modify the group operation in  $G$  by extending the domains of transformation compositions by continuity. The main advantage of the suggested interpretation of  $G$  is that then the pseudogroup  $G$  contains a subgroup  $G^{\text{ess}}$ , which is a (finite-dimensional) Lie group with  $\mathfrak{g}^{\text{ess}}$  as its Lie algebra. We call the subgroup  $G^{\text{ess}}$  the essential point symmetry group of the equation (2). Moreover, we surprisingly find out that the pseudogroup  $G$  contains only one independent, up to combining with elements from the identity component of  $G$ , discrete

<sup>1</sup>There was a misprint in the constant factor of this solution in [19], which was corrected later.

element. As such an element, one can choose the involution that only alternates the sign of  $u$ . One more implication of the construction of the pseudogroup  $G$  is that Kolmogorov's fundamental solution of the equation (2) is formally  $G^{\text{ess}}$ -equivalent to the function  $u(t, x, y) = 1 - H(t)$ .

The exhaustive classification of Lie reductions of the equation (2) on the basis of listing  $G^{\text{ess}}$ -inequivalent one- and two-dimensional subalgebras of its essential Lie invariance algebra  $\mathfrak{g}^{\text{ess}}$  leads to finding wide families of its exact solutions, which are in general  $G^{\text{ess}}$ -inequivalent to each other. The most interesting among these families are three families parameterised by single arbitrary solutions of the classical (1 + 1)-dimensional linear heat equation and one family parameterised by an arbitrary solution of the (1 + 1)-dimensional linear heat equation with a particular inverse square potential. One more family is expressed in terms of the general solutions of a one-parameter family of Kummer's equations and thus splits into two one-parameter families. We show how to construct more general families of solutions of the equation (2) using generalised reductions with respect to generalised symmetries generated from elements of  $\mathfrak{g}^{\text{ess}}$  via acting by recursion operators that are counterparts of elements of  $\mathfrak{g}^{\text{ess}}$  among first-order differential operators in total derivatives. Since the (1 + 1)-dimensional linear heat equation with a particular inverse square potential arises in the course of a Lie reduction of the equation (2), we exhaustively carry out the classical symmetry analysis of the (1 + 1)-dimensional linear heat equation with a general inverse square potential, including the construction of its complete point symmetry pseudogroup by the direct method and the comprehensive classification of its Lie reductions.

Among (1 + 2)-dimensional ultraparabolic Fokker–Planck equations of specific form, which are called (1 + 2)-dimensional Klein–Kramers equations or just Kramers equations, we consider those whose essential Lie invariance algebras are eight-dimensional. We map these equations to the equation (2) using point transformations and thus reduce the entire study of these equations within the framework of symmetry analysis, including the construction of exact solutions, to the study of the equation (2).

The structure of the paper is as follows. In Section 2, we present the maximal Lie invariance algebra of the equation (2) and describe its key properties. Using the direct method, in Section 3, we compute the complete point symmetry pseudogroup of the equation (2) and analyse its structure, including the decomposition of this pseudogroup and the description of its discrete elements. Section 4 is devoted to the classification of one- and two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  and of  $\mathfrak{g}$ . These are the dimensions that are relevant to the framework of Lie reductions. The classification of subalgebras lets us comprehensively study the codimension-one, codimension-two and codimension-three Lie reductions of the equation (2) in Sections 5, 6 and 7, respectively, and the subsequent construction of wide families of exact solutions of (2). In Section 8, we carry out its peculiar generalised reductions, which are associated with powers of certain Lie-symmetry operators, and show that the corresponding families of invariant solutions can be generated by iteratively acting with Lie-symmetry operators.<sup>2</sup> The similarity of Kramers equations from the class  $\bar{\mathcal{F}}$  with eight-dimensional essential Lie-symmetry algebras to the equation (2) with respect to point transformations is established explicitly in Section 9. It leads to easily finding wide families of exact solutions of such Kramers equations. In Section 10, we discuss how to develop results of the present paper. Appendix A is devoted to extended symmetry analysis of the (1 + 1)-dimensional linear heat equations with inverse square potentials, including the explicit construction of their (real) exact solutions. In view of results of Section 5, these solutions directly lead to the solutions of the equation (2). In Appendix B, we discuss an optimised procedure of constructing hidden invariant solutions of a general system of differential equations.

For readers' convenience, the constructed exact solutions of the equation (2) are marked by the bullet symbol ●.

<sup>2</sup>A Lie-symmetry operator of a homogeneous linear differential equation  $\mathcal{L}: \mathfrak{L}u = 0$  is a first-order linear differential operator  $\mathcal{Q}$  in total derivatives that commutes with the operator  $\mathfrak{L}$  on solutions of  $\mathcal{L}$  or, equivalently, such that the differential function  $\mathcal{Q}u$  is the characteristic of an (essential) Lie symmetry of  $\mathcal{L}$ .

### 2. Lie invariance algebra

The classical infinitesimal approach results in the well-known algorithm for computing the maximal Lie-symmetry algebras of systems of differential equations [9, 10, 25]. The maximal Lie invariance algebra of the equation (2) is (see, e.g. [20])

$$\mathfrak{g} := \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I}, \mathcal{Z}(f) \rangle,$$

where

$$\begin{aligned} \mathcal{P}^t &= \partial_t, \quad \mathcal{D} = 2t\partial_t + x\partial_x + 3y\partial_y - 2u\partial_u, \quad \mathcal{K} = t^2\partial_t + (tx + 3y)\partial_x + 3ty\partial_y - (x^2 + 2t)u\partial_u, \\ \mathcal{P}^3 &= 3t^2\partial_x + t^3\partial_y + 3(y - tx)u\partial_u, \quad \mathcal{P}^2 = 2t\partial_x + t^2\partial_y - xu\partial_u, \quad \mathcal{P}^1 = \partial_x + t\partial_y, \quad \mathcal{P}^0 = \partial_y, \\ \mathcal{I} &= u\partial_u, \quad \mathcal{Z}(f) = f(t, x, y)\partial_u. \end{aligned}$$

Here the parameter function  $f$  of  $(t, x, y)$  runs through the solution set of the equation (2).

The vector fields  $\mathcal{Z}(f)$  constitute the infinite-dimensional abelian ideal  $\mathfrak{g}^{\text{lin}}$  of  $\mathfrak{g}$  associated with the linear superposition of solutions of (2),  $\mathfrak{g}^{\text{lin}} := \{\mathcal{Z}(f)\}$ . Thus, the algebra  $\mathfrak{g}$  can be represented as a semidirect sum,  $\mathfrak{g} = \mathfrak{g}^{\text{ess}} \ltimes \mathfrak{g}^{\text{lin}}$ , where

$$\mathfrak{g}^{\text{ess}} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle \tag{4}$$

is an (eight-dimensional) subalgebra of  $\mathfrak{g}$ , called the essential Lie invariance algebra of (2).

Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations between the basis vector fields of  $\mathfrak{g}^{\text{ess}}$  are the following:

$$\begin{aligned} [\mathcal{P}^t, \mathcal{D}] &= 2\mathcal{P}^t, \quad [\mathcal{P}^t, \mathcal{K}] = \mathcal{D}, \quad [\mathcal{D}, \mathcal{K}] = 2\mathcal{K}, \\ [\mathcal{P}^t, \mathcal{P}^3] &= 3\mathcal{P}^2, \quad [\mathcal{P}^t, \mathcal{P}^2] = 2\mathcal{P}^1, \quad [\mathcal{P}^t, \mathcal{P}^1] = \mathcal{P}^0, \\ [\mathcal{D}, \mathcal{P}^3] &= 3\mathcal{P}^3, \quad [\mathcal{D}, \mathcal{P}^2] = \mathcal{P}^2, \quad [\mathcal{D}, \mathcal{P}^1] = -\mathcal{P}^1, \quad [\mathcal{D}, \mathcal{P}^0] = -3\mathcal{P}^0, \\ [\mathcal{K}, \mathcal{P}^2] &= -\mathcal{P}^3, \quad [\mathcal{K}, \mathcal{P}^1] = -2\mathcal{P}^2, \quad [\mathcal{K}, \mathcal{P}^0] = -3\mathcal{P}^1, \\ [\mathcal{P}^1, \mathcal{P}^2] &= -\mathcal{I}, \quad [\mathcal{P}^0, \mathcal{P}^3] = 3\mathcal{I}. \end{aligned}$$

The algebra  $\mathfrak{g}^{\text{ess}}$  is nonsolvable. Its Levi decomposition is given by  $\mathfrak{g}^{\text{ess}} = \mathfrak{f} \ltimes \mathfrak{r}$ , where the radical  $\mathfrak{r}$  of  $\mathfrak{g}^{\text{ess}}$  coincides with the nilradical of  $\mathfrak{g}^{\text{ess}}$  and is spanned by the vector fields  $\mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0$  and  $\mathcal{I}$ . The Levi factor  $\mathfrak{f} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K} \rangle$  of  $\mathfrak{g}^{\text{ess}}$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ , the radical  $\mathfrak{r}$  of  $\mathfrak{g}^{\text{ess}}$  is isomorphic to the rank-two Heisenberg algebra  $\mathfrak{h}(2, \mathbb{R})$ , and the real representation of the Levi factor  $\mathfrak{f}$  on the radical  $\mathfrak{r}$  coincides, in the basis  $(\mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I})$ , with the real representation  $\rho_3 \oplus \rho_0$  of  $\mathfrak{sl}(2, \mathbb{R})$ . Here  $\rho_n$  is the standard real irreducible representation of  $\mathfrak{sl}(2, \mathbb{R})$  in the  $(n + 1)$ -dimensional vector space. More specifically,

$$\rho_n(\mathcal{P}^t)_{ij} = (n - j)\delta_{i,j+1}, \quad \rho_n(\mathcal{D})_{ij} = (n - 2j)\delta_{ij}, \quad \rho_n(-\mathcal{K})_{ij} = j\delta_{i+1,j},$$

where  $i, j \in \{1, 2, \dots, n + 1\}$ ,  $n \in \mathbb{N} \cup \{0\}$ , and  $\delta_{kl}$  is the Kronecker delta, that is,  $\delta_{kl} = 1$  if  $k = l$  and  $\delta_{kl} = 0$  otherwise,  $k, l \in \mathbb{Z}$ . Thus, the entire algebra  $\mathfrak{g}^{\text{ess}}$  is isomorphic to the algebra  $L_{8,19}$  from the classification of indecomposable Lie algebras of dimensions up to eight with nontrivial Levi decompositions, which was carried out in [38].

Lie algebras whose Levi factors are isomorphic to the algebra  $\mathfrak{sl}(2, \mathbb{R})$  often arise within the field of group analysis of differential equations as Lie invariance algebras of parabolic partial differential equations. At the same time, the action of Levi factors on the corresponding radicals is usually described in terms of the representations  $\rho_0, \rho_1, \rho_2$  or their direct sums. To the best of our knowledge, algebras similar to  $\mathfrak{g}^{\text{ess}}$  were never considered in group analysis from the point of view of their subalgebra structure.

### 3. Complete point symmetry pseudogroup

We start computing the complete point symmetry pseudogroup  $G$  of the equation (2) by presenting the exhaustive description of the equivalence groupoid of the class  $\bar{\mathcal{F}}$ . Then we use special properties

of this groupoid for deriving an explicit representation for elements of  $G$ . See [4, 6, 26, 31, 33, 39] and references therein for definitions and theoretical results on various structures constituted by point transformations within classes of differential equations.

**Theorem 1.** *The class  $\tilde{\mathcal{F}}$  is normalised. Its (usual) equivalence pseudogroup  $G_{\tilde{\mathcal{F}}}^{\sim}$  consists of the point transformations with the components*

$$\tilde{t} = T(t, y), \quad \tilde{x} = X(t, x, y), \quad \tilde{y} = Y(t, y), \quad \tilde{u} = U^1(t, x, y)u + U^0(t, x, y), \tag{5a}$$

$$\tilde{A}^0 = \frac{A^0}{T_t + BT_y} - \frac{A^1}{T_t + BT_y} \frac{U_x^1}{U^1} + \frac{A^2}{T_t + BT_y} \left( \left( \frac{U_x^1}{U^1} \right)^2 - \left( \frac{U_x^1}{U^1} \right)_x \right) + \frac{1}{U^1} \frac{U_t^1 + BU_y^1}{T_t + BT_y}, \tag{5b}$$

$$\tilde{A}^1 = A^1 \frac{X_x}{T_t + BT_y} - \frac{X_t + BX_y}{T_t + BT_y} + A^2 \frac{X_{xx} - 2X_x U_x^1 / U^1}{T_t + BT_y}, \tag{5c}$$

$$\tilde{A}^2 = A^2 \frac{X_x^2}{T_t + BT_y}, \quad \tilde{B} = \frac{Y_t + BY_y}{T_t + BT_y}, \tag{5d}$$

$$\tilde{C} = \frac{U^1}{T_t + BT_y} \left( C - E \frac{U^0}{U^1} \right), \tag{5e}$$

where  $T, X, Y, U^1$  and  $U^0$  are arbitrary smooth functions of their arguments with  $(T_t Y_y - T_y Y_t) X_x U^1 \neq 0$ , and  $E := \partial_t + B\partial_y - A^2\partial_{xx} - A^1\partial_x - A^0$ .

The proof of this theorem is beyond the subject of the present paper and will be presented elsewhere.

The equation (2) corresponds to the value  $(x, 1, 0, 0, 0) =: \bar{\theta}_0$  of the arbitrary-element tuple  $\bar{\theta} = (B, A^2, A^1, A^0, C)$  of the class  $\tilde{\mathcal{F}}$ . The vertex group  $\mathcal{G}_{\bar{\theta}_0} := \mathcal{G}_{\tilde{\mathcal{F}}}^{\sim}(\bar{\theta}_0, \bar{\theta}_0)$  is the set of admissible transformations of the class  $\tilde{\mathcal{F}}$  with  $\bar{\theta}_0$  as both their source and target,  $\mathcal{G}_{\bar{\theta}_0} = \{(\bar{\theta}_0, \Phi, \bar{\theta}_0) \mid \Phi \in G\}$ . The normalisation of the class  $\tilde{\mathcal{F}}$  means that its equivalence groupoid coincides with the action groupoid of the pseudogroup  $G_{\tilde{\mathcal{F}}}^{\sim}$ , and thus, the latter groupoid necessarily contains the vertex group  $\mathcal{G}_{\bar{\theta}_0}$ . This argument allows us to use Theorem 1 in the course of computing the pseudogroup  $G$ .

**Theorem 2.** *The complete point symmetry pseudogroup  $G$  of the remarkable Fokker–Planck equation (2) consists of the transformations of the form*

$$\begin{aligned} \tilde{t} &= \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{\hat{x}}{\gamma t + \delta} - \frac{3\gamma\hat{y}}{(\gamma t + \delta)^2}, \quad \tilde{y} = \frac{\hat{y}}{(\gamma t + \delta)^3}, \\ \tilde{u} &= \sigma(\gamma t + \delta)^2 \exp\left(\frac{\gamma\hat{x}^2}{\gamma t + \delta} - \frac{3\gamma^2\hat{x}\hat{y}}{(\gamma t + \delta)^2} + \frac{3\gamma^3\hat{y}^2}{(\gamma t + \delta)^3}\right) \\ &\quad \times \exp(3\lambda_3(y - tx) - \lambda_2x - (3\lambda_3^2t^3 + 3\lambda_3\lambda_2t^2 + \lambda_2^2t)(u + f(t, x, y))), \end{aligned} \tag{6}$$

where  $\hat{x} := x + 3\lambda_3t^2 + 2\lambda_2t + \lambda_1$ ,  $\hat{y} := y + \lambda_3t^3 + \lambda_2t^2 + \lambda_1t + \lambda_0$ ;  $\alpha, \beta, \gamma$  and  $\delta$  are arbitrary constants with  $\alpha\delta - \beta\gamma = 1$ ;  $\lambda_0, \dots, \lambda_3$  and  $\sigma$  are arbitrary constants with  $\sigma \neq 0$ , and  $f$  is an arbitrary solution of (2).

*Proof.* We should integrate the system (5) with  $\bar{\theta} = \bar{\theta} = \bar{\theta}_0$  with respect to the parameter functions  $T, X, Y$  and  $U^1$ . The equations (5d) take the form

$$X = \frac{Y_t + xY_y}{T_t + xT_y}, \quad X_x^2 = T_t + xT_y.$$

In view of the first of these equations, the second equation reduces to  $(T_t Y_y - T_y Y_t)^2 = (T_t + xT_y)^5$ , which implies  $T_y = 0, T_t > 0$  and

$$Y = \varepsilon T_t^{3/2} y + Y^0(t),$$

where  $\varepsilon = \pm 1$  and  $Y^0$  is a function of  $t$  arising due to the integration with respect to  $y$ . This is why we have  $X = (Y_t + xY_y)/T_t$ , and hence  $X_{xx} = 0$ . Then the equation (5c) simplifies to  $X_t + xX_y = -2X_xU_x^1/U^1$  and thus integrates to

$$U^1 = V(t, y) \exp \left( -\frac{T_{tt}}{2T_t}x^2 - 3\frac{2T_{tt}T_t - T_{tt}^2}{8T_t^2}xy - \frac{(Y^0/T_t)_t}{2T_t^{5/2}}x \right),$$

where  $V$  is a nonvanishing smooth function of  $(t, y)$ , the explicit expression for which will be derived below. Substituting the expression for  $U^1$  into the restricted equation (5b),

$$\left(\frac{U_x^1}{U^1}\right)^2 - \left(\frac{U_x^1}{U^1}\right)_x + \frac{U_t^1 + xU_y^1}{U^1} = 0,$$

leads to an equation whose left-hand side is a quadratic polynomial in  $x$ . Collecting the coefficients of  $x^2$ , we derive the equation  $T_{tt}/T_t - \frac{3}{2}(T_{tt}/T_t)^2 = 0$ , meaning that the Schwarzian derivative of  $T$  is zero. Therefore,  $T$  is a linear fractional function of  $t$ ,  $T = (\alpha t + \beta)/(\gamma t + \delta)$ . Since the constants  $\alpha, \beta, \gamma$  and  $\delta$  are defined up to a constant nonzero multiplier and  $T_t > 0$ , we can assume that  $\alpha\delta - \beta\gamma = 1$ . Then, the result of collecting the coefficients of  $x$  in the above equation can be represented, up to an inessential multiplier, in the form

$$\frac{V_y}{V} = \frac{6\gamma^3}{(\gamma t + \delta)^3}y + ((\gamma t + \delta)^3 Y^0)_{tt} - 6\gamma^2 ((\gamma t + \delta) Y^0)_t.$$

The general solution of the last equation as an equation with respect to  $V$  is

$$V = \phi(t) \exp \left( \frac{3\gamma^3}{(\gamma t + \delta)^3}y^2 + ((\gamma t + \delta)^3 Y^0)_{tt}y - 6\gamma^2 ((\gamma t + \delta) Y^0)_t y \right),$$

where  $\phi$  is a nonvanishing smooth function of  $t$ . Analogously, we collect the summands without  $x$ , substitute the above representation for  $V$  into the obtained equation, split the result with respect to  $y$  and in addition neglect inessential multipliers. This gives the system of two equations

$$((\gamma t + \delta)^3 Y^0)_{ttt} = 0, \quad \frac{\phi_t}{\phi} - \frac{2\gamma}{\gamma t + \delta} + \frac{1}{4}(\gamma t + \delta)^4 ((\gamma t + \delta) Y^0)_{tt} = 0,$$

whose general solution can be represented as

$$Y^0 = \frac{\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0}{(\gamma t + \delta)^3},$$

$$\phi = \sigma(\gamma t + \delta)^2 \exp \left( \frac{\gamma \psi_t^2}{\gamma t + \delta} - \frac{3\gamma^2 \psi_t \psi}{(\gamma t + \delta)^2} + \frac{3\gamma^3 \psi^2}{(\gamma t + \delta)^3} \right) \exp(-\lambda_2^2 t - 3\lambda_3 \lambda_2 t^2 - 3\lambda_3^2 t^3),$$

where  $\psi$  is an arbitrary at most cubic polynomial of  $t$ ,  $\psi(t) := \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0$ , and  $\lambda_0, \dots, \lambda_3$  and  $\sigma$  are arbitrary constants with  $\sigma \neq 0$ . Since the constant parameters  $\alpha, \beta, \gamma$  and  $\delta$  are still defined up to the multiplier  $\pm 1$ , we can choose these parameters in such a way that  $\varepsilon|\gamma t + \delta| = \gamma t + \delta$  and then neglect the parameter  $\varepsilon$ .

Finally, the equation (5e) takes the form

$$\left(\frac{U^0}{U^1}\right)_t + x \left(\frac{U^0}{U^1}\right)_y = \left(\frac{U^0}{U^1}\right)_{xx},$$

and thus  $U^0 = U^1 f$ , where  $f = f(t, x, y)$  is an arbitrary solution of (2). □

If the transformations of the form (6) and their compositions are properly interpreted, then the structure of the pseudogroup  $G$  is simplified. Thus, the natural domain of a transformation  $\Phi$  of the form (6) is

$$\text{dom } \Phi = (\text{dom } f \times \mathbb{R}_u) \setminus M_{\gamma\delta},$$

that is, it can be assumed to coincide with the relative complement of the set  $M_{\gamma\delta} := \{(t, x, y, u) \in \mathbb{R}^4 \mid \gamma t + \delta = 0\}$  with respect to the set  $\text{dom } f \times \mathbb{R}_u$ , where  $\text{dom } f$  is the domain of the function  $f$ .

In view of the fact that parameters  $\gamma$  and  $\delta$  do not vanish simultaneously, the set  $M_{\gamma\delta}$  is empty if  $\gamma = 0$  and is the hyperplane defined by the equation  $t = -\delta/\gamma$  in the space  $\mathbb{R}^4_{t,x,y,u}$  otherwise. We modify the standard definition of composition in the case of transformations of the form (6). More specifically, according to the standard definition, the domain of the composition  $\Phi_1 \circ \Phi_2 =: \tilde{\Phi}$  of transformations  $\Phi_1$  and  $\Phi_2$  is the preimage of the domain of  $\Phi_1$  with respect of  $\Phi_2$ . If the transformations  $\Phi_1$  and  $\Phi_2$  are of the form (6), then  $\text{dom } \tilde{\Phi} = \Phi_2^{-1}(\text{dom } \Phi_1) = (\text{dom } f \times \mathbb{R}_u) \setminus (M_{\gamma_2\delta_2} \cup M_{\tilde{\gamma}\tilde{\delta}})$ , where  $\tilde{\gamma} = \gamma_1\alpha_2 + \delta_1\gamma_2$ ,  $\tilde{\delta} = \gamma_1\beta_2 + \delta_1\delta_2$ ,  $\text{dom } \tilde{f} = ((\pi_*\Phi_2)^{-1}\text{dom } f^1) \cap \text{dom } f^2$ ,  $\pi$  is the natural projection onto  $\mathbb{R}^3_{t,x,y}$  in  $\mathbb{R}^4_{t,x,y,u}$ , and the parameters with indices 1 and 2 and tildes correspond to  $\Phi_1$ ,  $\Phi_2$  and  $\tilde{\Phi}$ , respectively. As the *modified composition*  $\Phi_1 \circ^m \Phi_2$  of transformations  $\Phi_1$  and  $\Phi_2$ , we take the extension of  $\Phi_1 \circ \Phi_2$  by continuity to the set

$$\text{dom}^m \tilde{\Phi} := (\text{dom } \tilde{f} \times \mathbb{R}_u) \setminus M_{\tilde{\gamma}\tilde{\delta}},$$

that is,  $\text{dom}(\Phi_1 \circ^m \Phi_2) = \text{dom}^m \tilde{\Phi}$ . Therefore,  $\Phi_1 \circ^m \Phi_2$  is the transformation of the form (6) with the same parameters as in  $\Phi_1 \circ \Phi_2$  and with its natural domain. This means that  $\text{dom}^m \tilde{\Phi} = \text{dom } \tilde{\Phi}$  and the extension is trivial if  $\gamma_1\gamma_2 = 0$ , and otherwise we redefine  $\Phi_1 \circ \Phi_2$  on the set  $(\text{dom } \tilde{f} \times \mathbb{R}_u) \cap M_{\gamma_2\delta_2}$ .

Now we can analyse the structure of  $G$ . The point transformations of the form

$$\mathcal{Z}(f): \quad \tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{y} = y, \quad \tilde{u} = u + f(t, x, y),$$

where the parameter function  $f = f(t, x, y)$  is an arbitrary solution of the equation (2), are associated with the linear superposition of solutions of this equation and thus can be considered as trivial. They constitute the normal pseudosubgroup  $G^{\text{lin}}$  of the pseudogroup  $G$ . The pseudogroup  $G$  splits over  $G^{\text{lin}}$ ,  $G = G^{\text{ess}} \ltimes G^{\text{lin}}$ , where  $G^{\text{ess}}$  is the *subgroup* of  $G$  consisting of the transformations of the form (6) with  $f = 0$  and with their natural domains, and thus it is an eight-dimensional Lie group. The Lie group structure of  $G^{\text{ess}}$  is the main benefit of redefining the transformation composition above.<sup>3</sup> We call the subgroup  $G^{\text{ess}}$  the *essential point symmetry group* of the equation (2).

The subgroup  $G^{\text{ess}}$  itself splits over  $R$ ,  $G^{\text{ess}} = F \times R$ . Here  $R$  and  $F$  are the normal subgroup and the subgroup of  $G^{\text{ess}}$  that are singled out by the constraints  $\alpha = \delta = 1$ ,  $\beta = \gamma = 0$  and  $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$ ,  $\sigma = 1$ , respectively. They are isomorphic to the groups  $H(2, \mathbb{R}) \times \mathbb{Z}_2$  and  $SL(2, \mathbb{R})$ , and their Lie algebras coincide with  $\mathfrak{h}(2, \mathbb{R})$  and  $\mathfrak{sl}(2, \mathbb{R})$ . Here  $H(2, \mathbb{R})$  denotes the rank-two real Heisenberg group. The normal subgroups  $R_c$  and  $R_d$  of  $R$  that are isomorphic to  $H(2, \mathbb{R})$  and  $\mathbb{Z}_2$  are constituted by the elements of  $R$  with parameter values satisfying the constraints  $\sigma > 0$  and  $\lambda_3 = \lambda_2 = \lambda_1 = \lambda_0 = 0$ ,  $\sigma \in \{-1, 1\}$ , respectively. The isomorphisms of  $F$  to  $SL(2, \mathbb{R})$  and of  $R_c$  to  $H(2, \mathbb{R})$  are established by the correspondences

$$(\alpha, \beta, \gamma, \delta)_{\alpha\delta-\beta\gamma=1} \mapsto \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad (\lambda_3, \lambda_2, \lambda_1, \lambda_0, \sigma)_{\sigma>0} \mapsto \begin{pmatrix} 1 & 3\lambda_3 & -\lambda_2 & \ln \sigma \\ 0 & 1 & 0 & \lambda_0 \\ 0 & 0 & 1 & \lambda_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus,  $F$  and  $R_c$  are connected subgroups of  $G^{\text{ess}}$ , but  $R_d$  is not. The natural conjugacy action of the group  $F$  on the normal subgroup  $R$  is given by  $(\tilde{\lambda}_3, \tilde{\lambda}_2, \tilde{\lambda}_1, \tilde{\lambda}_0, \tilde{\sigma}) = (\lambda_3, \lambda_2, \lambda_1, \lambda_0, \sigma)A$  in the parameterisation (6) of  $G$ , where  $A = \varrho_3(\alpha, \beta, \gamma, \delta) \oplus (1)$ , and  $\varrho_3$  is the standard real irreducible four-dimensional representation of  $SL(2, \mathbb{R})$ , which can be identified with the action of  $SL(2, \mathbb{R})$  on binary cubics,

$$\varrho_3 : (\alpha, \beta, \gamma, \delta)_{\alpha\delta-\beta\gamma=1} \mapsto \begin{pmatrix} \alpha^3 & 3\alpha^2\beta & 3\alpha\beta^2 & \beta^3 \\ \alpha^2\gamma & 2\alpha\beta\gamma + \alpha^2\delta & 2\alpha\beta\delta + \beta^2\gamma & \beta^2\delta \\ \alpha\gamma^2 & 2\alpha\gamma\delta + \beta\gamma^2 & 2\beta\gamma\delta + \alpha\delta^2 & \beta\delta^2 \\ \gamma^3 & 3\gamma^2\delta & 3\gamma\delta^2 & \delta^3 \end{pmatrix}.$$

<sup>3</sup>For the standard transformation composition, we have only the representation  $G = \bar{G}^{\text{ess}} \ltimes G^{\text{lin}}$ , where  $\bar{G}^{\text{ess}}$  is the pseudosubgroup of  $G$  consisting of the transformations of the form (6) with  $f = 0$  and with all admitted domains. Thus, the pseudosubgroup  $\bar{G}^{\text{ess}}$  does not possess a group structure.

Summing up, the group  $G^{\text{ess}}$  is isomorphic to  $(\text{SL}(2, \mathbb{R}) \ltimes_{\varphi} \text{H}(2, \mathbb{R})) \times \mathbb{Z}_2$ , where the antihomomorphism  $\varphi : \text{SL}(2, \mathbb{R}) \rightarrow \text{Aut}(\text{H}(2, \mathbb{R}))$  is defined, in the chosen local coordinates, by  $\varphi(\alpha, \beta, \gamma, \delta) = (\lambda_3, \lambda_2, \lambda_1, \lambda_0, \sigma) \mapsto (\lambda_3, \lambda_2, \lambda_1, \lambda_0, \sigma)A$ .

Transformations from the one-parameter subgroups of  $G^{\text{ess}}$  that are generated by the basis elements of  $\mathfrak{g}^{\text{ess}}$  given in (4) are of the following form:

$$\begin{aligned} \mathcal{P}^t(\epsilon) : \quad & \tilde{t} = t + \epsilon, & \tilde{x} = x, & & \tilde{y} = y, & & \tilde{u} = u, \\ \mathcal{D}(\epsilon) : \quad & \tilde{t} = e^{2\epsilon}t, & \tilde{x} = e^{\epsilon}x, & & \tilde{y} = e^{3\epsilon}x, & & \tilde{u} = e^{-2\epsilon}u, \\ \mathcal{K}(\epsilon) : \quad & \tilde{t} = \frac{t}{1-\epsilon t}, & \tilde{x} = \frac{x}{1-\epsilon t} + \frac{3\epsilon y}{(1-\epsilon t)^2}, & & \tilde{y} = \frac{y}{(1-\epsilon t)^3}, & & \tilde{u} = (1-\epsilon t)^2 e^{\frac{\epsilon y^2}{1-\epsilon t} + \frac{3\epsilon^2 xy}{(1-\epsilon t)^2} + \frac{3\epsilon^3 y^2}{(1-\epsilon t)^3}} u, \\ \mathcal{P}^3(\epsilon) : \quad & \tilde{t} = t, & \tilde{x} = x + 3\epsilon t^2, & & \tilde{y} = y + \epsilon t^3, & & \tilde{u} = e^{3\epsilon(y-tx) - 3\epsilon^2 t^3} u, \\ \mathcal{P}^2(\epsilon) : \quad & \tilde{t} = t, & \tilde{x} = x + 2\epsilon t, & & \tilde{y} = y + \epsilon t^2, & & \tilde{u} = e^{-\epsilon x + \epsilon^2 t} u, \\ \mathcal{P}^1(\epsilon) : \quad & \tilde{t} = t, & \tilde{x} = x + \epsilon, & & \tilde{y} = y + \epsilon t, & & \tilde{u} = u, \\ \mathcal{P}^0(\epsilon) : \quad & \tilde{t} = t, & \tilde{x} = x, & & \tilde{y} = y + \epsilon, & & \tilde{u} = u, \\ \mathcal{J}(\epsilon) : \quad & \tilde{t} = t, & \tilde{x} = x, & & \tilde{y} = y, & & \tilde{u} = e^{\epsilon} u, \end{aligned}$$

where  $\epsilon$  is the group parameter. At the same time, using this basis of  $\mathfrak{g}^{\text{ess}}$  in the course of studying the structure of the group  $G^{\text{ess}}$  hides some of its important properties and complicates its study.

Although the pushforward of the group  $G^{\text{ess}}$  by the natural projection of  $\mathbb{R}^4_{t,x,y,u}$  onto  $\mathbb{R}_t$  coincides with the group of linear fractional transformations of  $t$  and is thus isomorphic to the group  $\text{PSL}(2, \mathbb{R})$ , the subgroup  $F$  of  $G^{\text{ess}}$  is isomorphic to the group  $\text{SL}(2, \mathbb{R})$ , and its Iwasawa decomposition is given by the one-parameter subgroups of  $G^{\text{ess}}$ , respectively, generated by the vector fields  $\mathcal{P}^t + \mathcal{K}$ ,  $\mathcal{D}$  and  $\mathcal{P}^t$ . The first subgroup, which is associated with  $\mathcal{P}^t + \mathcal{K}$ , consists of the point transformations

$$\begin{aligned} \tilde{t} &= \frac{t \cos \epsilon - \sin \epsilon}{t \sin \epsilon + \cos \epsilon}, & \tilde{x} &= \frac{x}{t \sin \epsilon + \cos \epsilon} - \frac{3y \sin \epsilon}{(t \sin \epsilon + \cos \epsilon)^2}, & \tilde{y} &= \frac{y}{(t \sin \epsilon + \cos \epsilon)^3}, \\ \tilde{u} &= (t \sin \epsilon + \cos \epsilon)^2 \exp \left( \frac{x^2 \sin \epsilon}{t \sin \epsilon + \cos \epsilon} - \frac{3xy \sin^2 \epsilon}{(t \sin \epsilon + \cos \epsilon)^2} + \frac{3y^2 \sin^3 \epsilon}{(t \sin \epsilon + \cos \epsilon)^3} \right) u \end{aligned} \tag{7}$$

parameterised by an arbitrary constant  $\epsilon$ , which is defined by the corresponding transformation up to a summand  $2\pi k$ ,  $k \in \mathbb{Z}$ .

The equation (2) is invariant with respect to the involution  $\mathcal{J}$  simultaneously alternating the sign of  $(x, y)$ ,

$$\mathcal{J} : (t, x, y, u) \mapsto (t, -x, -y, u).$$

In the context of the one-parameter subgroups of  $G^{\text{ess}}$  that are generated by the basis elements of  $\mathfrak{g}^{\text{ess}}$  listed in (4), the involution  $\mathcal{J}$  looks like a discrete point symmetry transformation of (2), but in fact this is not the case. It belongs to the one-parameter subgroup of  $G^{\text{ess}}$  generated by  $\mathcal{P}^t + \mathcal{K}$ . More precisely, it coincides with the transformation (7) with  $\epsilon = \pi$ . The value  $\epsilon = \pi/2$  corresponds to the transformation

$$\mathcal{K} : \tilde{t} = -\frac{1}{t}, \quad \tilde{x} = \frac{x}{t} - 3\frac{y}{t^2}, \quad \tilde{y} = \frac{y}{t^3}, \quad \tilde{u} = t^2 e^{\frac{t^2}{t} - \frac{3y}{t^2} + \frac{3y^2}{t^3}} u,$$

which also deceptively looks, in the above context, like a discrete point symmetry transformation of (2) being independent with  $\mathcal{J}$  since the factorisation  $\mathcal{J} \circ \mathcal{K} = \mathcal{P}^t(1) \circ \mathcal{K}(1) \circ \mathcal{P}^t(1)$  is not intuitive. This is why it is relevant to accurately describe discrete point symmetries of the equation (2).

**Corollary 3.** *A complete list of discrete point symmetry transformations of the remarkable Fokker-Planck equation (2) that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution  $\mathcal{J}$  alternating the sign of  $u$ ,*

$$\mathcal{J}' : (t, x, y, u) \mapsto (t, x, y, -u).$$



Thus, the quotient group of the complete point symmetry pseudogroup  $G$  with respect to its identity component is isomorphic to  $\mathbb{Z}_2$ .

*Proof.* It is obvious that the entire pseudosubgroup  $G^{\text{lin}}$  is contained in the connected component of the identity transformation in  $G$ . The same claim holds for the subgroups  $F$  and  $R_c$  in view of their isomorphisms to the groups  $SL(2, \mathbb{R})$  and  $H(2, \mathbb{R})$ , respectively. Therefore, without loss of generality, a complete list of independent discrete point symmetry transformations of (2) can be assumed to consist of elements of the subgroup  $R_d$ . Thus, the only discrete point symmetry transformation of (2) that is independent in the above sense is the transformation  $J'$ .  $\square$

Hereafter, the transformations  $\mathcal{P}^i(\epsilon)$ ,  $\mathcal{D}(\epsilon)$ ,  $\mathcal{K}(\epsilon)$ ,  $\mathcal{P}^3(\epsilon)$ ,  $\mathcal{P}^2(\epsilon)$ ,  $\mathcal{P}^1(\epsilon)$ ,  $\mathcal{P}^0(\epsilon)$ ,  $\mathcal{J}(\epsilon)$ ,  $\mathcal{J}$ ,  $\mathcal{K}'$ ,  $\mathcal{J}'$  and  $\mathcal{Z}(f)$  are called *elementary point symmetry transformations* of the equation (2).

In view of Theorem 2, the formal application of the point symmetry transformation  $\Phi := \mathcal{J} \left( \ln \frac{\sqrt{3}}{2\pi} \right) \circ \mathcal{P}^0(y') \circ \mathcal{P}^1(x') \circ \mathcal{P}^i(t')$  of the equation (2),

$$\begin{aligned} \Phi: \quad \tilde{t} &= -\frac{1}{t} + t', & \tilde{x} &= \frac{x}{t} - 3\frac{y}{t^2} + x', & \tilde{y} &= \frac{y}{t^3} + y', \\ \tilde{u} &= \frac{\sqrt{3}}{2\pi} t^2 \exp \left( \frac{x^2}{t} - 3\frac{xy}{t^2} + 3\frac{y^2}{t^3} + 3x'x - 3x'\frac{y}{t} + 3(x')^2 t \right) u, \end{aligned} \tag{8}$$

maps the function  $u(t, x, y) = 1 - H(t)$  to the fundamental solution (3) of the equation (2). Recall that  $H$  denotes the Heaviside step function. Note that attempts to interpret this fundamental solution within the framework of group analysis of differential equations were made in [17, 20].

#### 4. Classification of inequivalent subalgebras

In order to carry out the Lie reductions of codimension one and two for the equation (2) in the optimal way, we need to classify one- and two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  up to the  $G^{\text{ess}}$ -equivalence. In the course of this classification, we use the Levi decomposition  $\mathfrak{g}^{\text{ess}} = \mathfrak{f} \ltimes \mathfrak{r}$  and the fact that the Levi factor  $\mathfrak{f}$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{R})$ . This allows us to apply the technique of classifying the subalgebras of an algebra with a proper ideal suggested in [29]. This technique becomes simpler if the algebra under consideration can be represented as the semidirect sum of a subalgebra and an ideal, which are  $\mathfrak{f}$  and  $\mathfrak{r}$  for the algebra  $\mathfrak{g}^{\text{ess}}$ , respectively. An additional simplification is that an optimal list of subalgebras of  $\mathfrak{sl}(2, \mathbb{R})$  is well known (see, e.g. [28, 32]). Thus, for the realisation  $\mathfrak{f}$  of  $\mathfrak{sl}(2, \mathbb{R})$ , this list consists of the subalgebras  $\{0\}$ ,  $\langle \mathcal{P}^i \rangle$ ,  $\langle \mathcal{D} \rangle$ ,  $\langle \mathcal{P}^i + \mathcal{K} \rangle$ ,  $\langle \mathcal{P}^i, \mathcal{D} \rangle$  and  $\mathfrak{f}$  itself. The subalgebras  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  of  $\mathfrak{g}^{\text{ess}}$  are definitely  $G^{\text{ess}}$ -inequivalent if their projections  $\pi_{\mathfrak{f}}\mathfrak{s}_1$  and  $\pi_{\mathfrak{f}}\mathfrak{s}_2$  are  $F$ -inequivalent, see Section 3. Here and in what follows,  $\pi_{\mathfrak{f}}$  and  $\pi_{\mathfrak{r}}$  denote the natural projections of  $\mathfrak{g}^{\text{ess}}$  onto  $\mathfrak{f}$  and  $\mathfrak{r}$  according to the decomposition  $\mathfrak{g}^{\text{ess}} = \mathfrak{f} \ltimes \mathfrak{r}$  of the vector space  $\mathfrak{g}^{\text{ess}}$  as the direct sum of its subspaces  $\mathfrak{f}$  and  $\mathfrak{r}$ . In other words, we can use the projections  $\pi_{\mathfrak{f}}\mathfrak{s}$  of the subalgebras  $\mathfrak{s}$  of  $\mathfrak{g}^{\text{ess}}$  of the same dimension for partitioning the set of these subalgebras into subsets such that each subset contains no subalgebra being equivalent to a subalgebra from another subset.

Let us specifically describe applying the above technique to the classification of one- and two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  up to the  $G^{\text{ess}}$ -equivalence. We fix the dimension  $d$  of subalgebras  $\mathfrak{s}$  to be classified, either  $d = 1$  or  $d = 2$ , and consider the subalgebras  $\mathfrak{s}_i$  of  $\mathfrak{f}$  from the above list with dimension  $d'$  less than or equal to  $d$ . For each of these subalgebras, we take the set of  $d$ -dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  with  $\pi_{\mathfrak{f}}\mathfrak{s} = \mathfrak{s}_i$  and construct a complete list of  $G^{\text{ess}}$ -inequivalent subalgebras in this set.

One usually classifies subalgebras of a Lie algebra up to their equivalence generated by the group of inner automorphisms of this algebra, and this group is computed by summing up Lie series or solving Cauchy problems with the adjoint action of the algebra on itself, see, for example, [25, Section 3.3]. At the same time, we already know that the algebra  $\mathfrak{g}^{\text{ess}}$  is the Lie algebra of the group  $G^{\text{ess}}$ , and the actions of  $G^{\text{ess}}$  and of the group of inner automorphisms of  $\mathfrak{g}^{\text{ess}}$  on  $\mathfrak{g}^{\text{ess}}$  coincide. Moreover, recall that

for the purpose of finding Lie invariant solutions of the equation (2) subalgebras of  $\mathfrak{g}^{\text{ess}}$  have in fact to be classified up to the  $G^{\text{ess}}$ -equivalence. This is why following, for example, [15] we directly compute the action of  $G^{\text{ess}}$  on  $\mathfrak{g}^{\text{ess}}$  by pushing forward vector fields from  $\mathfrak{g}^{\text{ess}}$  by transformations from  $G^{\text{ess}}$ . The nonidentity pushforwards of basis elements of  $\mathfrak{g}^{\text{ess}}$  by the elementary transformations from  $G^{\text{ess}}$  are the following:

$$\begin{aligned} \mathcal{P}^i(\epsilon)_* \mathcal{D} &= \mathcal{D} - 2\epsilon \mathcal{P}^i, & \mathcal{K}(\epsilon)_* \mathcal{D} &= \mathcal{D} + 2\epsilon \mathcal{K}, \\ \mathcal{P}^i(\epsilon)_* \mathcal{K} &= \mathcal{K} - \epsilon \mathcal{D} + \epsilon^2 \mathcal{P}^i, & \mathcal{K}(\epsilon)_* \mathcal{P}^i &= \mathcal{P}^i + \epsilon \mathcal{D} + \epsilon^2 \mathcal{K}, \\ \mathcal{P}^i(\epsilon)_* \mathcal{P}^3 &= \mathcal{P}^3 - 3\epsilon \mathcal{P}^2 + 3\epsilon^2 \mathcal{P}^1 - \epsilon^3 \mathcal{P}^0, & \mathcal{K}(\epsilon)_* \mathcal{P}^2 &= \mathcal{P}^2 + \epsilon \mathcal{P}^3, \\ \mathcal{P}^i(\epsilon)_* \mathcal{P}^2 &= \mathcal{P}^2 - 2\epsilon \mathcal{P}^1 + \epsilon^2 \mathcal{P}^0, & \mathcal{K}(\epsilon)_* \mathcal{P}^1 &= \mathcal{P}^1 + 2\epsilon \mathcal{P}^2 + \epsilon^2 \mathcal{P}^3, \\ \mathcal{P}^i(\epsilon)_* \mathcal{P}^1 &= \mathcal{P}^1 - \epsilon \mathcal{P}^0, & \mathcal{K}(\epsilon)_* \mathcal{P}^0 &= \mathcal{P}^0 + 3\epsilon \mathcal{P}^1 + 3\epsilon^2 \mathcal{P}^2 + \epsilon^3 \mathcal{P}^3, \\ \mathcal{D}(\epsilon)_* \mathcal{P}^i &= e^{2\epsilon} \mathcal{P}^i, & \mathcal{D}(\epsilon)_* \mathcal{P}^3 &= e^{-3\epsilon} \mathcal{P}^3, & \mathcal{D}(\epsilon)_* \mathcal{P}^1 &= e^\epsilon \mathcal{P}^1, \\ \mathcal{D}(\epsilon)_* \mathcal{K} &= e^{-2\epsilon} \mathcal{K}, & \mathcal{D}(\epsilon)_* \mathcal{P}^2 &= e^{-\epsilon} \mathcal{P}^2, & \mathcal{D}(\epsilon)_* \mathcal{P}^0 &= e^{3\epsilon} \mathcal{P}^0, \\ \\ \mathcal{P}^3(\epsilon)_* \mathcal{P}^i &= \mathcal{P}^i + 3\epsilon \mathcal{P}^2, & \mathcal{P}^0(\epsilon)_* \mathcal{D} &= \mathcal{D} - 3\epsilon \mathcal{P}^0, \\ \mathcal{P}^3(\epsilon)_* \mathcal{D} &= \mathcal{D} + 3\epsilon \mathcal{P}^3, & \mathcal{P}^0(\epsilon)_* \mathcal{K} &= \mathcal{K} - 3\epsilon \mathcal{P}^1, \\ \mathcal{P}^3(\epsilon)_* \mathcal{P}^0 &= \mathcal{P}^0 + 3\epsilon \mathcal{I}, & \mathcal{P}^0(\epsilon)_* \mathcal{P}^3 &= \mathcal{P}^3 - 3\epsilon \mathcal{I}, \\ \\ \mathcal{P}^2(\epsilon)_* \mathcal{P}^i &= \mathcal{P}^i + 2\epsilon \mathcal{P}^1 - \epsilon^2 \mathcal{I}, & \mathcal{P}^1(\epsilon)_* \mathcal{P}^i &= \mathcal{P}^i + \epsilon \mathcal{P}^0, \\ \mathcal{P}^2(\epsilon)_* \mathcal{D} &= \mathcal{D} + \epsilon \mathcal{P}^2, & \mathcal{P}^1(\epsilon)_* \mathcal{D} &= \mathcal{D} - \epsilon \mathcal{P}^1, \\ \mathcal{P}^2(\epsilon)_* \mathcal{K} &= \mathcal{K} - \epsilon \mathcal{P}^3, & \mathcal{P}^1(\epsilon)_* \mathcal{K} &= \mathcal{K} - 2\epsilon \mathcal{P}^2 - \epsilon^2 \mathcal{I}, \\ \mathcal{P}^2(\epsilon)_* \mathcal{P}^1 &= \mathcal{P}^1 - \epsilon \mathcal{I}, & \mathcal{P}^1(\epsilon)_* \mathcal{P}^2 &= \mathcal{P}^2 + \epsilon \mathcal{I}, \\ \\ \mathcal{J}_*(\mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0) &= (-\mathcal{P}^3, -\mathcal{P}^2, -\mathcal{P}^1, -\mathcal{P}^0), \\ \mathcal{K}'_*(\mathcal{P}^i, \mathcal{D}, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0) &= (\mathcal{K}, -\mathcal{D}, \mathcal{P}^i, \mathcal{P}^0, -\mathcal{P}^1, \mathcal{P}^2, -\mathcal{P}^3). \end{aligned}$$

The result of the classification of one- and two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  up to the  $G^{\text{ess}}$ -equivalence is presented in the two subsequent lemmas.

**Lemma 4.** *A complete list of  $G^{\text{ess}}$ -inequivalent one-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  is exhausted by the subalgebras*

$$\begin{aligned} \mathfrak{s}_{1.1} &= \langle \mathcal{P}^i + \mathcal{P}^3 \rangle, & \mathfrak{s}_{1.2}^\delta &= \langle \mathcal{P}^i + \delta \mathcal{I} \rangle, & \mathfrak{s}_{1.3}^\nu &= \langle \mathcal{D} + \nu \mathcal{I} \rangle, & \mathfrak{s}_{1.4}^\mu &= \langle \mathcal{P}^i + \mathcal{K} + \mu \mathcal{I} \rangle, \\ \mathfrak{s}_{1.5}^\epsilon &= \langle \mathcal{P}^2 + \epsilon \mathcal{P}^0 \rangle, & \mathfrak{s}_{1.6} &= \langle \mathcal{P}^1 \rangle, & \mathfrak{s}_{1.7} &= \langle \mathcal{P}^0 \rangle, & \mathfrak{s}_{1.8} &= \langle \mathcal{I} \rangle, \end{aligned}$$

where  $\epsilon \in \{-1, 1\}$ ,  $\delta \in \{-1, 0, 1\}$ , and  $\mu$  and  $\nu$  are arbitrary real constants with  $\nu \geq 0$ .

*Proof.* A complete set of  $F$ -inequivalent  $d'$ -dimensional subalgebras of  $\mathfrak{f}$  with  $d' \leq 1$  consists of the subalgebras  $\langle \mathcal{P}^i \rangle$ ,  $\langle \mathcal{D} \rangle$ ,  $\langle \mathcal{P}^i + \mathcal{K} \rangle$  and  $\{0\}$ . Therefore, without loss of generality we can consider only one-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  with basis vector fields  $Q$  of the general form

$$Q = \hat{Q} + a_3 \mathcal{P}^3 + 3a_2 \mathcal{P}^2 + 3a_1 \mathcal{P}^1 + a_0 \mathcal{P}^0 + b \mathcal{I},$$

where  $\hat{Q} \in \{\mathcal{P}^i, \mathcal{D}, \mathcal{P}^i + \mathcal{K}, 0\}$  and  $a_0, \dots, a_3$  and  $b$  are real constants. We factor out the multiplier 3 in the coefficients of  $\mathcal{P}^1$  and  $\mathcal{P}^2$  since then the coefficients  $a_3, a_2, a_1$  and  $a_0$  are changed under the action of  $F$  in the same way as the coefficients of the real cubic binary form  $a_3x^3 + 3a_2x^2y + 3a_1xy^2 + a_0y^3$  are changed under the standard action of the group of linear fractional transformations on such forms, cf. [24, Example 2.22]. We need to further simplify the vector fields  $Q$  of the above form by acting with elements of  $G^{\text{ess}}$  on them and/or scaling them.

Let  $\hat{Q} = \mathcal{P}^i$ . If  $a_3 \neq 0$ , then we successively set  $a_3 = 1, a_2 = a_1 = a_0 = 0$  and  $b = 0$  using  $\mathcal{D}(e^{\epsilon_4})_*$  simultaneously with rescaling  $Q, \mathcal{P}^1(\epsilon_1)_* \circ \mathcal{P}^2(\epsilon_2)_* \circ \mathcal{P}^3(\epsilon_3)_*$  and  $\mathcal{P}^0(\epsilon_0)_*$ , with appropriate constants  $\epsilon_i, i = 0, \dots, 4$ , respectively. This gives the subalgebra  $\mathfrak{s}_{1.1}$ . Similarly, for  $a_3 = 0$ , we apply  $\mathcal{P}^3(\epsilon_2)_*, \mathcal{P}^2(\epsilon_1)_*$  and  $\mathcal{P}^1(\epsilon_0)_*$  with appropriate constants  $\epsilon_i, i = 1, 2, 3$ , to get rid of  $a_2, a_1$  and  $a_0$ . Then, the action of  $\mathcal{D}(e^{\epsilon_b})_*$  with appropriate  $\epsilon_b$  and rescaling  $Q$  allow us to set  $b \in \{-1, 1\}$  if  $b \neq 0$ . Thus, we obtain the subalgebras  $\mathfrak{s}_{1.2}^\delta$ .

For  $\hat{Q} = \mathcal{D}$  and  $\hat{Q} = \mathcal{P}^i + \mathcal{K}$ , removing the summands with  $\mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1$  and  $\mathcal{P}^0$  is analogous. The only possible further simplification is that in the case  $\hat{Q} = \mathcal{D}$ , the sign of the obtained value of  $b$  can be alternated using  $\mathcal{K}_*$  and alternating the sign of  $Q$ . This results in the collections of the subalgebras  $\mathfrak{s}_{1,3}^\nu$  and  $\mathfrak{s}_{1,4}^\mu$ , respectively.

Let  $\hat{Q} = 0$ . The classification of real binary cubics presented in the second table on page 26 in [24] implies that, up to the  $F$ -equivalence and rescaling  $Q$ , we have  $a_3 = 0$  and  $(3a_2, 3a_1, a_0) \in \{(1, \varepsilon, 0), (0, 1, 0), (0, 0, 1), (0, 0, 0)\}$ . If  $(a_3, a_2, a_1, a_0) \neq (0, 0, 0, 0)$ , then using one of the actions  $\mathcal{P}^3(\epsilon)_*$ ,  $\mathcal{P}^2(\epsilon)_*$ ,  $\mathcal{P}^1(\epsilon)_*$  and  $\mathcal{P}^0(\epsilon)_*$  we can set  $b = 0$ ; otherwise, we set  $b = 1$  by rescaling  $Q$ . Thus, we obtain the rest of subalgebras from the list given in the lemma's statement.  $\square$

**Lemma 5.** A complete list of  $G^{\text{ess}}$ -inequivalent two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  is given by

$$\begin{aligned} \mathfrak{s}_{2,1}^\mu &= \langle \mathcal{P}^i, \mathcal{D} + \mu\mathcal{I} \rangle, \quad \mathfrak{s}_{2,2}^\delta = \langle \mathcal{P}^i + \delta\mathcal{I}, \mathcal{P}^0 \rangle, \quad \mathfrak{s}_{2,3} = \langle \mathcal{P}^i, \mathcal{P}^0 + \mathcal{I} \rangle, \\ \mathfrak{s}_{2,4}^\mu &= \langle \mathcal{D} + \mu\mathcal{I}, \mathcal{P}^1 \rangle, \quad \mathfrak{s}_{2,5}^\mu = \langle \mathcal{D} + \mu\mathcal{I}, \mathcal{P}^0 \rangle, \\ \mathfrak{s}_{2,6} &= \langle \mathcal{P}^0, \mathcal{P}^1 \rangle, \quad \mathfrak{s}_{2,7} = \langle \mathcal{P}^0, \mathcal{P}^2 \rangle, \quad \mathfrak{s}_{2,8}^\varepsilon = \langle \mathcal{P}^1, \mathcal{P}^3 + \varepsilon\mathcal{P}^0 \rangle, \\ \mathfrak{s}_{2,9} &= \langle \mathcal{P}^i + \mathcal{P}^3, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,10}^\delta = \langle \mathcal{P}^i, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,11} = \langle \mathcal{D}, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,12} = \langle \mathcal{P}^i + \mathcal{K}, \mathcal{I} \rangle, \\ \mathfrak{s}_{2,13}^\varepsilon &= \langle \mathcal{P}^2 + \varepsilon\mathcal{P}^0, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,14} = \langle \mathcal{P}^1, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,15} = \langle \mathcal{P}^0, \mathcal{I} \rangle, \end{aligned}$$

where  $\varepsilon \in \{-1, 1\}$ ,  $\delta \in \{-1, 0, 1\}$ , and  $\mu$  is an arbitrary real constant.

*Proof.* A complete set of  $F$ -inequivalent  $d'$ -dimensional subalgebras of  $\mathfrak{f}$  with  $d' \leq 2$  consists of the subalgebras  $\langle \mathcal{P}^i, \mathcal{D} \rangle, \langle \mathcal{P}^i \rangle, \langle \mathcal{D} \rangle, \langle \mathcal{P}^i + \mathcal{K} \rangle$  and  $\{0\}$ . Therefore, without loss of generality, we can consider only two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  with basis vector fields of the general form  $Q_i = \hat{Q}_i + \check{Q}_i, i = 1, 2$ , where

$$\langle \hat{Q}_1, \hat{Q}_2 \rangle \in \{ \langle \mathcal{P}^i, \mathcal{D} \rangle, \langle \mathcal{P}^i, 0 \rangle, \langle \mathcal{D}, 0 \rangle, \langle \mathcal{P}^i + \mathcal{K}, 0 \rangle, \langle 0, 0 \rangle \},$$

and  $\check{Q}_i = a_{i3}\mathcal{P}^3 + 3a_{i2}\mathcal{P}^2 + 3a_{i1}\mathcal{P}^1 + a_{i0}\mathcal{P}^0 + b_i\mathcal{I}$  with real constants  $a_{ij}$  and  $b_i, i = 1, 2, j = 0, \dots, 3$ . We further need to simplify  $\check{Q}_i$  by linearly recombining the vector fields  $Q_i$  and acting on them by elements from  $G^{\text{ess}}$ , while simultaneously maintaining the closedness of the subalgebras with respect to the Lie bracket,  $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$ , under the condition  $\dim \langle Q_1, Q_2 \rangle = 2$  and preserving  $\hat{Q}_i$ .

We separately consider each of the values of  $\langle \hat{Q}_1, \hat{Q}_2 \rangle$  in the above set.

**$\langle \mathcal{P}^i, \mathcal{D} \rangle$ .** Then according to Lemma 4, up to the  $G^{\text{ess}}$ -equivalence, the vector field  $Q_1$  is equal to either  $\mathcal{P}^i + \mathcal{P}^3$  or  $\mathcal{P}^i + \delta\mathcal{I}$ . The first value is not appropriate, since then  $[Q_1, Q_2] = 2\mathcal{P}^i - 3\mathcal{P}^3 + \check{Q}$  with  $\check{Q} \in \langle \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle$  and thus  $[Q_1, Q_2] \notin \langle Q_1, Q_2 \rangle$  for any  $\check{Q}_2 \in \mathfrak{r}$ . For the second value, requiring the condition  $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$  implies  $a_{23} = a_{22} = a_{21} = \delta = 0$ . Acting by  $\mathcal{P}^0(a_{20})_*$  on  $Q_2$ , we can set  $a_{20} = 0$ . Thus, we obtain the subalgebra  $\mathfrak{s}_{2,1}^\mu$  with  $\mu = b_2$ .

In all the other cases, we have  $\hat{Q}_2 = 0$ . If, in addition,  $a_{2j} = 0, j = 0, \dots, 3$ , then  $Q_2 = \mathcal{I}$ , and thus, the basis elements  $Q_1$  and  $Q_2$  commute. The corresponding canonical forms of  $Q^1$  modulo the  $G^{\text{ess}}$ -equivalence are given by the basis elements of the one-dimensional algebras  $\mathfrak{s}_{1,1}, \dots, \mathfrak{s}_{1,7}$  from Lemma 4 up to neglecting the summands with  $\mathcal{I}$  due to the possibility of linearly combining  $Q_1$  with  $Q_2$ . This results in the subalgebras  $\mathfrak{s}_{2,9}, \dots, \mathfrak{s}_{2,15}$ . Hence, below we assume that  $(a_{23}, a_{22}, a_{21}, a_{20}) \neq (0, 0, 0, 0)$  and, moreover,  $\text{rank}(a_{ij})_{j=0, \dots, 3}^{i=1,2} = 2$  if  $\hat{Q}_1 = \hat{Q}_2 = 0$ .

**$\langle \mathcal{P}^i, 0 \rangle$ .** The closedness of  $\langle Q_1, Q_2 \rangle$  with respect to the Lie bracket implies  $a_{23} = a_{22} = a_{21} = a_{13}a_{20} = 0$  and thus  $a_{13} = 0$  since then  $a_{20} \neq 0$ . Hence,  $Q_1 = \mathcal{P}^i + b_1\mathcal{I}$  and  $Q_2 = \mathcal{P}^0 + b_2\mathcal{I}$ .

If  $b_2 = 0$ , we can rescale  $b_1$  to  $\delta$  by combining  $\mathcal{D}(\epsilon)_*$  with scaling the entire vector field  $Q^1$ . This results in the subalgebras  $\mathfrak{s}_{2,2}^\delta$ .

Let  $b_2 \neq 0$ . Simultaneously acting by  $\mathcal{D}(\epsilon)_*$  with the suitable value of  $\epsilon$ , rescaling  $b_2$  and, if  $b_2 < 0$ , acting by  $\mathcal{J}_*$  and alternating the sign of  $Q^2$ , we can set  $b_2 = 1$ . Then, the pushforward  $\mathcal{P}^1(b_1)_*$  preserves  $Q^2$ , and  $\mathcal{P}^1(b_1)_*Q_1 = \mathcal{P}^i + b_1Q^2$ . After linearly combining  $Q_1$  with  $Q^2$ , we obtain the subalgebras  $\mathfrak{s}_{2,3}$ .

**$\langle \mathcal{D}, 0 \rangle$ .** In view of Lemma 4, we can reduce  $Q_1$  to  $\mathcal{D} + \mu\mathcal{I}$  modulo the  $G^{\text{ess}}$ -equivalence. The condition  $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$  and the commutation relations of  $\mathcal{D}$  with basis elements of  $\mathfrak{r}$  imply that up to scaling of  $Q^2$ , we have

$$(a_{23}, 3a_{22}, 3a_{21}, a_{20}, b_2) \in \{(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0)\},$$

where the first and the second values are reduced by  $\mathcal{K}'_*$  to the third and the fourth values, respectively. Thus, we obtain the subalgebras  $\mathfrak{s}_{2,4}^\mu$  and  $\mathfrak{s}_{2,5}^\mu$ .

**( $\mathcal{P}^i + \mathcal{K}, \mathbf{0}$ ).** This pair is not appropriate since, from the closedness of  $\langle Q_1, Q_2 \rangle$  with respect to the Lie bracket, we straightforwardly derive  $a_{2j} = 0, j = 0, \dots, 3$ .

**( $\mathbf{0}, \mathbf{0}$ ).** Using Lemma 4, from the very beginning we can set  $Q_1 \in \{\mathcal{P}^2 + \varepsilon\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^0\}$ . Since  $\text{rank}(a_{ij})_{j=0,\dots,3}^{i=1,2} = 2$  here, simultaneously with  $b_1 = 0$  we can set  $b_2 = 0$  using the suitable pushforward  $\mathcal{P}^1(\epsilon)_*$ .

For  $Q = a_3\mathcal{P}^3 + 3a_2\mathcal{P}^2 + 3a_1\mathcal{P}^1 + a_0\mathcal{P}^0$  with  $\bar{a} := (a_0, a_1, a_2, a_3) \neq (0, 0, 0, 0)$ , we denote

$$\text{Discr}(Q) := a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 - 3a_1^2a_2^2 + 4a_1^3a_3.$$

The association with cubic binary forms implies that  $Q \in \{\mathcal{P}^0, \mathcal{P}^1\}$  modulo the  $G^{\text{ess}}$ -equivalence if and only if  $\text{Discr}(Q) = 0$ , cf. [24, Example 2.22]. To distinguish inequivalent subalgebras in the present classification case, we will consider lines in the space run by  $\bar{a}$ , where  $\text{Discr}(Q) = 0$ . We call such lines singular. It is obvious that two-dimensional subalgebras in  $\langle \mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \mathcal{P}^3 \rangle$  are  $G^{\text{ess}}$ -inequivalent if they possess different numbers of singular lines associated with their elements or, more precisely, different sets of multiplicities of such singular lines.

If  $Q_1 = \mathcal{P}^0$ , then the condition  $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$  implies  $a_{23} = 0$ , and we can set  $a_{20} = 0$  at any step by linearly combining  $Q_2$  with  $Q_1$ .

The case  $a_{22} = 0$  corresponds to the subalgebra  $\mathfrak{s}_{2,6}$ .  $\text{Discr}(Q) = 0$  for any  $Q \in \mathfrak{s}_{2,6}$ , that is, the subalgebra  $\mathfrak{s}_{2,6}$  possesses an infinite number of singular lines.

For  $a_{22} \neq 0$ , we divide  $Q_2$  by  $a_{22}$ , act by  $\mathcal{P}^1(\epsilon)_*$  with  $\epsilon = a_{21}/(2a_{22})$  and reset  $a_{20} = 0$ , linearly combining  $Q_2$  with  $Q_1$ , which gives the basis of the subalgebra  $\mathfrak{s}_{2,7}$ . For an arbitrary vector field  $Q = a_0\mathcal{P}^0 + 3a_2\mathcal{P}^2$  in  $\mathfrak{s}_{2,7}$ , we obtain  $\text{Discr}(Q) = 4a_0a_2^3$ , and thus, the subalgebra  $\mathfrak{s}_{2,7}$  possesses one singular line of multiplicity one and one singular line of multiplicity three.

If  $Q_1 = \mathcal{P}^1$ , then the condition  $[Q_1, Q_2] \in \langle Q_1, Q_2 \rangle$  implies  $a_{22} = 0$ , and we can set  $a_{21} = 0$  at any step, linearly combining  $Q_2$  with  $Q_1$ . We can assume  $a_{23} \neq 0$  since otherwise we again obtain the subalgebra  $\mathfrak{s}_{2,6}$ . Rescaling the basis element  $Q_2$ , we set  $a_{23} = 1$ . The subalgebra  $\langle \mathcal{P}^1, \mathcal{P}^3 \rangle$  is mapped by  $\mathcal{K}'_*$  to the subalgebra  $\mathfrak{s}_{2,7}$ . This is why the coefficient  $a_{20}$  should be nonzero as well. The action by  $\mathcal{D}(\epsilon)_*$  with the suitable value of  $\epsilon$  allows us to set  $a_{20} = \varepsilon$ , that is, we derive the subalgebras  $\mathfrak{s}_{2,8}^\varepsilon$ . For an arbitrary vector field  $Q = 3a_1\mathcal{P}^1 + a_3(\mathcal{P}^3 + \varepsilon\mathcal{P}^0)$  in  $\mathfrak{s}_{2,8}^\varepsilon$ , we obtain  $\text{Discr}(Q) = a_3(4a_1^3 + a_3^3)$ . This means that each of the subalgebras  $\mathfrak{s}_{2,8}^\varepsilon$  possesses two singular line of multiplicity one.

The consideration of the case  $Q_1 = \mathcal{P}^2 + \varepsilon\mathcal{P}^0$  is the most complicated. The span  $\langle Q_1, Q_2 \rangle$  is closed with respect to the Lie bracket if and only if  $a_{21} = -\varepsilon a_{23}$ . The coefficient  $a_{22}$  is set to zero by linearly combining  $Q_2$  with  $Q_1$ . If  $a_{23} = 0$ , then  $a_{21} = 0$  as well, and we again obtain the subalgebra  $\mathfrak{s}_{2,7}$ . Let then further  $a_{23} \neq 0$ . We scale  $Q_2$  to set  $a_{23} = 1$ , which leads to  $Q_2 = \mathcal{P}^3 - 3\varepsilon\mathcal{P}^1 + \alpha\mathcal{P}^0$  with  $\alpha := a_{20}$ . We intend to show that depending on the value of  $(\varepsilon, \alpha)$ , the subalgebra  $\mathfrak{s} := \langle Q_1, Q_2 \rangle$  is  $G^{\text{ess}}$ -equivalent to either the subalgebra  $\mathfrak{s}_{2,7}$  or the subalgebra  $\mathfrak{s}_{2,8}$ .

We compute that

$$\text{Discr}(Q) = 12\varepsilon a_2^4 + 4\alpha a_3 a_2^3 + 24a_3^2 a_2^2 + 12\varepsilon \alpha a_3^3 a_2 + a_3^4 \alpha^2 - 4a_3^4 \varepsilon$$

for a nonzero vector field  $Q = 3a_2Q_1 + a_3Q_2$  in  $\mathfrak{s}$ . If  $a_3 = 0$  and thus  $a_2 \neq 0$ , then  $\text{Discr}(Q) = 12\varepsilon a_2^4 \neq 0$ , that is, the corresponding line is not singular. Hence, we can further set  $a_3 \neq 0$  and assume without loss of generality that  $a_3 = 1$ . Thus, finding singular lines associated with the subalgebra  $\mathfrak{s}$  reduces to solving the quartic equation  $\text{Discr}(Q) = 0$  with respect to  $a_2$ . The substitution  $a_2 = z - \varepsilon\alpha/12$  and the division of the left-hand side by  $12\varepsilon$  reduce this equation to its monic depressed counterpart  $z^4 + pz^2 + qz + r = 0$ , where

$$p := -\frac{1}{24}\alpha^2 + 2\varepsilon, \quad q := \frac{\varepsilon}{216}\alpha^3 + \frac{2}{3}\alpha, \quad r := -\frac{1}{6912}\alpha^4 + \frac{\varepsilon}{72}\alpha^2 - \frac{1}{3}.$$

To find the number of (real) roots of this equation, following [3, p. 53] we compute

$$\delta := 256r^3 - 128p^2r^2 + 144pq^2r + 16p^4r - 27q^4 - 4p^3q^2 = -\frac{1}{432}(\alpha^2 + 16\varepsilon)^4,$$

$$L := 8pr - 9q^2 - 2p^3 = -\frac{\varepsilon}{12}(\alpha^2 + 16\varepsilon)^2.$$

Thus, there are only two cases for the number of roots of the equation  $\text{Discr}(Q) = 0$ . Note that in both cases, this number is not zero. Therefore, as indicated above, the subalgebra  $\mathfrak{s}$  contains, modulo the  $G^{\text{ess}}$ -equivalence, at least one of the vector fields  $\mathcal{P}^0$  and  $\mathcal{P}^1$ , and thus we can use the considerations of the cases where  $Q_1 = \mathcal{P}^0$  or  $Q_1 = \mathcal{P}^1$ . This is why the recognition of the corresponding representatives among subalgebras listed in the lemma’s statement depends only on the root structure of the equation  $\text{Discr}(Q) = 0$ .

1. If either  $\varepsilon = -1$  and  $\alpha \neq \pm 4$  or  $\varepsilon = 1$ , then  $\delta < 0$ , and hence the equation  $\text{Discr}(Q) = 0$  has two roots of multiplicity one, which corresponds to two singular lines of multiplicity one for the subalgebra  $\mathfrak{s}$ . Therefore, this algebra is  $G^{\text{ess}}$ -equivalent to one of the subalgebras  $\mathfrak{s}_{2,8}^\varepsilon$ .

2. If  $\varepsilon = -1$  and  $\alpha = \pm 4$ , then  $\delta = L = 0$  and  $p < 0$ , and thus, the equation  $\text{Discr}(Q) = 0$  has one root of multiplicity one and one root of multiplicity three, which corresponds to one singular line of multiplicity one and one singular line of multiplicity three for the subalgebra  $\mathfrak{s}$ . Hence, this algebra is  $G^{\text{ess}}$ -equivalent to the subalgebra  $\mathfrak{s}_{2,7}$ . □

We can also classify one and two-dimensional subalgebras of the entire algebra  $\mathfrak{g}$  up to the  $G$ -equivalence and show that only subalgebras of  $\mathfrak{g}^{\text{ess}}$  are essential in the course of classifying Lie reductions of the equation (2). Thus, according to the decomposition  $G = G^{\text{ess}} \times G^{\text{lin}}$  (see Section 3), an arbitrary transformation  $\Phi$  from  $G$  can be represented in the form  $\Phi = \mathcal{F} \circ \mathcal{Z}(f)$ , where  $\mathcal{F} \in G^{\text{ess}}$  and  $\mathcal{Z}(f) \in G^{\text{lin}}$ . To exhaustively describe the adjoint action of  $G$  on the algebra  $\mathfrak{g}$ , in view of the decomposition  $\mathfrak{g} = \mathfrak{g}^{\text{ess}} \ltimes \mathfrak{g}^{\text{lin}}$  it suffices to supplement the adjoint action of  $G^{\text{ess}}$  on  $\mathfrak{g}^{\text{ess}}$  with the adjoint actions of  $G^{\text{ess}}$  on  $\mathfrak{g}^{\text{lin}}$  and of  $G^{\text{lin}}$  on  $\mathfrak{g}^{\text{ess}}$ ,  $\mathcal{Z}(f)_*Q = Q + Q[f]\partial_u$  and  $\mathcal{F}_*\mathcal{Z}(f) = \mathcal{Z}(\mathcal{F}_*f)$ , whereas the adjoint action of  $G^{\text{lin}}$  on  $\mathfrak{g}^{\text{lin}}$  is trivial. Here  $Q$  is an arbitrary vector field from  $\mathfrak{g}^{\text{ess}}$ ,  $Q[f]$  denotes the evaluation of the characteristic  $Q[u]$  of  $Q$  at  $u = f$ , and  $\mathcal{F}$  is an arbitrary transformation from  $G^{\text{ess}}$ .

The classification of subalgebras of  $\mathfrak{g}$  is based on the classification of subalgebras of  $\mathfrak{g}^{\text{ess}}$ . This is due to the fact that subalgebras  $\mathfrak{s}_1$  and  $\mathfrak{s}_2$  of  $\mathfrak{g}$  are definitely  $G$ -inequivalent if  $\pi_{\mathfrak{g}^{\text{ess}}}\mathfrak{s}_1$  and  $\pi_{\mathfrak{g}^{\text{ess}}}\mathfrak{s}_2$  are  $G^{\text{ess}}$ -inequivalent. Here  $\pi_{\mathfrak{g}^{\text{ess}}}$  denotes the natural projection of  $\mathfrak{g}$  onto  $\mathfrak{g}^{\text{ess}}$  under the decomposition  $\mathfrak{g} = \mathfrak{g}^{\text{ess}} \dot{+} \mathfrak{g}^{\text{lin}}$  in the sense of vector spaces. In the course of classifying the subalgebras  $\mathfrak{s}$  of  $\mathfrak{g}$  of a fixed (finite) dimension, we partition the set of these subalgebras into subsets such that each subset consists of the subalgebras with the same (up to the  $G^{\text{ess}}$ -equivalence) projection  $\pi_{\mathfrak{g}^{\text{ess}}}\mathfrak{s}$ . As a result, each of these subsets contains no subalgebra being equivalent to a subalgebra from another subset.

The classification of one- and two-dimensional subalgebras of  $\mathfrak{g}$  up to the  $G$ -equivalence is given in the following two assertions.

**Lemma 6.** *A complete list of  $G$ -inequivalent one-dimensional subalgebras of  $\mathfrak{g}$  consists of the one-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  listed in Lemma 4 and the subalgebras of the form  $\langle \mathcal{Z}(f) \rangle$ , where the function  $f$  belongs to a fixed complete set of  $G^{\text{ess}}$ -inequivalent nonzero solutions of the equation (2).*

*Proof.* Consider an arbitrary one-dimensional subalgebra  $\mathfrak{s}$  of  $\mathfrak{g}$ . Let  $Q$  be its basis element. We can represent  $Q$  as  $Q = \hat{Q} + \mathcal{Z}(f)$  for some  $\hat{Q} \in \mathfrak{g}^{\text{ess}}$  and some solution  $f$  of the equation (2).

If  $\hat{Q} \neq 0$ , then Lemma 4 implies that, modulo the  $G^{\text{ess}}$ -equivalence, the vector field  $\hat{Q}$  can be assumed to be the basis element of one of the subalgebras  $\mathfrak{s}_{1,1}, \dots, \mathfrak{s}_{1,8}$  of  $\mathfrak{g}^{\text{ess}}$ . Then, we set  $f$  identically equal to zero by pushing  $Q$  forward with  $\mathcal{Z}(h)$ , where  $h$  is a solution of the equation (2) that in addition satisfies the constraint  $Q[h] + f = 0$ .<sup>4</sup>

<sup>4</sup>For  $\hat{Q} = \mathcal{I}$ , the constraint  $\hat{Q}[h] + f = 0$  just means  $h = -f$ . For the other values of  $\hat{Q}$  and locally analytical solutions  $f$  of the equation (2), the existence of locally analytical solutions of the system  $h_t + xh_y = h_{xx}, \hat{Q}[h] + f = 0$  with respect to  $h$  follows from Riquier’s theorem and the fact that  $\hat{Q}$  is a Lie-symmetry vector field of (2). We suppose this existence in the smooth case as well.

If  $\hat{Q} = 0$ , then  $Q = \mathcal{Z}(f)$ . It suffices to note that the  $G$ - and  $G^{\text{ess}}$ -equivalences coincide on  $\mathfrak{g}^{\text{lin}}$ .  $\square$

**Lemma 7.** *A complete list of  $G$ -inequivalent two-dimensional subalgebras of  $\mathfrak{g}$  consists of*

1. *the two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$  listed in Lemma 5,*
2. *the subalgebras of the form  $\langle \hat{Q}, \mathcal{Z}(f) \rangle$ , where  $\hat{Q}$  is the basis element of one of the one-dimensional subalgebras  $\mathfrak{s}_{1,1}, \dots, \mathfrak{s}_{1,7}$  of  $\mathfrak{g}^{\text{ess}}$  listed in Lemma 4, and the function  $f$  belongs to a fixed complete set of  $\text{St}_{G^{\text{ess}}}(\langle \hat{Q} \rangle)$ -inequivalent nonzero  $(\hat{Q} + \lambda \mathcal{I})$ -invariant solutions of the equation (2) with  $\text{St}_{G^{\text{ess}}}(\langle \hat{Q} \rangle)$  denoting the stabiliser subgroup of  $G^{\text{ess}}$  with respect to  $\langle \hat{Q} \rangle$  under the action of  $G^{\text{ess}}$  on  $\mathfrak{g}^{\text{ess}}$  and with  $\lambda \in \{0, 1\}$  if  $\hat{Q} \in \{\mathcal{P}^1, \mathcal{P}^0\}$ ,  $\lambda \in \{-1, 0, 1\}$  if  $\hat{Q} = \mathcal{P}^1$ ,  $\lambda \geq 0$  if  $\hat{Q} = \mathcal{P}^2 + \varepsilon \mathcal{P}^0$  and  $\lambda \in \mathbb{R}$  otherwise,*
3. *the subalgebras of the form  $\langle \mathcal{I}, \mathcal{Z}(f) \rangle$ , where the function  $f$  belongs to a fixed complete set of  $G^{\text{ess}}$ -inequivalent nonzero solutions of the equation (2), and*
4. *the subalgebras of the form  $\langle \mathcal{Z}(f^1), \mathcal{Z}(f^2) \rangle$ , where the function pair  $(f^1, f^2)$  belongs to a fixed complete set of  $G^{\text{ess}}$ -inequivalent (up to linearly recombining components) pairs of linearly independent solutions of the equation (2).*

*Proof.* Consider an arbitrary two-dimensional subalgebra  $\mathfrak{s}$  of  $\mathfrak{g}$ , and let  $Q_1$  and  $Q_2$  be its basis elements,  $\mathfrak{s} = \langle Q_1, Q_2 \rangle$  with  $[Q_1, Q_2] \in \mathfrak{s}$ . Due to the decomposition  $\mathfrak{g} = \mathfrak{g}^{\text{ess}} \ltimes \mathfrak{g}^{\text{lin}}$ , each  $Q_i$ ,  $i = 1, 2$ , can be represented in the form  $Q_i = \hat{Q}_i + \mathcal{Z}(f^i)$ , where  $\hat{Q}_i \in \mathfrak{g}^{\text{ess}}$  and the function  $f^i$  is a solution of the equation (2). As the principal  $G$ -invariant value in the course of classifying two-dimensional subalgebras of  $\mathfrak{g}$ , we choose the dimension  $d := \dim \pi_{\mathfrak{g}^{\text{ess}}} \mathfrak{s}$  of the projection of  $\mathfrak{s}$  onto  $\mathfrak{g}^{\text{ess}}$ .

**$d = 2$ .** Up to the  $G^{\text{ess}}$ -equivalence, the pair  $(\hat{Q}_1, \hat{Q}_2)$  can be assumed to be the chosen basis of one of the subalgebras  $\mathfrak{s}_{2,1}^\mu, \dots, \mathfrak{s}_{2,15}$  of  $\mathfrak{g}^{\text{ess}}$  listed in Lemma 5.

For the subalgebras  $\mathfrak{s}_{2,9}, \dots, \mathfrak{s}_{2,15}$ , we have  $Q_2 = \mathcal{I}$ . Pushing  $\mathfrak{s}$  forward with  $\mathcal{Z}(h)$ , where  $h = -f^2$ , we can set  $f^2 = 0$ . Then, the commutation relation  $[Q_1, Q_2] = 0$  implies  $f^1 = 0$ .

For the subalgebras  $\mathfrak{s}_{2,1}^\mu, \dots, \mathfrak{s}_{2,8}^\varepsilon$ , let  $\kappa_1$  and  $\kappa_2$  be the constants such that  $[\hat{Q}_1, \hat{Q}_2] = \kappa_1 \hat{Q}_1 + \kappa_2 \hat{Q}_2$ . Then, the functions  $f^1$  and  $f^2$  satisfy the constraint  $\hat{Q}_1[f^2] - \hat{Q}_2[f^1] = \kappa_1 f^1 + \kappa_2 f^2$ . Hence, we can set them to be identically zero using the pushforward  $\mathcal{Z}(h)_*$ , where  $h$  is a solution of the overdetermined system

$$h_x + xh_y = h_{xx}, \quad \hat{Q}_1[h] + f^1 = 0, \quad \hat{Q}_2[h] + f^2 = 0$$

with respect to  $h$ .<sup>5</sup>

In total, this results in the first family of subalgebras from the lemma’s statement.

**$d = 1$ .** Without loss of generality, we can assume  $\hat{Q}_2 = 0$  up to linearly recombining the basis elements  $Q_1$  and  $Q_2$  and thus  $f^2 \neq 0$ , whereas modulo the  $G^{\text{ess}}$ -equivalence, the vector field  $\hat{Q}_1$  can be assumed to be the basis element of one of the subalgebras  $\mathfrak{s}_{1,1}, \dots, \mathfrak{s}_{1,8}$  of  $\mathfrak{g}^{\text{ess}}$  that are listed in Lemma 4. Then Lemma 4 also implies that  $f^1 = 0$  up to the  $G^{\text{ess}}$ -equivalence.

For the subalgebras  $\mathfrak{s}_{1,1}, \dots, \mathfrak{s}_{1,7}$ , we have  $\hat{Q}_1 \notin \langle \mathcal{I} \rangle$ . Following the proof of Lemma 6, we can set the function  $f^1$  to be identically zero. Then, the closedness of the subalgebra  $\mathfrak{s}$  with respect to the Lie bracket implies the constraint  $\hat{Q}_1[f^2] = \lambda f^2$ . In other words, the function  $f^2$  is a  $(\hat{Q}_1 + \lambda \mathcal{I})$ -invariant solution of the equation (2). Transformations from the group  $\text{St}_{G^{\text{ess}}}(\langle \hat{Q}_1 \rangle)$  allow us to set, depending on the value of  $\hat{Q}_1$ , the restrictions on  $\lambda$  that are presented in item 2 of the lemma’s statement. The transformations from  $\text{St}_{G^{\text{ess}}}(\langle \hat{Q}_1 \rangle)$  that preserve  $\lambda$  also induce an equivalence relation on the corresponding set run by  $f^2$ .

Since  $\hat{Q}_1 = \mathcal{I}$  for the subalgebra  $\mathfrak{s}_{1,8}$ , in the way described in the proof of Lemma 6, we can set the function  $f^1$  to be identically zero. Then the only commutation relation of the algebra  $\mathfrak{s}$  is  $[Q_1, Q_2] = Q_2$ ,

In fact, the proof of this existence reduces to the existence of a solution of linear inhomogeneous partial differential equation in two independent variables, whose homogeneous counterpart coincides with a reduction of the equation (2) with respect to  $\langle \hat{Q} \rangle$ .

<sup>5</sup>Similarly to the proof of Lemma 6, for locally analytical solutions  $f^1$  and  $f^2$  of the equation (2) that satisfy the above constraint, the existence of locally analytical solutions of the last system follows from Riquier’s theorem and the fact that  $\hat{Q}_1$  and  $\hat{Q}_2$  are Lie-symmetry vector fields of (2), and we assume this existence in the smooth case as well.

**Table 1.** *G*-inequivalent Lie reductions of codimension one

| no.                | $u$                                              | $z_1$                                      | $z_2$                                        | Reduced equation                                                                   |
|--------------------|--------------------------------------------------|--------------------------------------------|----------------------------------------------|------------------------------------------------------------------------------------|
| 1.1                | $e^{\frac{3}{10}t(t^4-5tx+10y)}w$                | $y - \frac{1}{4}t^4$                       | $x - t^3$                                    | $z_2w_1 = w_{22} - 3z_1w$                                                          |
| 1.2 $^\delta$      | $e^{\delta t}w$                                  | $y$                                        | $x$                                          | $z_2w_1 = w_{22} - \delta w$                                                       |
| 1.3 $^v$           | $ t ^{\frac{1}{2}v-1}w$                          | $ t ^{-\frac{3}{2}}y$                      | $ t ^{-\frac{1}{2}}x$                        | $(2z_2 - 3\varepsilon'z_1)w_1 = 2w_{22} + z_2\varepsilon'w_2 - (v-2)\varepsilon'w$ |
| 1.4 $^\mu$         | $\frac{e^{-\psi(t,x,y,\mu)}}{t^2+1}w$            | $\frac{y}{(t^2+1)^{\frac{3}{2}}}$          | $\frac{(t^2+1)x-3ty}{(t^2+1)^{\frac{3}{2}}}$ | $z_2w_1 = 3z_1w_2 + w_{22} + (\mu + z_2^2)w$                                       |
| 1.5 $^\varepsilon$ | $ t ^{-\frac{1}{2}}e^{-\frac{\varepsilon}{4t}}w$ | $\frac{1}{3}t^3 + 2\varepsilon t - t^{-1}$ | $2y - (t + \varepsilon t^{-1})x$             | $w_1 = w_{22}$                                                                     |
| 1.6                | $w$                                              | $\frac{1}{3}t^3$                           | $y - tx$                                     | $w_1 = w_{22}$                                                                     |
| 1.7                | $w$                                              | $t$                                        | $x$                                          | $w_1 = w_{22}$                                                                     |

Here  $\varepsilon' = \text{sgn } t$ ,  $\varepsilon \in \{-1, 1\}$ ,  $\delta \in \{-1, 0, 1\}$ ,  $w = w(z_1, z_2)$  is the new unknown function of the new independent variables  $(z_1, z_2)$ , and

$$\psi(t, x, y, \mu) = \frac{3t^3y^2 + t(2x(t^2 + 1) - 3ty)^2}{4(t^2 + 1)^3} + \mu \arctan t.$$

which implies no constraint for the function  $f^2$ , and  $\text{St}_{G^{\text{ess}}}(\mathcal{I}) = G^{\text{ess}}$ . Hence in this case, the parameter function  $f^2$  should run through a fixed complete set of  $G^{\text{ess}}$ -inequivalent nonzero solutions of the equation (2).

Thus, we obtained the second and third families of subalgebras listed in the lemma’s statement.

$d = 0$ . This case is obvious since then  $\hat{Q}_1 = \hat{Q}_2 = 0$ . In other words, the subalgebra  $\langle Q_1, Q_2 \rangle$  is of the form  $\langle \mathcal{Z}(f^1), \mathcal{Z}(f^2) \rangle$ , where  $f^1$  and  $f^2$  are linearly independent solutions of the equation (2), which should be chosen modulo the  $G^{\text{ess}}$ -equivalence and linearly recombining.  $\square$

### 5. Lie reductions of codimension one

The classification of Lie reductions of the equation (2) to partial differential equations with two independent variables is based on the classification of one-dimensional subalgebras of the algebra  $\mathfrak{g}^{\text{ess}}$ , which is given in Lemma 4. In Table 1, for each of the subalgebras listed therein, we present an associated ansatz for  $u$  and the corresponding reduced equation. Throughout this section, the subscripts 1 and 2 of functions depending on  $(z_1, z_2)$  denote derivatives with respect to  $z_1$  and  $z_2$ , respectively. In this and the next sections, we mark Lie reductions and all related objects with complex labels  $d.i^*$ , where  $d, i$  and  $*$  are the dimension of the corresponding subalgebra, its number in the list of  $d$ -dimensional subalgebras of the algebra  $\mathfrak{g}^{\text{ess}}$ , which is given in Lemma 4 for  $d = 1$  or in Lemma 5 for  $d = 2$ , and the list of subalgebra parameters for a subalgebra family, respectively. We omit the superscript in the label when it is not essential.

Each of the reduced equations 1. $i$  presented in Table 1,  $i \in \{1, 2^\delta, 3^v, 4^\mu, 5^\varepsilon, 6, 7\}$ , is a linear homogeneous partial differential equation in two independent variables. Hence, its maximal Lie invariance algebra  $\mathfrak{g}_{1.i}$  contains the infinity-dimensional abelian ideal  $\{h(z_1, z_2)\partial_w\}$ , which is associated with the linear superposition of solutions of reduced equation 1. $i$  and thus assumed to be a trivial part of  $\mathfrak{g}_{1.i}$ . Here the parameter function  $h = h(z_1, z_2)$  runs through the solution set of reduced equation 1. $i$ . Moreover, the algebra  $\mathfrak{g}_{1.i}$  is the semidirect sum of the above ideal and a finite-dimensional subalgebra  $\mathfrak{a}_i$  called the essential Lie invariance algebra of reduced equation 1. $i$ , cf. Section 2. The algebras  $\mathfrak{a}_i$  are the following:

$$\begin{aligned} \mathfrak{a}_1 = \mathfrak{a}_3 = \mathfrak{a}_4 &= \langle w\partial_w \rangle, \\ \mathfrak{a}_2^0 &= \langle \partial_{z_1}, 3z_1\partial_{z_1} + z_2\partial_{z_2}, 9z_1^2\partial_{z_1} + 6z_1z_2\partial_{z_2} - (6z_1 + z_2^3)w\partial_w, w\partial_w \rangle, \\ \mathfrak{a}_2^\delta &= \langle \partial_{z_1}, w\partial_w \rangle \quad \text{if } \delta \neq 0, \\ \mathfrak{a}_5 = \mathfrak{a}_6 = \mathfrak{a}_7 &= \langle \partial_{z_1}, \partial_{z_2}, 2z_1\partial_{z_1} + z_2\partial_{z_2}, 2z_1\partial_{z_2} - z_2w\partial_w, \\ &\quad 4z_1^2\partial_{z_1} + 4z_1z_2\partial_{z_2} - (2z_1 + z_2^2)w\partial_w, w\partial_w \rangle. \end{aligned}$$

The equation (2) admits hidden symmetries with respect to a Lie reduction if the corresponding reduced equation possesses Lie symmetries that are not induced by Lie symmetries of the original equation (2). To check which Lie symmetries of reduced equation 1.i are induced by Lie symmetries of (2), we compute the normaliser of the subalgebra  $\mathfrak{s}_{1,i}$  in the algebra  $\mathfrak{g}^{\text{ess}}$ :

$$\begin{aligned} N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,1}) &= \langle \mathcal{P}^t + \mathcal{P}^3, \mathcal{I} \rangle, \quad N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,2}^0) = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{P}^0, \mathcal{I} \rangle, \quad N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,2}^\delta) = \langle \mathcal{P}^t, \mathcal{P}^0, \mathcal{I} \rangle \text{ if } \delta \neq 0, \\ N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,3}^v) &= \langle \mathcal{D}, \mathcal{I} \rangle, \quad N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,4}^\mu) = \langle \mathcal{P}^t + \mathcal{K}, \mathcal{I} \rangle, \quad N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,5}^\varepsilon) = \langle \mathcal{P}^2, \mathcal{P}^0, \mathcal{P}^3 - 3\varepsilon\mathcal{P}^1, \mathcal{I} \rangle, \\ N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,6}) &= \langle \mathcal{D}, \mathcal{P}^3, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle, \quad N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,7}) = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I} \rangle. \end{aligned}$$

The algebra of induced Lie symmetries of reduced equation 1.i is isomorphic to the quotient algebra  $N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,i})/\mathfrak{s}_{1,i}$ . Thus, all Lie symmetries of the reduced equation 1.i are induced by Lie symmetries of the original equation (2) if and only if  $\dim \mathfrak{a}_i = \dim N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,i}) - 1$ . Comparing the dimensions of  $N_{\mathfrak{g}^{\text{ess}}}(\mathfrak{s}_{1,i})$  and  $\mathfrak{a}_i$ , we conclude that all Lie symmetries of reduced equations 1.1, 1.3 and 1.4 are induced by Lie symmetries of (2). The subalgebras of induced symmetries in the algebras  $\mathfrak{a}_2^\delta$  and  $\mathfrak{a}_5, \dots, \mathfrak{a}_7$  are

$$\begin{aligned} \tilde{\mathfrak{a}}_2^0 &= \langle \partial_{z_1}, 3z_1\partial_{z_1} + z_2\partial_{z_2}, w\partial_w \rangle, \quad \tilde{\mathfrak{a}}_2^\delta = \langle \partial_{z_1}, w\partial_w \rangle \text{ if } \delta \neq 0, \\ \tilde{\mathfrak{a}}_5 &= \langle \partial_{z_2}, 2z_1\partial_{z_2} - z_2w\partial_w, w\partial_w \rangle, \\ \tilde{\mathfrak{a}}_6 &= \langle \partial_{z_2}, 2z_1\partial_{z_1} + z_2\partial_{z_2}, 2z_1\partial_{z_2} - z_2w\partial_w, w\partial_w \rangle, \\ \tilde{\mathfrak{a}}_7 &= \langle \partial_{z_1}, \partial_{z_2}, 2z_1\partial_{z_1} + z_2\partial_{z_2}, 2z_1\partial_{z_2} - z_2w\partial_w, w\partial_w \rangle. \end{aligned}$$

For each  $i \in \{2^0, 5, 6, 7\}$ , the elements of the complement  $\mathfrak{a}_i/\tilde{\mathfrak{a}}_i$  are nontrivial hidden symmetries of the equation (2) that are associated with reduction 1.i. See Appendix B on an optimised procedure for constructing exact solutions using hidden symmetries.

We discuss reduced equations 1.1–1.7 and present solutions of (2) that can be found using the known solutions of the reduced equations.

Using point transformations, we can further map reduced equations 1.1–1.4 to (1 + 1)-dimensional linear heat equations with potentials, which are of the general form  $w_1 = w_{22} + Vw$ , where  $V$  is an arbitrary function of  $(z_1, z_2)$ . These transformations and mapped equations are

$$\begin{aligned} 1.1. \quad & \tilde{z}_1 = \frac{9}{4}\tilde{\varepsilon}z_1, \quad \tilde{z}_2 = |z_2|^{\frac{3}{2}}, \quad \tilde{w} = |z_2|^{\frac{1}{4}}w \quad \text{with } \tilde{\varepsilon} := \text{sgn } z_2: \quad \tilde{w}_1 = \tilde{w}_{22} - \frac{1}{9} \left( \frac{16}{3}\tilde{\varepsilon}\tilde{z}_1\tilde{z}_2^{-\frac{2}{3}} - \frac{5}{4}\tilde{z}_2^{-2} \right) \tilde{w}; \\ 1.2^\delta. \quad & \tilde{z}_1 = \frac{9}{4}\tilde{\varepsilon}z_1, \quad \tilde{z}_2 = |z_2|^{\frac{3}{2}}, \quad \tilde{w} = |z_2|^{\frac{1}{4}}w \quad \text{with } \tilde{\varepsilon} := \text{sgn } z_2: \quad \tilde{w}_1 = \tilde{w}_{22} - \frac{1}{9} \left( 4\delta\tilde{z}_2^{-\frac{2}{3}} - \frac{5}{4}\tilde{z}_2^{-2} \right) \tilde{w}; \\ 1.3^v. \quad & \tilde{z}_1 = \frac{9}{4}\tilde{\varepsilon}z_1, \quad \tilde{z}_2 = |z_2 - \frac{3}{2}\varepsilon'z_1|^{\frac{3}{2}}, \quad \tilde{w} = |2z_2 - 3\varepsilon'z_1|^{\frac{1}{4}}e^{\frac{1}{8}z_2(4\varepsilon'z_2 - 9z_1)}w \quad \text{with } \tilde{\varepsilon} := \text{sgn}(z_2 - \frac{3}{2}\varepsilon'z_1): \\ & \tilde{w}_1 = \tilde{w}_{22} + \frac{1}{144}\tilde{z}_2^{-\frac{2}{3}} \left( 16\tilde{z}_1^2 - 40(\tilde{z}_2^{\frac{2}{3}} + \frac{2}{3}\varepsilon'\tilde{z}_1)^2 + 20\tilde{z}_2^{-\frac{4}{3}} - 32\varepsilon'v \right) \tilde{w}; \\ 1.4^\mu. \quad & \tilde{z}_1 = \frac{9}{4}\tilde{\varepsilon}z_1, \quad \tilde{z}_2 = |z_2|^{\frac{3}{2}}, \quad \tilde{w} = |z_2|^{\frac{1}{4}}e^{\frac{3}{2}z_1z_2}w \quad \text{with } \tilde{\varepsilon} := \text{sgn } z_2: \\ & \tilde{w}_1 = \tilde{w}_{22} + \frac{1}{9} \left( \frac{5}{4}\tilde{z}_2^{-2} + 10\tilde{z}_2^{\frac{2}{3}} + (4\mu - 9z_1^2)\tilde{z}_2^{-\frac{2}{3}} \right) \tilde{w}. \end{aligned}$$

The essential Lie invariance algebras of reduced equations 1.1, 1.3 and 1.4 are trivial, and thus, the further consideration of these equations within the classical framework gives no results.

Mapped reduced equation 1.2<sup>0</sup> coincides with the linear heat equation with potential  $V = \mu z_2^{-2}$ , where  $\mu = \frac{5}{36}$ . Hence, we construct the following family of solutions of the equation (2):

$$\bullet \quad u = |x|^{-\frac{1}{4}}\theta^\mu \left( \frac{9}{4}\tilde{\varepsilon}y, |x|^{\frac{3}{2}} \right) \quad \text{with } \mu = \frac{5}{36},$$

where  $\theta^\mu = \theta^\mu(z_1, z_2)$  is an arbitrary solution of the equation  $\theta_1^\mu = \theta_{22}^\mu + \mu z_2^{-2}\theta^\mu$ , and  $\tilde{\varepsilon} := \text{sgn } x$ . A complete collection of inequivalent Lie invariant solutions of equations of such form with general  $\mu \neq 0$  is presented in Appendix A.

The essential Lie invariance algebra of reduced equation 1.2<sup>δ</sup> with  $\delta \neq 0$  is induced by the subalgebra  $\langle \mathcal{P}^0, \mathcal{I} \rangle$  of  $\mathfrak{g}^{\text{ess}}$ . The subalgebras of  $\mathfrak{a}_2^\delta$  that are appropriate for Lie reduction of reduced equation 1.2<sup>δ</sup> are exhausted by the subalgebras of the form  $\langle \partial_{z_1} + \nu w\partial_w \rangle$ , which are respectively induced by the subalgebras  $\langle \mathcal{P}^0 + \nu\mathcal{I} \rangle$  of  $\mathfrak{g}^{\text{ess}}$ . This is why the family of solutions of reduced equation 1.2<sup>δ</sup> that are invariant with



**Table 2.** *G*-inequivalent Lie reductions of codimension two

| no.                | $u$                                                        | $\omega$                     | Reduced equation                                                                             |
|--------------------|------------------------------------------------------------|------------------------------|----------------------------------------------------------------------------------------------|
| 2.1 $^\mu$         | $ y ^{\frac{1}{3}(\mu-2)}\varphi$                          | $y^{-1}x^3$                  | $9\omega\varphi_{\omega\omega} + (\omega + 6)\varphi_\omega = \frac{1}{3}(\mu - 2)\varphi$   |
| 2.2 $^\delta$      | $e^{\delta t}\varphi$                                      | $x$                          | $\varphi_{\omega\omega} = \delta\varphi$                                                     |
| 2.3                | $e^t\varphi$                                               | $x$                          | $\varphi_{\omega\omega} = \omega\varphi$                                                     |
| 2.4 $^\mu$         | $ t ^{\frac{1}{2}\mu-1}\varphi$                            | $ t ^{-\frac{3}{2}}(y - tx)$ | $2\varphi_{\omega\omega} = \varepsilon'(\mu - 2)\varphi - 3\varepsilon'\omega\varphi_\omega$ |
| 2.5 $^\mu$         | $ t ^{\frac{1}{2}\mu-1}\varphi$                            | $ t ^{-\frac{1}{2}}x$        | $2\varphi_{\omega\omega} = \varepsilon'(\mu - 2)\varphi - \varepsilon'\omega\varphi_\omega$  |
| 2.6                | $\varphi$                                                  | $t$                          | $\varphi_\omega = 0$                                                                         |
| 2.7                | $e^{-\frac{x^2}{4t}}\varphi$                               | $t$                          | $2\omega\varphi_\omega + \varphi = 0$                                                        |
| 2.8 $^\varepsilon$ | $e^{-\frac{3}{2}\frac{(y-tx)^2}{2t^3-\varepsilon}}\varphi$ | $t$                          | $(2\omega^3 - \varepsilon)\varphi_\omega + 3\omega^2\varphi = 0$                             |

Here  $\varepsilon' = \text{sgn } t$ ,  $\varphi = \varphi(\omega)$  is the new unknown function of the invariant independent variable  $\omega$ .

respect to  $\langle \partial_{z_1} + \nu w \partial_w \rangle$  gives a family of solutions of the equation (2) that is contained, up to the  $G^{\text{ess}}$ -equivalence, in the family of  $\langle \mathcal{P}^0 \rangle$ -invariant solutions, see reduction 1.7.

Each of reduced equations 1.5–1.7 itself coincides with the classical (1 + 1)-dimensional linear heat equation, which leads to the following families of solutions of the equation (2):

- $u = |t|^{-\frac{1}{2}}e^{-\frac{x^2}{4t}}\theta(z_1, z_2)$ , where  $z_1 = \frac{1}{3}t^3 + 2\varepsilon t - t^{-1}$ ,  $z_2 = 2y - (t + \varepsilon t^{-1})x$ ,
- $u = \theta(z_1, z_2)$ , where  $z_1 = \frac{1}{3}t^3$ ,  $z_2 = y - tx$ ,
- $u = \theta(z_1, z_2)$ , where  $z_1 = t$ ,  $z_2 = x$ .

Here  $\theta = \theta(z_1, z_2)$  is an arbitrary solution of the (1 + 1)-dimensional linear heat equation,  $\theta_1 = \theta_{22}$ . An enhanced complete collection of inequivalent Lie invariant solutions of this equation was presented in [40, Section A], following Examples 3.3 and 3.17 in [25]. These solutions exhaust, up to combining the  $G^{\text{ess}}$ -equivalence and linear superposition with each other, the set of known exact solutions of this equation that are expressed in a closed form in terms of elementary and special functions.

### 6. Lie reductions of codimension two

Based on the list of  $G^{\text{ess}}$ -inequivalent two-dimensional subalgebras from Lemma 5, we carry out the exhaustive classification of codimension-two Lie reductions of the equation (2). In Table 2, for each of these subalgebras, we present an associated ansatz for  $u$  and the corresponding reduced ordinary differential equation.

We find (nonzero) exact solutions of the constructed reduced equations and present the associated exact solutions of the equation (2). Hereafter,  $C_0, C_1$  and  $C_2$  are arbitrary constants.

**2.1.** Only this codimension-two Lie reduction results in new exact solutions of (2) in comparison with those constructed using Lie reductions with respect to one-dimensional subalgebras. The substitution  $\varphi = |\omega|^{\frac{1}{3}}e^{-\frac{\omega}{3}}\tilde{\varphi}(\omega)$  maps reduced equation 2.1 to a Kummer’s equation

$$9\omega\tilde{\varphi}_{\omega\omega} + (12 - \omega)\tilde{\varphi}_\omega = \frac{\mu + 1}{3}\tilde{\varphi},$$

whose general solution is

$$\tilde{\varphi} = C_1M\left(\frac{\mu + 1}{3}, \frac{4}{3}, \frac{\omega}{9}\right) + C_2U\left(\frac{\mu + 1}{3}, \frac{4}{3}, \frac{\omega}{9}\right),$$

where  $M(a, b, z)$  and  $U(a, b, z)$  denote the standard solutions of general Kummer’s equation  $z\varphi_{zz} + (b - z)\varphi_z - a\varphi = 0$ . After taking  $\kappa := \frac{1}{3}(\mu + 1)$  instead of  $\mu$  as an arbitrary constant parameter, the

corresponding family of particular solutions of the equation (2) can be represented in the form

- $u = x|y|^{\kappa-\frac{4}{3}}e^{-\tilde{\omega}} \left( C_1M\left(\kappa, \frac{4}{3}, \tilde{\omega}\right) + C_2U\left(\kappa, \frac{4}{3}, \tilde{\omega}\right) \right)$  with  $\tilde{\omega} := \frac{x^3}{9y}$ .

It also admits the equivalent representation

- $u = x^{-1}|y|^{\kappa-\frac{2}{3}}e^{-\frac{\tilde{\omega}}{2}} \left( C_1M_{\frac{2}{3}-\kappa, \frac{1}{6}}(\tilde{\omega}) + C_2W_{\frac{2}{3}-\kappa, \frac{1}{6}}(\tilde{\omega}) \right)$  with  $\tilde{\omega} := \frac{x^3}{9y}$

in terms of the Whittaker functions  $M_{a,b}(z)$  and  $W_{a,b}(z)$ , which constitute the fundamental solution set of the Whittaker equation (16), due to a relation of these functions to Kummer’s functions,

$$M_{a,b}(z) := e^{-\frac{z}{2}}z^{b+\frac{1}{2}}M\left(b-a+\frac{1}{2}, 1+2b, z\right),$$

$$W_{a,b}(z) := e^{-\frac{z}{2}}z^{b+\frac{1}{2}}U\left(b-a+\frac{1}{2}, 1+2b, z\right).$$

The rest of the reductions only lead to solutions of the equation (2), each of which is  $G^{\text{ess}}$ -equivalent to a solution of form 1.6 or 1.7 given in Section 5. Indeed, each of the corresponding subalgebras  $\mathfrak{s}_{2,2}, \dots, \mathfrak{s}_{2,8}$  contains, up to the  $G^{\text{ess}}$ -equivalence for the subalgebra  $\mathfrak{s}_{2,3}$ , at least one of the vector fields  $\mathcal{P}_0$  and  $\mathcal{P}_1$ . Nevertheless, we discuss these reductions below in order to explicitly present solutions of (2) that are invariant with respect to two-dimensional algebras of Lie-symmetry vector fields of (2).

**2.2<sup>δ</sup>.** Reduced equation 2.2<sup>δ</sup> trivially integrates to

$$\varphi = \begin{cases} C_1e^{\omega} + C_2e^{-\omega} & \text{if } \delta = 1, \\ C_1\omega + C_2 & \text{if } \delta = 0, \\ C_1\sin\omega + C_2\cos\omega & \text{if } \delta = -1. \end{cases}$$

**2.3.** Reduced equation 2.3 is the Airy equation, whose general solution is  $\varphi = C_1\text{Ai}(\omega) + C_2\text{Bi}(\omega)$ .

**2.4, 2.5.** Substituting  $\varphi = \omega e^{-\frac{3}{4}\varepsilon'\omega^2}\tilde{\varphi}(\omega)$  and  $\varphi = \omega e^{-\frac{1}{4}\varepsilon'\omega^2}\tilde{\varphi}(\omega)$  into reduced equations 2.4 and 2.5, respectively, we obtain the equations

$$2\omega\tilde{\varphi}_{\omega\omega} + (4 - 3\varepsilon'\omega^2)\tilde{\varphi}_{\omega} = \varepsilon'(\mu + 4)\omega\tilde{\varphi},$$

$$2\omega\tilde{\varphi}_{\omega\omega} + (4 - \varepsilon'\omega^2)\tilde{\varphi}_{\omega} = \varepsilon'\mu\omega\tilde{\varphi},$$

whose general solutions are

$$\tilde{\varphi} = C_1M\left(\frac{\mu}{6} + \frac{2}{3}, \frac{3}{2}, \frac{3}{4}\varepsilon'\omega^2\right) + C_2U\left(\frac{\mu}{6} + \frac{2}{3}, \frac{3}{2}, \frac{3}{4}\varepsilon'\omega^2\right),$$

$$\tilde{\varphi} = C_1M\left(\frac{\mu}{2}, \frac{3}{2}, \frac{1}{4}\varepsilon'\omega^2\right) + C_2U\left(\frac{\mu}{2}, \frac{3}{2}, \frac{1}{4}\varepsilon'\omega^2\right).$$

**2.6–2.8.** Here the reduced equations are ordinary first-order differential equations, which are trivially integrated to

$$2.6. \varphi = C_0, \quad 2.7. \varphi = \frac{C_0}{\sqrt{|\omega|}}, \quad 2.8. \varphi = \frac{C_0}{\sqrt{|2\omega^3 - \varepsilon|}}.$$

Substituting the above general solutions of reduced equations 2.2–2.8 into the corresponding ansatzes, we construct the following families of particular solutions of the equation (2), which are contained (as mentioned earlier, up to the  $G^{\text{ess}}$ -equivalence for the fourth family) in the solution families obtained in Section 5 using Lie reductions 1.6 and 1.7:

- $u = C_1e^{t+x} + C_2e^{t-x},$
- $u = C_1x + C_2,$
- $u = C_1e^{-t}\sin x + C_2e^{-t}\cos x,$
- $u = C_1e^y\text{Ai}(x) + C_2e^y\text{Bi}(x),$

- $u = |t|^{\frac{\mu-5}{2}}(y-tx)e^{-\tilde{\omega}} \left( C_1M\left(\frac{\mu}{6} + \frac{2}{3}, \frac{3}{2}, \tilde{\omega}\right) + C_2U\left(\frac{\mu}{6} + \frac{2}{3}, \frac{3}{2}, \tilde{\omega}\right) \right)$  with  $\tilde{\omega} := \frac{3(y-tx)^2}{4t^3}$ ,
- $u = |t|^{\frac{\mu-3}{2}}xe^{-\tilde{\omega}} \left( C_1M\left(\frac{\mu}{2}, \frac{3}{2}, \tilde{\omega}\right) + C_2U\left(\frac{\mu}{2}, \frac{3}{2}, \tilde{\omega}\right) \right)$  with  $\tilde{\omega} := \frac{x^2}{4t}$ ,
- $u = C_0$ ,      •  $u = \frac{C_0}{\sqrt{|t|}}e^{-\frac{x^2}{4t}}$ ,      •  $u = \frac{C_0}{\sqrt{|2t^3 - \varepsilon|}}e^{-\frac{3}{2}\frac{(y-tx)^2}{2t^3 - \varepsilon}}$ .

Considering nonzero solutions from this section up to the  $G^{\text{ess}}$ -equivalence, we can set the constant  $C_0$  and one of the constants  $C_1$  and  $C_2$  to one. Taking into account the linear superposition principle for solutions of the equation (2), it suffices to present only linearly independent solutions.

### 7. Lie reductions to algebraic equations

As for an equation in three independent variables, the maximal codimension of Lie reductions for the equation (2) is equal to three, and such reductions lead to algebraic equations. Let us show that these reductions do not give new explicit solutions of the equation (2).

Any three-dimensional subalgebra of the algebra  $\mathfrak{g}$  that is appropriate for Lie reduction is  $G$ -equivalent to a subalgebra  $\mathfrak{s}$  of  $\mathfrak{g}^{\text{ess}}$ . If the subalgebra  $\mathfrak{s}$  nontrivially intersects the radical  $\mathfrak{r}$  of  $\mathfrak{g}^{\text{ess}}$ , then any  $\mathfrak{s}$ -invariant solution of (2) is  $G^{\text{ess}}$ -equivalent to a solution from one of families 1.5–1.7 constructed in Section 5. Otherwise, the natural projection of the subalgebra  $\mathfrak{s}$  onto the Levi factor  $\mathfrak{f} = \langle \mathcal{P}^t, \mathcal{D}, \mathcal{K} \rangle$  under the Levi decomposition  $\mathfrak{g}^{\text{ess}} = \mathfrak{f} \ltimes \mathfrak{r}$  is three-dimensional. Therefore, the subalgebra  $\mathfrak{s}$  is nonsolvable, and thus it is simple, that is, it is a Levi factor of the algebra  $\mathfrak{g}^{\text{ess}}$ . According to the Levi–Malcev theorem, stating that any two Levi factors of a finite-dimensional Lie algebra are conjugate by an inner automorphism generated by elements of the corresponding nilradical, the subalgebra  $\mathfrak{s}$  is  $G^{\text{ess}}$ -equivalent to  $\mathfrak{f}$ . An ansatz constructed using the subalgebra  $\mathfrak{f}$

$$\bullet \quad u = C_0y^{-\frac{2}{3}}e^{-\frac{x^3}{9y}},$$

where  $C_0$  plays the role of the unknown constant, identically satisfies the equation (2). Up to the  $G^{\text{ess}}$ -equivalence, we can set the constant  $C_0$  equal to one. This solution belongs to the family of particular solutions obtained with reduction 2.1 $^\mu$  in Section 6, where  $\mu = 0$  or, equivalently,  $\kappa = 1/3$ , since  $U\left(\frac{1}{3}, \frac{4}{3}, z\right) = z^{-\frac{1}{3}}$ .

### 8. Generalised reductions and generating solutions with symmetry operators

We can construct more general families of exact solutions of the equation (2) using generalised symmetries. Since the equation (2) is linear and homogeneous, each first-order linear differential operator in total derivatives that is associated with an essential Lie symmetry of this equation is its recursion operator. Thus, it is obvious that the equation (2) admits, in particular, the generalised vector fields  $((\mathfrak{P}^2 + \varepsilon\mathfrak{P}^0)^n u)_\partial u$ ,  $((\mathfrak{P}^1)^n u)_\partial u$  and  $((\mathfrak{P}^0)^n u)_\partial u$  as its generalised symmetries. Here and in what follows,

$$\mathfrak{P}^3 := 3t^2D_x + t^3D_y - 3(y-tx), \quad \mathfrak{P}^2 := 2tD_x + t^2D_y + x, \quad \mathfrak{P}^1 := D_x + tD_y, \quad \mathfrak{P}^0 := D_y$$

are the differential operators in total derivatives that are associated with the Lie-symmetry vector fields  $-\mathcal{P}^3, -\mathcal{P}^2, -\mathcal{P}^1$  and  $-\mathcal{P}^0$  of the equation (2), and  $D_x$  and  $D_y$  denote the operators of total derivatives with respect to  $x$  and  $y$ , respectively. The corresponding invariant solutions are constructed by means of solving the equation (2) with the associated invariant surface condition,  $(\mathfrak{P}^2 + \varepsilon\mathfrak{P}^0)^n u = 0$ ,  $(\mathfrak{P}^1)^n u = 0$  or  $(\mathfrak{P}^0)^n u = 0$ , respectively. For  $n = 2$ , it is easy to derive the representation for such solutions in terms of solutions of the (1 + 1)-dimensional linear heat equation,

$$\bullet \quad u = |t|^{-\frac{1}{2}}e^{-\frac{x^2}{4t}} \left( (t - \varepsilon t^{-1})\theta_2^1 - \frac{x}{2t}\theta^1 + \theta^0 \right),$$

where  $z_1 = \frac{1}{3}t^3 + 2\epsilon t - t^{-1}$ ,  $z_2 = 2y - (t + \epsilon t^{-1})x$ ,

- $u = x\theta^1 - t^2\theta_2^1 + \theta^0$ , where  $z_1 = \frac{1}{3}t^3$ ,  $z_2 = y - tx$ ,
- $u = 4y\theta_{22}^1 + x^2\theta_2^1 - x\theta^1 + \theta^0$ , where  $z_1 = t$ ,  $z_2 = x$ .

Here  $\theta^i = \theta^i(z_1, z_2)$  is an arbitrary solution of the  $(1 + 1)$ -dimensional linear heat equation,  $\theta_i^i = \theta_{22}^i$ ,  $i = 0, 1$ . As in Section 5, the subscripts 1 and 2 of functions depending on  $(z_1, z_2)$  denote derivatives with respect to  $z_1$  and  $z_2$ , respectively.

In fact, for each of the above generalised symmetries, we can construct the corresponding family of invariant solutions using algebraic tools and the structure of the algebra  $\mathfrak{g}$ , or, more specifically, of its radical  $\mathfrak{r}$ . Since  $[\mathcal{P}^2 + \epsilon\mathcal{P}^0, \mathcal{P}^1] = \mathcal{I}$ ,  $[\mathcal{P}^1, \mathcal{P}^2] = -\mathcal{I}$  and  $[\mathcal{P}^0, \mathcal{P}^3] = 3\mathcal{I}$ , then

$$\begin{aligned} (\mathfrak{P}^2 + \epsilon\mathfrak{P}^0)^n (\mathfrak{P}^1)^{n-1} \left( |t|^{-\frac{1}{2}} e^{-\frac{t^2}{4t}} \theta(z_1, z_2) \right) &= 0, \\ (\mathfrak{P}^1)^n (\mathfrak{P}^2)^{n-1} (\theta(z_1, z_2)) &= 0, \\ (\mathfrak{P}^0)^n (\mathfrak{P}^3)^{n-1} (\theta(z_1, z_2)) &= 0 \end{aligned} \tag{9}$$

for arbitrary solutions  $\theta = \theta(z_1, z_2)$  of the  $(1 + 1)$ -dimensional linear heat equation,  $\theta_1 = \theta_{22}$ , whereas the expression in the left-hand side of each equality in (9) in general becomes nonvanishing after replacing the  $n$ th power of the first operator by its  $(n - 1)$ th power. The expressions for the variables  $z_1$  and  $z_2$  should be taken from the corresponding item in the above list. Recall that the operators  $\mathfrak{P}^0, \dots, \mathfrak{P}^3$  preserve the solution set of the equation (2). Therefore, the families of its solutions that are invariant with respect to its generalised symmetries  $((\mathfrak{P}^2 + \epsilon\mathfrak{P}^0)^n u)_{\partial_u}$ ,  $((\mathfrak{P}^1)^n u)_{\partial_u}$  or  $((\mathfrak{P}^0)^n u)_{\partial_u}$ , respectively, take the form

- $u = \sum_{k=0}^{n-1} (\mathfrak{P}^1)^k \left( |t|^{-\frac{1}{2}} e^{-\frac{t^2}{4t}} \theta^k(z_1, z_2) \right)$  with  $z_1 = \frac{1}{3}t^3 + 2\epsilon t - t^{-1}$ ,  $z_2 = 2y - (t + \epsilon t^{-1})x$ ,
- $u = \sum_{k=0}^{n-1} (\mathfrak{P}^2)^k \theta^k(z_1, z_2)$  with  $z_1 = \frac{1}{3}t^3$ ,  $z_2 = y - tx$ ,
- $u = \sum_{k=0}^{n-1} (\mathfrak{P}^3)^k \theta^k(z_1, z_2)$  with  $z_1 = t$ ,  $z_2 = x$ ,

where  $\theta^k = \theta^k(z_1, z_2)$  are arbitrary solutions of the  $(1 + 1)$ -dimensional linear heat equation,  $\theta_1^k = \theta_{22}^k$ ,  $k = 0, \dots, n - 1$ . Note that for  $n = 2$ , the first two families in this list have exactly the same representations as the first two families in the previous list, and

$$\mathfrak{P}^3 \tilde{\theta}^1 + \tilde{\theta}^0 = 4y\theta_{22}^1 + x^2\theta_2^1 - x\theta^1 + \theta^0$$

if  $\tilde{\theta}^1 = -\frac{4}{3}\theta_1^1$  and  $\tilde{\theta}^0 = \theta^0 + 4t^2\theta_{12}^1 + 4tx\theta_{22}^1 + (x^2 - 2t)\theta_2^1 - 2t\theta_2^1 - x\theta^1$ .

Up to the  $G^{\text{ess}}$ -equivalence, the generalised vector fields  $((\mathfrak{P}^2 + \epsilon\mathfrak{P}^0)^n u)_{\partial_u}$ ,  $((\mathfrak{P}^1)^n u)_{\partial_u}$  and  $((\mathfrak{P}^0)^n u)_{\partial_u}$  exhaust those generalised symmetries of the equation (2) that are obtained using powers of operators associated with vector fields from  $\mathfrak{r}$ . It is much more challenging to use vector fields from  $\mathfrak{g} \setminus \mathfrak{r}$  for generalised reductions in a similar way. The question of whether operators associated with vector fields from  $\mathfrak{g} \setminus \mathfrak{r}$  can generate solutions that are  $G^{\text{ess}}$ -inequivalent to the above solutions needs a careful analysis.

### 9. On Kramers equations

An important subclass of the class  $\bar{\mathcal{F}}$  of  $(1 + 2)$ -dimensional ultraparabolic Fokker–Planck equations, which are of the general form (1), is the class  $\mathcal{K}$  of Kramers equations (more commonly called the Klein–Kramers equations)

$$u_t + xu_y = F(y)u_x + \gamma(xu + u_x)_x.$$

These equations describe the evolution of the probability density function  $u(t, x, y)$  of a Brownian particle in the phase space  $(y, x)$  in one spatial dimension. Here, the variables  $t, y$  and  $x$  play the role of the time, the position and the momentum, respectively,  $F$  is an arbitrary smooth function of  $y$  that is the derivative of the external potential with respect to  $y$ , and  $\gamma$  is an arbitrary nonzero constant, which is related to the friction coefficient. Up to a simple scale equivalence of Kramers equations, without loss of generality, one can assume  $\gamma = 1$ . The subclass  $\mathcal{K}$  is singled out from the class  $\bar{\mathcal{F}}$  by the auxiliary differential constraints  $C = 0, A^0 = A_x^1 = A^2, A_t^0 = A_x^0 = A_y^0 = 0$  and  $A_t^1 = 0$ .

A preliminary group classification of the class  $\mathcal{K}$  was presented in [37]. In particular, it was proved that the essential Lie invariance algebra of an equation from the class  $\mathcal{K}$  is eight-dimensional if and only if  $F(y) = ky$  up to shifts of  $y$ , where  $k = -\frac{3}{4}\gamma^2$  or  $k = \frac{3}{16}\gamma^2$ .

Any equation from the class  $\bar{\mathcal{F}}$  with eight-dimensional essential Lie symmetry algebra is  $\mathcal{G}_{\bar{\mathcal{F}}}$ -equivalent to the remarkable Fokker–Planck equation (2). (We will present the proof of this fact as well as the complete group classification of the class  $\bar{\mathcal{F}}$  in a future paper.) Therefore, there exist point transformations that map the equations

$$u_t + xu_y = \gamma u_{xx} + \gamma \left( x - \frac{3}{4}\gamma y \right) u_x + \gamma u, \tag{10}$$

$$u_t + xu_y = \gamma u_{xx} + \gamma \left( x + \frac{3}{16}\gamma y \right) u_x + \gamma u \tag{11}$$

to the equation (2).

Bases of the essential Lie invariance algebras  $\mathfrak{g}_{(10)}^{\text{ess}}$  and  $\mathfrak{g}_{(11)}^{\text{ess}}$  of the equations (10) and (11) consist of the vector fields

$$\begin{aligned} \hat{\mathcal{P}}^t &= e^{-\gamma t} \left( \frac{1}{\gamma} \partial_t - \frac{3}{2}y\partial_y + \frac{1}{2}(3\gamma y - x)\partial_x - \left( \frac{1}{4}x^2 + \frac{3}{4}\gamma xy + \frac{9}{16}\gamma^2 y^2 - \frac{3}{2} \right) u\partial_u \right), \\ \hat{\mathcal{D}} &= \frac{2}{\gamma} \partial_t + u\partial_u, \quad \hat{\mathcal{K}} = e^{\gamma t} \left( \frac{1}{\gamma} \partial_t + \frac{3}{2}y\partial_y + \frac{1}{2}(3\gamma y + x)\partial_x - \left( \frac{3}{4}x^2 + \frac{3}{4}\gamma xy - \frac{9}{16}\gamma^2 y^2 + \frac{1}{2} \right) u\partial_u \right), \\ \hat{\mathcal{P}}^3 &= e^{\frac{3}{2}\gamma t} \left( \frac{1}{\gamma} \partial_y + \frac{3}{2}\partial_x + \frac{3}{4}(\gamma y - 2x)u\partial_u \right), \quad \hat{\mathcal{P}}^2 = e^{\frac{1}{2}\gamma t} \left( \frac{1}{\gamma} \partial_y + \frac{1}{2}\partial_x \right), \\ \hat{\mathcal{P}}^1 &= e^{-\frac{1}{2}\gamma t} \left( \frac{1}{\gamma} \partial_y - \frac{1}{2}\partial_x + \frac{1}{4}(3\gamma y + 2x)u\partial_u \right), \quad \hat{\mathcal{P}}^0 = e^{-\frac{3}{2}\gamma t} \left( \frac{1}{\gamma} \partial_y - \frac{3}{2}\partial_x \right), \quad \hat{\mathcal{I}} = u\partial_u \end{aligned}$$

and

$$\begin{aligned} \check{\mathcal{P}}^t &= e^{-\frac{1}{2}\gamma t} \left( \frac{1}{\gamma} \partial_t - \frac{3}{4}y\partial_y + \frac{1}{8}(3\gamma y - 2x)\partial_x + u\partial_u \right), \quad \check{\mathcal{D}} = \frac{4}{\gamma} \partial_t + 2u\partial_u, \\ \check{\mathcal{K}} &= e^{\frac{1}{2}\gamma t} \left( \frac{4}{\gamma} \partial_t + 3y\partial_y + \frac{1}{2}(3\gamma y + 2x)\partial_x - \left( x^2 + \frac{3}{2}\gamma xy + \frac{9}{16}\gamma^2 y^2 \right) u\partial_u \right), \\ \check{\mathcal{P}}^3 &= e^{\frac{3}{4}\gamma t} \left( \frac{8}{\gamma} \partial_y + 6\partial_x - \frac{3}{2}(4x + \gamma y)u\partial_u \right), \quad \check{\mathcal{P}}^2 = e^{\frac{1}{4}\gamma t} \left( \frac{4}{\gamma} \partial_y + \partial_x - \left( x + \frac{3}{4}\gamma y \right) u\partial_u \right), \\ \check{\mathcal{P}}^1 &= e^{-\frac{1}{4}\gamma t} \left( \frac{2}{\gamma} \partial_y - \frac{1}{2}\partial_x \right), \quad \check{\mathcal{P}}^0 = e^{-\frac{3}{4}\gamma t} \left( \frac{1}{\gamma} \partial_y - \frac{3}{4}\partial_x \right), \quad \check{\mathcal{I}} = u\partial_u, \end{aligned}$$

respectively. We choose the basis elements of the algebra  $\mathfrak{g}_{(10)}^{\text{ess}}$  (resp.  $\mathfrak{g}_{(11)}^{\text{ess}}$ ) in such a way that the commutation relations between them coincide with those between the basis elements of the algebra  $\mathfrak{g}^{\text{ess}}$  presented in Section 2. Thus, the fact that the algebras  $\mathfrak{g}^{\text{ess}}, \mathfrak{g}_{(10)}^{\text{ess}}$  and  $\mathfrak{g}_{(11)}^{\text{ess}}$  are isomorphic becomes obvious.

Point transformations  $\Phi_{(10)}$  and  $\Phi_{(11)}$  that, respectively, map the equations (10) and (11) to the remarkable Fokker–Planck equation (2) are constructed using the conditions that  $\Phi_{(10)*} \hat{\mathcal{V}} = \mathcal{V}$  and  $\Phi_{(11)*} \check{\mathcal{V}} = \mathcal{V}$  for  $\mathcal{V} \in \{\mathcal{P}^t, \mathcal{D}, \mathcal{K}, \mathcal{P}^3, \mathcal{P}^2, \mathcal{P}^1, \mathcal{P}^0, \mathcal{I}\}$ . Thus, the pushforwards  $\Phi_{(10)*}$  and  $\Phi_{(11)*}$  establish the

isomorphisms of the algebras  $\mathfrak{g}_{(10)}^{\text{ess}}$  and  $\mathfrak{g}_{(11)}^{\text{ess}}$  to the algebra  $\mathfrak{g}^{\text{ess}}$ ,  $\Phi_{(10)*}\mathfrak{g}_{(10)}^{\text{ess}} = \mathfrak{g}^{\text{ess}}$  and  $\Phi_{(11)*}\mathfrak{g}_{(11)}^{\text{ess}} = \mathfrak{g}^{\text{ess}}$ . As a result, we obtain the point transformations

$$\Phi_{(10)} : \tilde{t} = e^{\gamma t}, \quad \tilde{x} = e^{\frac{1}{2}\gamma t} \left( \frac{3}{2}\gamma y + x \right), \quad \tilde{y} = \gamma e^{\frac{3}{2}\gamma t} y, \quad \tilde{u} = e^{-\frac{1}{4}(\frac{3}{2}\gamma y + x)^2 - \frac{3}{2}\gamma t} u,$$

$$\Phi_{(11)} : \tilde{t} = 2e^{\frac{1}{2}\gamma t}, \quad \tilde{x} = e^{\frac{1}{4}\gamma t} \left( \frac{3}{4}\gamma y + x \right), \quad \tilde{y} = \gamma e^{\frac{3}{4}\gamma t} y, \quad \tilde{u} = e^{-\gamma t} u,$$

where the variables with tildes correspond to the equation (2), and the variables without tildes correspond to the source equation (10) or (11). By direct computation, we check that indeed the transformations  $\Phi_{(10)}$  and  $\Phi_{(11)}$  respectively map the equations (10) and (11) to the equation (2). In other words, an arbitrary solution  $\tilde{u} = f(t, x, y)$  of (2) is pulled back by  $\Phi_{(10)}$  and  $\Phi_{(11)}$  to the solutions  $u = \Phi_{(10)}^* f$  and  $u = \Phi_{(11)}^* f$  of the equations (10) and (11), respectively,

$$u = \Phi_{(10)}^* f = e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} f \left( e^{\gamma t}, e^{\frac{1}{2}\gamma t} \left( \frac{3}{2}\gamma y + x \right), \gamma e^{\frac{3}{2}\gamma t} y \right),$$

$$u = \Phi_{(11)}^* f = e^{\gamma t} f \left( 2e^{\frac{1}{2}\gamma t}, e^{\frac{1}{4}\gamma t} \left( \frac{3}{4}\gamma y + x \right), \gamma e^{\frac{3}{4}\gamma t} y \right).$$

Using the families of solutions of the equation (2) that have been constructed in Sections 5 and 6, we find the following families of solutions for the equation (10):

$$u = e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} \left| \frac{3}{2}\gamma y + x \right|^{-\frac{1}{4}} \theta^\mu \left( \frac{9}{4}\gamma \tilde{\varepsilon} e^{\frac{3}{2}\gamma t} y, e^{\frac{3}{2}\gamma t} \left| \frac{3}{2}\gamma y + x \right|^{\frac{3}{2}} \right) \text{ with } \mu = \frac{5}{36}, \tilde{\varepsilon} := \text{sgn} \left( x + \frac{3}{2}\gamma y \right),$$

$$u = e^{\gamma t} \theta \left( \frac{1}{3}e^{3\gamma t} + 2\varepsilon e^{\gamma t} - e^{-\gamma t}, e^{-\frac{1}{2}\gamma t} \left( \frac{1}{2}\gamma(e^{2\gamma t} - 3\varepsilon)y - (e^{2\gamma t} + \varepsilon)x \right) \right),$$

$$u = e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} \theta \left( \frac{4}{3}e^{3\gamma t}, e^{\frac{3}{2}\gamma t}(\gamma y + 2x) \right),$$

$$u = e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} \theta \left( 4e^{\gamma t}, e^{\frac{1}{2}\gamma t}(3\gamma y + 2x) \right),$$

$$u = e^{\frac{1}{2}\tilde{\zeta} - \frac{\kappa^2}{24\gamma y} + \frac{3}{2}\gamma \kappa t} \zeta |y|^{\kappa - \frac{4}{3}} \left( C_1 M \left( \kappa, \frac{4}{3}, \tilde{\zeta} \right) + C_2 U \left( \kappa, \frac{4}{3}, \tilde{\zeta} \right) \right)$$

$$\text{with } \zeta := 3\gamma y + 2x, \quad \tilde{\zeta} := \frac{1}{72}(\gamma y)^{-1} \zeta^3.$$

In the same way, we can also obtain solutions of the equation (11),

$$u = e^{\frac{15}{16}\gamma t} \left| \frac{3}{4}\gamma y + x \right|^{-\frac{1}{4}} \theta^\mu \left( \frac{9}{4}\gamma \tilde{\varepsilon} e^{\frac{3}{4}\gamma t} y, e^{\frac{3}{4}\gamma t} \left| \frac{3}{4}\gamma y + x \right|^{\frac{3}{2}} \right) \text{ with } \mu = \frac{5}{36}, \tilde{\varepsilon} := \text{sgn} \left( x + \frac{3}{4}\gamma y \right),$$

$$u = e^{-\frac{1}{8}(\frac{3}{4}\gamma y + x)^2 + \frac{3}{4}\gamma t} \theta \left( \frac{8}{3}e^{\frac{3}{2}\gamma t} + 4\varepsilon e^{\frac{1}{2}\gamma t} - \frac{1}{2}e^{-\frac{1}{2}\gamma t}, e^{-\frac{1}{4}\gamma t} \left( \frac{1}{8}\gamma(4e^{\gamma t} - 3\varepsilon)y - \frac{1}{2}(4e^{\gamma t} + \varepsilon)x \right) \right),$$

$$u = e^{\gamma t} \theta \left( \frac{32}{3}e^{\frac{3}{2}\gamma t}, e^{\frac{3}{4}\gamma t}(\gamma y + 4x) \right), \quad u = e^{\gamma t} \theta \left( 32e^{\frac{1}{2}\gamma t}, e^{\frac{1}{4}\gamma t}(3\gamma y + 4x) \right),$$

$$u = e^{-\zeta^3 + \frac{1}{4}\gamma t(3\kappa + 1)} \zeta |y|^{\kappa - 1} \left( C_1 M \left( \kappa, \frac{4}{3}, \zeta^3 \right) + C_2 U \left( \kappa, \frac{4}{3}, \zeta^3 \right) \right)$$

$$\text{with } \zeta := (9\gamma y)^{-\frac{1}{3}} \left( \frac{3}{4}\gamma y + x \right).$$

Recall that  $\theta^\mu = \theta^\mu(z_1, z_2)$  and  $\theta = \theta(z_1, z_2)$  denote arbitrary solutions of the (1 + 1)-dimensional linear heat equation with potential  $\mu z_2^{-2}$  and of the (1 + 1)-dimensional linear heat equation,  $\theta_1^\mu = \theta_{22}^\mu + \mu z_2^{-2} \theta^\mu$

and  $\theta_1 = \theta_{22}$ , respectively.  $M(a, b, \omega)$  and  $U(a, b, \omega)$  are the standard solutions of Kummer's equation  $\omega\varphi_{\omega\omega} + (b - \omega)\varphi_{\omega} - a\varphi = 0$ .

Similarly to Section 8, we can also construct more general families of solutions of the equations (10) and (11), carrying out their generalised reductions or acting by Lie-symmetry operators on known solutions. At the same time, we can again use the transformations  $\Phi_{(10)}$  and  $\Phi_{(11)}$  just to pull back the solutions of the equation (2) that are obtained in Section 8.

Thus, the families of solutions of the equation (10) that are invariant with respect to the generalised symmetries  $((\hat{\mathfrak{P}}^2 + \varepsilon\hat{\mathfrak{P}}^0)^n u)_{\partial_u}$ ,  $((\hat{\mathfrak{P}}^1)^n u)_{\partial_u}$  or  $((\hat{\mathfrak{P}}^0)^n u)_{\partial_u}$  are, respectively, constituted by the solutions of the form

$$u = \sum_{k=0}^{n-1} (\hat{\mathfrak{P}}^1)^k (e^{\gamma t} \theta^k(z_1, z_2))$$

with  $z_1 = \frac{1}{3}e^{3\gamma t} + 2\varepsilon e^{\gamma t} - e^{-\gamma t}$ ,  $z_2 = e^{-\frac{1}{2}\gamma t} \left( \frac{1}{2}\gamma(e^{2\gamma t} - 3\varepsilon)y - (e^{2\gamma t} + \varepsilon)x \right)$ ,

$$u = \sum_{k=0}^{n-1} (\hat{\mathfrak{P}}^2)^k \left( e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} \theta^k(z_1, z_2) \right) \quad \text{with} \quad z_1 = \frac{4}{3}e^{3\gamma t}, \quad z_2 = e^{\frac{3}{2}\gamma t}(\gamma y + 2x),$$

$$u = \sum_{k=0}^{n-1} (\hat{\mathfrak{P}}^3)^k \left( e^{\frac{1}{16}(3\gamma y + 2x)^2 + \frac{3}{2}\gamma t} \theta^k(z_1, z_2) \right) \quad \text{with} \quad z_1 = 4e^{\gamma t}, \quad z_2 = e^{\frac{1}{2}\gamma t}(3\gamma y + 2x).$$

Here and in what follows,  $\theta^k = \theta^k(z_1, z_2)$  are arbitrary solutions of the (1 + 1)-dimensional linear heat equation,  $\theta_1^k = \theta_{22}^k$ ,  $k = 0, \dots, n - 1$ . The differential operators in total derivatives

$$\hat{\mathfrak{P}}^3 := e^{\frac{3}{2}\gamma t} \left( \frac{1}{\gamma}D_y + \frac{3}{2}D_x - \frac{3}{4}(\gamma y - 2x) \right),$$

$$\hat{\mathfrak{P}}^2 := e^{\frac{1}{2}\gamma t} \left( \frac{1}{\gamma}D_y + \frac{1}{2}D_x \right),$$

$$\hat{\mathfrak{P}}^1 := e^{-\frac{1}{2}\gamma t} \left( \frac{1}{\gamma}D_y - \frac{1}{2}D_x - \frac{1}{4}(3\gamma y + 2x) \right),$$

$$\hat{\mathfrak{P}}^0 := e^{-\frac{3}{2}\gamma t} \left( \frac{1}{\gamma}D_y - \frac{3}{2}D_x \right)$$

are Lie-symmetry operators of the equation (10) and are associated with the Lie-symmetry vector fields  $-\hat{\mathcal{P}}^3, -\hat{\mathcal{P}}^2, -\hat{\mathcal{P}}^1, -\hat{\mathcal{P}}^0$ , respectively.

Analogously, the solutions of the equation (11) that are invariant with respect to the generalised symmetries  $((\check{\mathfrak{P}}^2 + \varepsilon\check{\mathfrak{P}}^0)^n u)_{\partial_u}$ ,  $((\check{\mathfrak{P}}^1)^n u)_{\partial_u}$  or  $((\check{\mathfrak{P}}^0)^n u)_{\partial_u}$  are of the form

$$u = \sum_{k=0}^{n-1} (\check{\mathfrak{P}}^1)^k \left( e^{-\frac{1}{8}(\frac{3}{4}\gamma y + x)^2 + \frac{3}{4}\gamma t} \theta^k(z_1, z_2) \right)$$

with  $z_1 = \frac{8}{3}e^{\frac{3}{2}\gamma t} + 4\varepsilon e^{\frac{1}{2}\gamma t} - \frac{1}{2}e^{-\frac{1}{2}\gamma t}$ ,  $z_2 = e^{-\frac{1}{4}\gamma t} \left( \frac{1}{8}\gamma(4e^{\gamma t} - 3\varepsilon)y - \frac{1}{2}(4e^{\gamma t} + \varepsilon)x \right)$ ,

$$u = \sum_{k=0}^{n-1} (\check{\mathfrak{P}}^2)^k (e^{\gamma t} \theta^k(z_1, z_2)) \quad \text{with} \quad z_1 = \frac{32}{3}e^{\frac{3}{2}\gamma t}, \quad z_2 = e^{\frac{3}{4}\gamma t}(\gamma y + 4x),$$

$$u = \sum_{k=0}^{n-1} (\check{\mathfrak{P}}^3)^k (e^{\gamma t} \theta^k(z_1, z_2)) \quad \text{with} \quad z_1 = 32e^{\frac{1}{2}\gamma t}, \quad z_2 = e^{\frac{1}{4}\gamma t}(3\gamma y + 4x),$$

where the differential operators in total derivatives

$$\begin{aligned} \check{\mathfrak{P}}^3 &:= e^{\frac{3}{4}\gamma t} \left( \frac{8}{\gamma} D_y + 6D_x + \frac{3}{2}(4x + \gamma y) \right), & \check{\mathfrak{P}}^2 &:= e^{\frac{1}{4}\gamma t} \left( \frac{4}{\gamma} D_y + D_x + x + \frac{3}{4}\gamma y \right), \\ \check{\mathfrak{P}}^1 &:= e^{-\frac{1}{4}\gamma t} \left( \frac{2}{\gamma} D_y - \frac{1}{2} D_x \right), & \check{\mathfrak{P}}^0 &:= e^{-\frac{3}{4}\gamma t} \left( \frac{1}{\gamma} D_y - \frac{3}{4} D_x \right) \end{aligned}$$

are Lie-symmetry operators of the equation (11) and are associated with the Lie-symmetry vector fields  $-\check{\mathcal{P}}^3, -\check{\mathcal{P}}^2, -\check{\mathcal{P}}^1, -\check{\mathcal{P}}^0$ .

Note that  $\Phi_{(10)*}\hat{\Omega} = \Omega$  and  $\Phi_{(11)*}\check{\Omega} = \Omega$  for  $\Omega \in \{\check{\mathfrak{P}}^3, \check{\mathfrak{P}}^2, \check{\mathfrak{P}}^1, \check{\mathfrak{P}}^0, 1\}$ .

### 10. Conclusion

The remarkable ultraparabolic Fokker–Planck equation (2) had been intensively studied in the mathematical literature from various points of view, including the classical group analysis of differential equations. Nevertheless, there were no exhaustive and accurate results even on its complete point symmetry pseudogroup and on classification of its Lie reductions, and the present paper has filled up this gap. Moreover, some of the obtained results were unexpected and may affect the entire field of classical group analysis of differential equations.

The complete point symmetry pseudogroup  $G$  of the equation (2) is given in Theorem 2. To simplify the computation of this pseudogroup, we have applied the two-step version of the direct method. In the first step, we have considered the class  $\bar{\mathcal{F}}$  of  $(1 + 2)$ -dimensional ultraparabolic Fokker–Planck equations of the general form (1), which contains the equation (2), have proven its normalisation in the usual sense and have found its equivalence pseudogroup  $G_{\bar{\mathcal{F}}}$ , see Theorem 1. Exhaustively describing the equivalence groupoid of the class  $\bar{\mathcal{F}}$  in this way, we have derived the principal constraints for point symmetries of the equation (2). In the second step, we have in fact looked for admissible transformations of the class  $\bar{\mathcal{F}}$  that preserve the equation (2), that is, that constitute its vertex group. This has led to a highly coupled overdetermined system of nonlinear partial differential equations for the transformation components. We have successfully found its general solution and constructed a nice representation (6) for elements of  $G$ . A similar splitting in the course of computing point symmetries by the direct method was used earlier, for example, in [5], but there the equivalence groupoid of the corresponding class had been known. In the present paper, we have first found the equivalence groupoid of a class of differential equations in order to compute the point symmetry (pseudo)group of a single element of this class, and this is indeed the optimal way of computing in spite of looking peculiar.

The description of  $G$  in Theorem 2 implies that its elements with  $f = 0$  and with their natural domains are not necessarily defined on the entire space  $\mathbb{R}^4_{t,x,y,u}$ . This requires considering a pseudogroup constituted by all possible restrictions of such transformations, when the usual composition of point transformations is taken as the group operation in  $G$ , complicates the entire analysis of the structure of  $G$ . In Section 3, we have avoided the indicated problem by defining the multiplication of elements in  $G$  as their composition modified with the extension of its domain by continuity. Under the suggested formalism, the representation (6) for the elements of  $G$  has allowed us to comprehensively analyse the structure of  $G$ . The pseudogroup  $G$  contains the abelian normal pseudosubgroup  $G^{\text{lin}}$ , which is associated with the linear superposition of solutions. Moreover, the pseudogroup  $G$  splits over  $G^{\text{lin}}$ ,  $G = G^{\text{ess}} \times G^{\text{lin}}$ . Here  $G^{\text{ess}}$  is a subgroup of  $G$ , which is a (finite-dimensional) Lie group and admits the factorisation  $G^{\text{ess}} = (F \times R_c) \times R_d$ , where  $F \simeq \text{SL}(2, \mathbb{R})$ ,  $R_c \simeq \text{H}(2, \mathbb{R})$  and  $R_d \simeq \mathbb{Z}_2$ . Since  $\text{SL}(2, \mathbb{R})$  and  $\text{H}(2, \mathbb{R})$  are connected Lie groups and  $G^{\text{lin}}$  is a connected Lie pseudogroup, the above factorisations directly imply Corollary 3, which states the surprising fact that the equation (2) admits a single independent discrete point symmetry transformation. The simplest choice for such a transformation is the involution  $\mathcal{J} : (t, x, y, u) \mapsto (t, x, y, -u)$ , which generates the group  $R_d$ .



In addition, we have constructed the point transformation (8), which maps the function  $u = 1 - H(t)$  to the fundamental solution of the equation (2). A similar construction arises in the Lie-symmetry analysis of linear (1 + 1)-dimensional heat equation.

The accurate consideration of the representation of the subgroup  $F$  of  $G$  on the radical  $\mathfrak{r}$  of the algebra  $\mathfrak{g}^{\text{ess}}$ , which coincides with the representation of  $\text{SL}(2, \mathbb{R})$  on the space of real binary cubic forms, has allowed us to successfully classify  $G^{\text{ess}}$ -inequivalent one- and two-dimensional subalgebras of  $\mathfrak{g}^{\text{ess}}$ , which for a long time had been a stumbling block in the course of constructing inequivalent Lie invariant solutions of the equation (2) before.

Using this classification, we have found all inequivalent Lie reductions of codimensions one and two and constructed Lie invariant solutions of the equation (2). It has been reduced to the (1 + 1)-dimensional linear heat equation, the heat equation with an inverse quadratic potential, Kummer’s equation, the Airy equation and first-order differential equations. The (1 + 1)-dimensional linear heat equation had been comprehensively studied within the framework of Lie reduction. In Appendix A, we have constructed many particular solution of linear (1 + 1)-dimensional heat equation with inverse square potential over the real field. For each of the reduced ordinary differential equations, we have found its general solution. It has also been shown that Lie reductions to algebraic equations, which are of codimension three, give no new solutions in comparison with the ones constructed using Lie reductions of less codimensions. One can further extend the constructed families of solutions, iteratively acting on them by Lie-symmetry operators. In this way, we have obtained families of exact solutions of the equation (2) that are non-trivially parameterised by an arbitrary finite number of arbitrary solutions of the (1 + 1)-dimensional linear heat equation. Modulo the  $G^{\text{ess}}$ -equivalence, these families exhaust solutions of the equation (2) that are invariant with respect to its generalised symmetries associated with powers of operators from  $\langle \mathfrak{P}^0, \mathfrak{P}^1, \mathfrak{P}^2, \mathfrak{P}^3, 1 \rangle$ . Mapping solutions of reduced equations to those of the original equation (2) has led to finding wide families of its explicit closed-form solutions. Up to the  $G^{\text{ess}}$ -equivalence, essential among these families are three families parameterised by single arbitrary solutions of the classical (1 + 1)-dimensional linear heat equation, one family parameterised by an arbitrary solution of the (1 + 1)-dimensional linear heat equation with a particular inverse square potential and one more family expressed in terms of the general solutions of a one-parameter family of Kummer’s equations. We have showed that the equation (2) admits hidden symmetries that are associated with reductions to the (1 + 1)-dimensional linear heat equations with the zero or inverse square potentials. This is why, in Appendix B, we have presented a general optimised algorithm for constructing hiddenly invariant solutions.

Studying Kramers equations, which constitute the subclass  $\mathcal{K}$  of the class  $\tilde{\mathcal{F}}$ , we have found nontrivial point transformations that map all equations from the class  $\mathcal{K}$  with eight-dimensional essential Lie invariance algebras, that is, the equations (10) and (11) up to shifts of  $y$ , to the equation (2). Moreover, the presented formulas for mapping solutions of the equation (2) to solutions of the equations (10) and (11) give the simplest way of obtaining explicit solutions of the latter equations.

The results obtained in the paper can be extended in many directions. In particular, one can compute reduction modules [12] of the equation (2) or, more promisingly, extend the consideration from Section 8 on generalised symmetries of this equation and on generating its new solutions by acting with Lie-symmetry differential operators. Conservation laws and potential symmetries of the equation (2) also require a comprehensive study.

Instead of considering various properties of the equation (2), we can extend results of the present paper by studying other linear (1 + 2)-dimensional ultraparabolic equations, including the group classification of classes of such equations, all of which are reduced by point transformations to the form (1). We are working on completing the solution of the group classification problem for the subclass of  $\tilde{\mathcal{F}}$  that consists of the Fokker–Planck equations of the form  $u_t + xu_y = |x|^\beta u_{xx}$  parameterised by an arbitrary real constant  $\beta$ . This subclass contains the equation (2) and is of interest from the point of view of applications and due to its mathematical properties. In particular, it admits a non-obvious discrete point equivalence transformation

$$\tilde{t} = y, \quad \tilde{x} = \frac{1}{x}, \quad \tilde{y} = t, \quad \tilde{u} = \frac{u}{x}, \quad \tilde{\beta} = 5 - \beta.$$

Of course, the most prominent among the above group classification problems is the group classification of the entire class of linear  $(1 + 2)$ -dimensional ultraparabolic equations, which reduces to the group classification of the class  $\bar{\mathcal{F}}$  of the linear  $(1 + 2)$ -dimensional ultraparabolic Fokker–Planck equations, which are of the form (1). The first step of the latter classification is the description of the equivalence groupoid of the class  $\bar{\mathcal{F}}$  by proving its normalisation in the usual sense and constructing its usual equivalence pseudogroup, which is presented in Theorem 1. The proof of this theorem and the complete solution of the group classification problem for the class  $\bar{\mathcal{F}}$  will be a subject of another paper.

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## References

- [1] Abraham-Shrauner, B. & Govinder, K. S. (2008) Master partial differential equations for a type II hidden symmetry. *J. Math. Anal. Appl.* **343** (1), 525–530.
- [2] Abramowitz, M. J. & Stegun, I. (1970). *Handbook on Mathematical Functions, National Bureau of Standards Appl. Math. Series*, Vol. 55, U.S. Govt. Printing Office, Washington, D.C.
- [3] Arnon, D. S. (1988) Geometric reasoning with logic and algebra. *Artif. Intell.* **37** (1-3), 37–60.
- [4] Bihlo, A., Dos Santos Cardoso-Bihlo, E. & Popovych, R. O. (2012) Complete group classification of a class of nonlinear wave equations. *J. Math. Phys.* **53** (12), 123515.
- [5] Bihlo, A. & Popovych, R. O. (2010) Point symmetry group of the barotropic vorticity equation. *Proceedings of 5th Workshop “Group Analysis of Differential Equations & Integrable Systems” (June 6–10, 2010, Protaras, Cyprus)*, Nicosia: University of Cyprus, 15–27.
- [6] Bihlo, A. & Popovych, R. O. (2017) Group classification of linear evolution equations. *J. Math. Anal. Appl.* **448** (2), 982–1005.
- [7] Bluman, G. (1980) On the transformation of diffusion processes into the Wiener process. *SIAM J. Appl. Math.* **39** (2), 238–247.
- [8] Bluman, G. & Shtelen, V. (2004) Nonlocal transformations of Kolmogorov equations into the backward heat equation. *J. Math. Anal. Appl.* **291** (2), 419–437.
- [9] Bluman, G. W., Cheviakov, A. F. & Anco, S. C. (2010). *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York.
- [10] Bluman, G. W. & Kumei, S. (1989). *Symmetries and Differential Equations*, Springer-Verlag, New York.
- [11] Bluman, G. W. & Yüzbaşı, Z. K. (2020) How symmetries yield non-invertible mappings of linear partial differential equations. *J. Math. Anal. Appl.* **491** (2), 124354.
- [12] Boyko, V. M., Kunzinger, M. & Popovych, R. O. (2016) Singular reduction modules of differential equations. *J. Math. Phys.* **57** (10), 101503.
- [13] Cherkasov, I. D. (1957) On the transformation of the diffusion process to a Wiener process. *Theory Probab. Appl.* **2**: 373–377.
- [14] Davydovych, V. (2015) Preliminary group classification of  $(2 + 1)$ -dimensional linear ultraparabolic Kolmogorov–Fokker–Planck equations. *Ufimsk. Mat. Zh.* **7** (3), 38–46.
- [15] Dos Santos Cardoso-Bihlo, E., Bihlo, A. & Popovych, R. O. (2011) Enhanced preliminary group classification of a class of generalized diffusion equations. *Commun. Nonlinear Sci. Numer. Simul.* **16** (9), 3622–3638.
- [16] Güngör, F. (2018) Equivalence and symmetries for variable coefficient linear heat type equations. I. *J. Math. Phys.* **59** (5), 051507.
- [17] Güngör, F. (2018) Equivalence and symmetries for variable coefficient linear heat type equations. II. Fundamental solutions. *J. Math. Phys.* **59** (5), 061507.
- [18] Johnpillai, I. K. & Mahomed, F. M. (2001) Singular invariant equation for the  $(1+1)$  Fokker–Planck equation. *J. Phys. A* **34** (49), 11033–11051.
- [19] Kolmogoroff, A. (1934) Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. Math. (2)* **35**: 116–117.
- [20] Kovalenko, S. S., Kopas, I. M. & Stogniy, V. I. (2013) Preliminary group classification of a class of generalized linear Kolmogorov equations. *Res. Bull. Natl. Tech. Univ. Ukraine “Kyiv Polytechnic Institute”* **4**: 67–72.

- [21] Lie, S. (1881). Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichungen, *Arch. for Math.* **6**, 328–368. [Translated by Ibragimov, N. H.: Lie, S. (1995) On integration of a class of linear partial differential equations by means of definite integrals, *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2, CRC Press, Boca Raton, FL, 328–368.]
- [22] Matveev, V. B. & Salle, M. A. (1991). *Darboux Transformations and Solitons*, Springer-Verlag, Berlin.
- [23] Morozov, O. (2003) Contact equivalence problem for linear parabolic equations, [arXiv: math-ph/0304045](https://arxiv.org/abs/math-ph/0304045).
- [24] Olver, P. J. (1999). *Classical Invariant Theory*, Cambridge University Press, Cambridge.
- [25] Olver, P. J. (2000). *Application of Lie Groups to Differential Equations*, Springer, New York.
- [26] Opanasenko, S., Bihlo, A. & Popovych, R. O. (2017) Group analysis of general Burgers–Korteweg–de Vries equations. *J. Math. Phys.* **58** (8), 081511.
- [27] Opanasenko, S. & Popovych, R. O. (2022) Mapping method of group classification. *J. Math. Anal. Appl.* **513** (2), 126209.
- [28] Patera, J. & Winternitz, P. (1977) Subalgebras of real three and four-dimensional Lie algebras. *J. Math. Phys.* **18** (7), 1449–1455.
- [29] Patera, J., Winternitz, P. & Zassenhaus, H. (1975) Continuous subgroups of the fundamental groups of physics. I. General method and the Poincaré group. *J. Math. Phys.* **16**: 1597–1614.
- [30] Pocheketa, O. A. & Popovych, R. O. (2017) Extended symmetry analysis of generalized Burgers equations. *J. Math. Phys.* **58** (10), 101501.
- [31] Popovych, R. O. (2006) Classification of admissible transformations of differential equations. *Collection Works Inst. Math. NAS Ukraine* **3** (2), 239–254.
- [32] Popovych, R. O., Boyko, V. M., Nesterenko, M. O. & Lutfullin, M. W. (2003) Realizations of real low-dimensional Lie algebras. *J. Phys. A* **36** (26), 7337–7360.
- [33] Popovych, R. O., Kunzinger, M. & Eshraghi, H. (2010) Admissible transformations and normalized classes of nonlinear Schrödinger equations. *Acta Appl. Math.* **109** (2), 315–359.
- [34] Popovych, R. O., Kunzinger, M. & Ivanova, N. M. (2008) Conservation laws and potential symmetries of linear parabolic equations. *Acta Appl. Math.* **2** (2), 113–185.
- [35] Spichak, S. & Stognii, V. (1999) Symmetry classification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion. *J. Phys. A* **32** (47), 8341–8353.
- [36] Spichak, S., Stogniy, V. & Kopas, I. (2011). Symmetry properties and exact solutions of the linear Kolmogorov equation, *Res. Bull. Natl. Technic. Univ. Ukraine “Kyiv Polytechnic Institute”*, **4**, 93–97 (in Ukrainian).
- [37] Spichak, S. & Stogny, V. (1997) Symmetry analysis of the Kramers equation. *Rep. Math. Phys.* **40** (1), 125–130.
- [38] Turkowski, P. (1988) Low-dimensional real Lie algebras. *J. Math. Phys.* **29** (10), 2139–2144.
- [39] Vaneeva, O. O., Bihlo, A. & Popovych, R. O. (2020) Generalization of the algebraic method of group classification with application to nonlinear wave and elliptic equations. *Commun. Nonlinear Sci. Numer. Simul.* **91**: 105419.
- [40] Vaneeva, O. O., Popovych, R. O. & Sophocleous, C. (2021) Extended symmetry analysis of two-dimensional degenerate Burgers equation. *J. Geom. Phys.* **169**: 104336.
- [41] Zhang, Z.-Y., Zheng, J., Guo, L.-L. & Wu, H.-F. (2020) Lie symmetries and conservation laws of the Fokker–Planck equation with power diffusion. *Math. Methods Appl. Sci.* **43** (15), 8894–8905.

## A. Exact solutions of (1+1)-dimensional linear heat equations with inverse square potential

Lie reduction of the equation (2) with respect to the subalgebra  $\mathfrak{g}_{1,2}^0 = \langle \mathcal{P}^t \rangle$  leads to reduced equation 1.2<sup>0</sup>, which is the linear heat equation with potential  $V = \mu z_2^{-2}$ , where  $\mu = \frac{5}{36}$ . The linear heat equations with general (nonzero) inverse square potentials  $V = \mu z_2^{-2}$ , where  $\mu \neq 0$ , constitute an important case of Lie-symmetry extension in the class of linear second-order partial differential equations in two independent variables. This result was obtained by Sophus Lie himself [21]. Lie invariant solutions of such heat equations over the complex field were considered in [17]. The essential point symmetry group of these equations was constructed as a by-product in the course of proving Theorems 18 in [27], and a background that can be used for exhaustively classifying Lie invariant solutions of these equations over the real field was developed in the proof of Theorems 51 therein. We revisit the above results, present them in an enhanced and closed form and complete the study of Lie invariant solutions of linear heat equations with (nonzero) inverse square potentials. Recall that Lie reductions of the (1+1)-dimensional linear heat equation, which corresponds to the value  $\mu = 0$ , were comprehensively studied in [40, Section A], following Examples 3.3 and 3.17 in [25]. Hereafter, we use notations that differ from those in the other sections of this paper.

Consider a linear heat equation with inverse square potential,

$$u_t = u_{xx} + \frac{\mu}{x^2} u, \quad (12)$$

where  $\mu \neq 0$ . It belongs to the class  $\mathcal{E}$  of linear (1+1)-dimensional second-order evolution equations of the general form

$$u_t = A(t, x)u_{xx} + B(t, x)u_x + C(t, x)u + D(t, x) \quad \text{with } A \neq 0. \tag{13}$$

Here the tuple of arbitrary elements of  $\mathcal{E}$  is  $\theta := (A, B, C, D) \in \mathcal{S}_{\mathcal{E}}$ , where  $\mathcal{S}_{\mathcal{E}}$  is the solution set of the auxiliary system consisting of the single inequality  $A \neq 0$  with no constraints on  $B, C$  and  $D$ .

Since the equation (12) is a linear homogeneous equation, its maximal Lie invariance algebra  $\mathfrak{g}$  contains the infinite-dimensional abelian ideal  $\mathfrak{g}^{\text{lin}} := \{h(t, x)\partial_u\}$ , where the parameter function  $h$  runs through its solution set, cf. Section 2. This ideal is associated with the linear superposition of solutions of (12). The algebra  $\mathfrak{g}$  splits over the ideal  $\mathfrak{g}^{\text{lin}}$ ,  $\mathfrak{g} = \mathfrak{g}^{\text{ess}} \ltimes \mathfrak{g}^{\text{lin}}$ , where the complement subalgebra  $\mathfrak{g}^{\text{ess}}$  is four-dimensional,

$$\mathfrak{g}^{\text{ess}} = \left\langle \mathcal{P}' = \partial_t, \mathcal{D} = t\partial_t + \frac{1}{2}x\partial_x - \frac{1}{4}u\partial_u, \mathcal{K} = t^2\partial_t + tx\partial_x - \frac{1}{4}(x^2 + 2t)u\partial_u, \mathcal{I} = u\partial_u \right\rangle.$$

The subalgebra  $\mathfrak{g}^{\text{ess}}$  is called the essential Lie invariance algebra of (12). It is the algebra  $\mathfrak{a}_2^{\delta=0}$  in the notation of Section 5. Up to the skew-symmetry of the Lie bracket, the nonzero commutation relations of this algebra are exhausted by  $[\mathcal{P}', \mathcal{D}] = \mathcal{P}'$ ,  $[\mathcal{D}, \mathcal{K}] = \mathcal{K}$ ,  $[\mathcal{P}', \mathcal{K}] = 2\mathcal{D}$ . Therefore, the algebra  $\mathfrak{g}^{\text{ess}}$  is isomorphic to  $\mathfrak{sl}(2, \mathbb{R}) \oplus A_1$ .

To find the complete point symmetry pseudogroup  $G$  of the equation (12), we start with considering the equivalence groupoid of the class  $\mathcal{E}$ , which in its turn is the natural choice for a (normalised) superclass for the equation (12). We use the papers [27, 34] as reference points for known results on admissible transformations of the class  $\mathcal{E}$ .

**Proposition 8.** ([34]) *The class  $\mathcal{E}$  is normalised in the usual sense. Its usual equivalence pseudogroup  $G_{\mathcal{E}}^{\sim}$  consists of the transformations of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = U^1(t, x)u + U^0(t, x), \tag{14a}$$

$$\tilde{A} = \frac{X_x^2}{T_t}A, \quad \tilde{B} = \frac{X_x}{T_t} \left( B - 2\frac{U_x^1}{U^1}A \right) - \frac{X_t - X_{xx}A}{T_t}, \quad \tilde{C} = -\frac{U^1}{T_t}E\frac{1}{U^1}, \tag{14b}$$

$$\tilde{D} = \frac{U^1}{T_t} \left( D + E\frac{U^0}{U^1} \right), \tag{14c}$$

where  $T, X, U^0$  and  $U^1$  are arbitrary smooth functions of their arguments with  $T_t X_x U^1 \neq 0$ , and  $E := \partial_t - A\partial_{xx} - B\partial_x - C$ .

The normalisation of the class  $\mathcal{E}$  means that its equivalence groupoid coincides with the action groupoid of the pseudogroup  $G_{\mathcal{E}}^{\sim}$ .

**Theorem 9.** *The complete point symmetry pseudogroup  $G$  of the (1+1)-dimensional linear heat equation with inverse square potential (12) consists of the point transformations of the form*

$$\tilde{t} = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \tilde{x} = \frac{x}{\gamma t + \delta}, \quad \tilde{u} = \sigma \sqrt{|\gamma t + \delta|} (u + h(t, x)) \exp \frac{\gamma x^2}{4(\gamma t + \delta)},$$

where  $\alpha, \beta, \gamma, \delta$  and  $\sigma$  are arbitrary constants with  $\alpha\delta - \beta\gamma = 1$  and  $\sigma \neq 0$ , and  $h$  is an arbitrary solution of (12).

*Proof.* The linear heat equation with inverse square potential (12) corresponds to the value  $(1, 0, \mu x^{-2}, 0) =: \theta^{\mu}$  of the arbitrary-element tuple  $\theta = (A, B, C, D)$  of class  $\mathcal{E}$ . Its vertex group  $\mathcal{G}_{\theta^{\mu}} := \mathcal{G}_{\mathcal{E}}^{\sim}(\theta^{\mu}, \theta^{\mu})$  is the set of admissible transformations of the class  $\mathcal{E}$  with  $\theta^{\mu}$  as both their source and target,  $\mathcal{G}_{\theta^{\mu}} = \{(\theta^{\mu}, \Phi, \theta^{\mu}) \mid \Phi \in G\}$ . This argument allows us to use Proposition 8 in the course of computing the pseudogroup  $G$ .

We should integrate the equations (14), where both the source value  $\theta$  of the arbitrary-element tuple and its target value  $\tilde{\theta}$  coincide with  $\theta^\mu$ , with respect to the parameter functions  $T, X, U^1$  and  $U^0$ . After a simplification, the equations (14b) take the form

$$X_x^2 = T_t, \quad \frac{U_x^1}{U^1} = -\frac{X_t}{2X_x}, \quad \frac{\mu}{X^2} = -\frac{U^1}{T_t} E \frac{1}{U^1}, \tag{15}$$

where  $E := \partial_t - \partial_{xx} - \mu x^{-2}$ . The first equation in (15) implies that  $T_t > 0$ , and the first two equations in (15) can be easily integrated to

$$X = \varepsilon \sqrt{T_t} x + X^0(t), \quad U^1 = \phi(t) \exp\left(-\frac{T_{tt}}{8T_t} x^2 - \frac{\varepsilon}{2} \frac{X_t^0}{\sqrt{T_t}} x\right),$$

where  $\varepsilon := \pm 1$  and  $\phi$  is a nonvanishing smooth function of  $t$ . By substituting these expressions for  $X$  and  $U^1$  into the third equation from (15) and splitting the obtained equation with respect to powers of  $x$ , we derive that  $X^0 = 0$ ,  $4T_t \phi_t + T_{tt} \phi = 0$  and  $T_{ttt}/T_t - \frac{3}{2}(T_{tt}/T_t)^2 = 0$ . The last equation means that the Schwarzian derivative of  $T$  is zero. Therefore,  $T$  is a linear fractional function of  $t$ ,  $T = (\alpha t + \beta)/(\gamma t + \delta)$ . Since the constant parameters  $\alpha, \beta, \gamma$  and  $\delta$  are defined up to a constant nonzero multiplier and  $T_t > 0$ , we can assume that  $\alpha\delta - \beta\gamma = 1$ . Then, these parameters are still defined up to a multiplier in  $\{-1, 1\}$ , and hence, we can choose them in such a way that  $\varepsilon|\gamma t + \delta| = \gamma t + \delta$ , thus neglecting the parameter  $\varepsilon$ . The equation  $4T_t \phi_t + T_{tt} \phi = 0$  takes the form  $2(\gamma t + \delta)\phi_t - \gamma\phi = 0$  and integrates, in view of  $\phi \neq 0$ , to  $\phi = \sigma \sqrt{|\gamma t + \delta|}$  with  $\sigma \in \mathbb{R} \setminus \{0\}$ .

Finally, the equation (14c) takes the form

$$\left(\frac{U^0}{U^1}\right)_t = \left(\frac{U^0}{U^1}\right)_{xx} + \frac{\mu}{x^2} \left(\frac{U^0}{U^1}\right).$$

Therefore,  $U^0 = U^1 h$ , where  $h = h(t, x)$  is an arbitrary solution of (12). □

Accurate analysis of the structure of the pseudogroup  $G$  needs a proper interpretation of the corresponding group operation, which we assume to be done in the same way as that in Section 3. The point transformations of the form  $\tilde{t} = t, \tilde{x} = x, \tilde{u} = u + h(t, x)$ , where the parameter function  $h = h(t, x)$  is an arbitrary solution of the equation (12), constitute the normal pseudosubgroup  $G^{\text{lin}}$  of  $G$ , which is associated with the linear superposition of solutions of (12), cf. Section 3. Moreover, the pseudogroup  $G$  splits over  $G^{\text{lin}}$ ,  $G = G^{\text{ess}} \ltimes G^{\text{lin}}$ , where the subgroup  $G^{\text{ess}}$  consists of the elements of  $G$  with  $h = 0$  and is a four-dimensional Lie group within the framework of the above interpretation. We call this subgroup the *essential point symmetry group* of the equation (12). In turn, the group  $G^{\text{ess}}$  contains the normal subgroup  $R$  that is constituted by the point transformations of the form  $\tilde{t} = t, \tilde{x} = x, \tilde{u} = \sigma u$  with  $\sigma \in \mathbb{R} \setminus \{0\}$  and is isomorphic to the multiplicative group  $\mathbb{R}^\times$  of the real field. The group  $G^{\text{ess}}$  splits over  $R$ ,  $G^{\text{ess}} = F \ltimes R$ , where the subgroup  $F$  is singled out from  $G^{\text{ess}}$  by the constraint  $\sigma = 1$  and is isomorphic to the group  $SL(2, \mathbb{R})$ .

**Corollary 10.** *A complete list of discrete point symmetry transformations of the equation (12) that are independent up to combining with each other and with continuous point symmetry transformations of this equation is exhausted by the single involution  $\mathcal{J}$  alternating the sign of  $u$ ,  $\mathcal{J} : (t, x, u) \mapsto (t, x, -u)$ . Thus, the quotient group of the complete point symmetry pseudogroup  $G$  of (12) with respect to its identity component is isomorphic to  $\mathbb{Z}_2$ .*

*Proof.* It is obvious that  $G^{\text{lin}}$  and  $F$  are connected pseudosubgroup and subgroup of  $G$ , respectively. Jointly with the splitting of  $G$  over  $R$ , this implies that elements of the required list can be selected from the subgroup  $R$ . Factoring out the elements of the identity component of  $R$ , which is isomorphic to  $\mathbb{R}_{>0}^\times$ , we obtain either the identity transformation or  $\mathcal{J}$ . □

**Lemma 11.** ([28, 32]) *A complete list of  $G^{\text{ess}}$ -inequivalent subalgebras of  $\mathfrak{g}^{\text{ess}}$  is exhausted by the subalgebras*

$$\begin{aligned} \mathfrak{s}_{1,1}^\delta &= \langle \mathcal{P}' + \delta \mathcal{I} \rangle, \quad \mathfrak{s}_{1,2}^\nu = \langle \mathcal{D} + \nu \mathcal{I} \rangle, \nu \geq 0, \quad \mathfrak{s}_{1,3}^\nu = \langle \mathcal{P}' + \mathcal{K} + 2\nu \mathcal{I} \rangle, \quad \mathfrak{s}_{1,4} = \langle \mathcal{I} \rangle, \\ \mathfrak{s}_{2,1}^\nu &= \langle \mathcal{P}', \mathcal{D} + \nu \mathcal{I} \rangle, \quad \mathfrak{s}_{2,2} = \langle \mathcal{P}', \mathcal{I} \rangle, \quad \mathfrak{s}_{2,3} = \langle \mathcal{D}, \mathcal{I} \rangle, \quad \mathfrak{s}_{2,4} = \langle \mathcal{P}' + \mathcal{K}, \mathcal{I} \rangle, \\ \mathfrak{s}_{3,1} &= \langle \mathcal{P}', \mathcal{D}, \mathcal{K} \rangle, \quad \mathfrak{s}_{3,1} = \langle \mathcal{P}', \mathcal{D}, \mathcal{I} \rangle, \quad \mathfrak{s}_{4,1} = \mathfrak{g}^{\text{ess}}, \end{aligned}$$

where  $\delta \in \{-1, 0, 1\}$ , and  $\nu$  is an arbitrary real constant satisfying indicated constraints. The first number in the subscript of  $\mathfrak{s}$  denotes the dimension of the corresponding subalgebra.

We present the exhaustive classification of Lie invariant solutions of the equation (12) over the field of real numbers using results from the proof of Theorem 51 in [27] for integrating the obtained reduced equations. Hereafter,  $C_1$  and  $C_2$  are arbitrary real constants,  $\kappa := \frac{1}{2}\sqrt{1 - 4\mu}$ ,  $\kappa' := \frac{1}{2}\kappa = \frac{1}{4}\sqrt{1 - 4\mu}$ , and  $\nu$  is a constant parameter.

The reduction with respect to the subalgebra  $\mathfrak{s}_{1,1}^\varepsilon$  with  $\varepsilon \in \{-1, 1\}$  results in the solution

$$u = e^{\varepsilon t} \sqrt{|x|} Z_{|\kappa|}(x),$$

where the cylinder function  $Z_{|\kappa|}(x)$  is  $\mathcal{Z}_\kappa$ ,  $\mathcal{C}_\kappa$ ,  $\tilde{\mathcal{Z}}_{|\kappa|}$  or  $\tilde{\mathcal{C}}_{|\kappa|}$  if  $4\mu \leq 1$  and  $\varepsilon = -1$ ,  $4\mu \leq 1$  and  $\varepsilon = 1$ ,  $4\mu > 1$  and  $\varepsilon = -1$  or  $4\mu > 1$  and  $\varepsilon = 1$ , respectively. Here,  $\mathcal{Z}_\kappa$  and  $\mathcal{C}_\kappa$  are linear combination of Bessel functions and linear combination of modified Bessel functions, respectively,  $\mathcal{Z}_\kappa = C_1 J_\kappa(x) + C_2 Y_\kappa(x)$ ,  $\mathcal{C}_\kappa = C_1 I_\kappa(x) + C_2 K_\kappa(x)$ . The cylinder function  $\tilde{\mathcal{Z}}_{|\kappa|}$  is a linear combination of modifications of the Hankel functions  $H_\kappa^{(1)}$  and  $H_\kappa^{(2)}$ ,

$$\begin{aligned} \tilde{\mathcal{Z}}_{|\kappa|} &= C_1 \tilde{H}_\kappa^{(1)}(x) + C_2 \tilde{H}_\kappa^{(2)}(x) \\ &= \frac{C_1}{2} \left( e^{-\frac{1}{2}\kappa\pi} H_\kappa^{(1)}(x) + e^{\frac{1}{2}\kappa\pi} H_\kappa^{(2)}(x) \right) + \frac{C_2}{2i} \left( e^{-\frac{1}{2}\kappa\pi} H_\kappa^{(1)}(x) - e^{\frac{1}{2}\kappa\pi} H_\kappa^{(2)}(x) \right), \end{aligned}$$

and the cylinder function  $\tilde{\mathcal{C}}_{|\kappa|}$  is a linear combination of the modified Bessel function  $K_\kappa$  and the function  $\tilde{I}_\kappa$ , which is a modification of the modified Bessel function  $I_\kappa$  (for  $\kappa \neq 0$ ),

$$\tilde{\mathcal{C}}_{|\kappa|} = C_1 \tilde{I}_\kappa(x) + C_2 K_\kappa(x), \quad \tilde{I}_\kappa(x) := \frac{\pi i}{2 \sin(\kappa\pi)} (I_\kappa(x) + I_{-\kappa}(x)).$$

Each of the functions  $\mathcal{Z}_\kappa$ ,  $\mathcal{C}_\kappa$ ,  $\tilde{\mathcal{Z}}_{|\kappa|}$  or  $\tilde{\mathcal{C}}_{|\kappa|}$  is real-valued and represents, for the corresponding values of the parameters  $\kappa$  and  $\varepsilon$ , the general solution of the equation  $\varphi_{\omega\omega} + \mu\omega^{-2}\varphi - \varepsilon\varphi = 0$ , which is obtained by the Lie reduction of (12) with respect to the subalgebra  $\langle \mathcal{P}' + \varepsilon \mathcal{I} \rangle$  using the ansatz  $u = e^{\varepsilon t} \varphi(\omega)$  with  $\omega = x$ .

The Lie reduction of the equation (12) with respect to the subalgebra  $\mathfrak{s}_{1,1}^0$  leads to the ansatz  $u = \varphi(\omega)$  with  $\omega = x$  and the Euler equation  $\omega^2 \varphi_{\omega\omega} + \mu\varphi = 0$  as the corresponding reduced equation. We integrate this equation depending on the value of  $\text{sgn}(1 - 4\mu) \in \{-1, 0, 1\}$  and obtain the following solutions of (12):

$$\begin{aligned} u &= \sqrt{|x|} (C_1 \cos(|\kappa| \ln |x|) + C_2 \sin(|\kappa| \ln |x|)), \quad u = \sqrt{|x|} (C_1 + C_2 \ln |x|), \\ u &= \sqrt{|x|} (C_1 |x|^\kappa + C_2 |x|^{-\kappa}). \end{aligned}$$

For each of the subalgebras  $\mathfrak{s}_{1,2}^\nu$  and  $\mathfrak{s}_{1,3}^\nu$ , we first construct an associated ansatz for  $u$ , derive the corresponding reduced equation, map this equation to a canonical form, which turns out to be the Whittaker equation

$$\varphi_{\omega\omega} + \left( -\frac{1}{4} + \frac{a}{\omega} + \frac{1/4 - b^2}{\omega^2} \right) \varphi = 0 \tag{16}$$

with certain  $a$  and  $b$ , by a point transformation of the invariant independent and dependent variables, and then use this transformation for modifying the ansatz. The general solution of the above Whittaker equation is the general linear combination of the Whittaker functions  $W_{a,b}(z)$  and  $M_{a,b}(z)$ , which are linearly independent and whose properties are comprehensively described, for example, in [2].

Thus, for the subalgebra  $\mathfrak{s}_{1,2}^\nu$  with a fixed  $\nu \geq 0$ , the used modified ansatz and the values of  $a$  and  $b$  are

$$u = |t|^\nu |x|^{-\frac{1}{2}} e^{-\frac{x^2}{4|t|}} \varphi(\omega) \quad \text{with} \quad \omega := \frac{x^2}{4|t|}, \quad a = -\varepsilon'\nu \quad \text{with} \quad \varepsilon' := \text{sgn } t, \quad b = \kappa'.$$

The representation of the corresponding solutions of (12) over the real field depends on the value of  $\text{sgn}(1 - 4\mu)$ , either  $\text{sgn}(1 - 4\mu) \geq 0$  or  $\text{sgn}(1 - 4\mu) < 0$ , and respectively is

$$u = |t|^\nu |x|^{-\frac{1}{2}} e^{-\frac{x^2}{4|t|}} \left( C_1 M_{-\varepsilon'\nu, \kappa'} \left( \frac{x^2}{4|t|} \right) + C_2 W_{-\varepsilon'\nu, \kappa'} \left( \frac{x^2}{4|t|} \right) \right),$$

$$u = |t|^\nu |x|^{-\frac{1}{2}} e^{-\frac{x^2}{4|t|}} \text{Re} \left( (C_1 - iC_2) W_{-\varepsilon'\nu, i|\kappa'|} \left( \frac{x^2}{4|t|} \right) \right).$$

We distinguish the case  $\text{sgn}(1 - 4\mu) < 0$  since then the Whittaker function  $W_{-\varepsilon'\nu, \kappa'}(\omega)$  is complex-valued. However, its real and imaginary parts are linearly independent real solutions of the corresponding Whittaker equation.

For each of the subalgebras  $\mathfrak{s}_{1,3}^\nu$ , we use the modified ansatz

$$u = |x|^{-\frac{1}{2}} e^{-\frac{1}{4} \frac{x^2}{t^2+1} + 2\nu \arctan t} \varphi(\omega) \quad \text{with} \quad \omega := \frac{ix^2}{2(t^2 + 1)}.$$

The associated reduced equation is of the form (16) with  $a = i\nu$  and  $b = \kappa'$ . Hence, the corresponding solutions of (12) over the real field are

$$u = |x|^{-\frac{1}{2}} e^{-\frac{1}{4} \frac{x^2}{t^2+1} + 2\nu \arctan t} \text{Re} \left( (C_1 - iC_2) W_{i\nu, \kappa'} \left( \frac{ix^2}{2(t^2 + 1)} \right) \right).$$

If  $\kappa \in 2\mathbb{N}_0 + 1$ , these solutions can be represented in terms of regular and irregular Coulomb functions,

$$u = |x|^{-\frac{1}{2}} e^{-\frac{1}{4} \frac{x^2}{t^2+1} + 2\nu \arctan t} \left( C_1 F_{\kappa'-\frac{1}{2}} \left( \nu, \frac{x^2}{4(t^2 + 1)} \right) + C_2 G_{\kappa'-\frac{1}{2}} \left( \nu, \frac{x^2}{4(t^2 + 1)} \right) \right).$$

**B. On construction and generation of new exact solutions using hidden symmetries**

Let  $\mathfrak{g}$  and  $G$  be the maximal Lie invariance algebra and the point symmetry (pseudo)group of a system of differential equations  $\mathcal{L}$ , and let  $\mathfrak{s}$  be a subalgebra of  $\mathfrak{g}$  that is appropriate for Lie reduction of the system  $\mathcal{L}$ . By  $\hat{\mathcal{L}}$ ,  $\hat{\mathfrak{g}}$  and  $\hat{G}$ , we denote a reduced system of  $\mathcal{L}$  with respect to a Lie reduction  $\rho$  associated with  $\mathfrak{s}$ , its maximal Lie invariance algebra and its point symmetry group, respectively. The system  $\hat{\mathcal{L}}$  has the same number of the dependent variables as the system  $\mathcal{L}$ , and its number of the independent variables is that of  $\mathcal{L}$  reduced by  $\dim \mathfrak{s}$ . Here  $\rho$  denotes an  $\mathfrak{s}$ -invariant map of maximal rank from the space whose coordinates are the independent and dependent variables of  $\mathcal{L}$  to the analogous space for the system  $\hat{\mathcal{L}}$ , such that  $\rho_* \mathcal{L}$  is equal to  $\hat{\mathcal{L}}$  up to a nondegenerate differential-functional matrix multiplier.

Lie-symmetry vector fields of the system  $\mathcal{L}$  from the normaliser  $N_{\mathfrak{g}}(\mathfrak{s})$  of the subalgebra  $\mathfrak{s}$  in the algebra  $\mathfrak{g}$  induce Lie-symmetry vector fields of the system  $\hat{\mathcal{L}}$ , which constitute a subalgebra  $\tilde{\mathfrak{g}}$  of  $\hat{\mathfrak{g}}$ ,  $\tilde{\mathfrak{g}} = \rho_* N_{\mathfrak{g}}(\mathfrak{s})$ . Similarly,  $\tilde{G} = \rho_* St_G(\mathfrak{s})$  is the group of induced point symmetries of the system  $\hat{\mathcal{L}}$ . Any element of  $\tilde{\mathfrak{g}} \setminus \hat{\mathfrak{g}}$  is a genuine hidden Lie-symmetry vector field of the original system  $\mathcal{L}$  with respect to the Lie reduction obtained with the subalgebra  $\mathfrak{s}$ , see [25, Example 3.5], [1] and the discussion in [30, Remark 15].

Subalgebras  $\hat{\mathfrak{s}}$  of  $\hat{\mathfrak{g}}$  can be used for further Lie reductions of the system  $\hat{\mathcal{L}}$ . The question is which subalgebras of  $\hat{\mathfrak{g}}$  may lead to solutions of  $\hat{\mathcal{L}}$ , whose pullbacks by  $\rho$  are *hiddenly invariant solutions* of the system  $\mathcal{L}$ , that is, they are not invariant with respect to subalgebras of  $\mathfrak{g}$  of greater dimensions than  $\dim \mathfrak{s}$ . It is obvious that this is definitely not the case for any subalgebra of  $\hat{\mathfrak{g}}$  each element of whose  $\hat{G}$ -orbit is contained in  $\tilde{\mathfrak{g}}$ . Moreover, if solutions  $\hat{f}^1$  and  $\hat{f}^2$  of the system  $\hat{\mathcal{L}}$  are  $\hat{G}$ -equivalent, then their counterparts  $\rho^* \hat{f}^1$  and  $\rho^* \hat{f}^2$  among solutions of the system  $\mathcal{L}$  are necessarily  $G$ -equivalent. Taking into

account these remarks, we can formulate the following optimised procedure for constructing hiddenly invariant solutions of  $\mathcal{L}$ :

- Construct a complete list of  $\hat{G}$ -inequivalent subalgebras of the algebra  $\hat{\mathfrak{g}}$ .
- From this list, exclude all such subalgebras  $\hat{\mathfrak{s}}$  that any element of the  $\hat{G}$ -orbit of  $\hat{\mathfrak{s}}$  is contained in  $\tilde{\mathfrak{g}}$ .
- For each of the remaining subalgebras in the list, carry out, if possible, the Lie reduction of the system  $\hat{\mathcal{L}}$  with respect to it and find the corresponding invariant solutions of  $\hat{\mathcal{L}}$ .
- Extend the obtained solutions using a complete set of  $\tilde{G}$ -inequivalent transformations from  $\hat{G}$  under the left action of  $\tilde{G}$  on  $\hat{G}$ .
- Pull back by  $\rho$  the extended set of solutions.
- Make a final arrangement of the constructed set of solutions of  $\mathcal{L}$ , selecting, up to the  $G$ -equivalence, only those solutions that are really hiddenly invariant.