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CENTRALIZING MAPPINGS OF SEMIPRIME RINGS

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ABSTRACT. Let R be a ring with center Z, and S a nonempty subset of R. A mapping F from R to R is called centralizing on S if $[x, F(x)] \in Z$ for all $x \in S$. We show that a semiprime ring R must have a nontrivial central ideal if it admits an appropriate endomorphism or derivation which is centralizing on some nontrivial one-sided ideal. Under similar hypotheses, we prove commutativity in prime rings.

1. **Introduction**. Let R denote a ring with center Z, and let S be a nonempty subset of R. A mapping F from R to R is called *centralizing on* S if $[x, F(x)] \in Z$ for all $x \in S$; in the special case where $[x, F(x)] = 0$ for all $x \in S$, the mapping F is described as *commuting on* S. Over the last fifteen years, several authors [5, 7, 8, 9, 10] have proved commutativity theorems for prime rings admitting automorphisms or derivations which are centralizing on appropriate subsets of R. The culminating theorems in this series, due to Mayne [9], assert that if a prime ring R admits either a nonidentity automorphism or a nonzero derivation which is centralizing on some nonzero ideal U of R, then R is commutative.

Our purpose is to study comparable problems in the setting of semiprime rings, to study centralizing endomorphisms which are not necessarily automorphisms, and to explore the consequences of the assumption that our mappings are centralizing on a one-sided ideal. For the case of prime rings, we establish Mayne's result under the weaker hypothesis that our mapping is centralizing on a nonzero left ideal U. In the case of derivations, our result for prime rings is a direct corollary of a theorem on semiprime rings. In the case of endomorphisms, we do achieve a result for semiprime rings, but the prime case requires an additional argument involving Martindale's extended centroid.

Our methods, which are somewhat different from those employed by other authors, make extensive use of the basic commutator identities

(I) $[x, yz] = y[x, z] + [x, y]z$ and $[xz, y] = x[z, y] + [x, y]z$

and several well-known facts about prime and semiprime rings:

(II) the center of a semiprime ring contains no nonzero nilpotent elements;

(III) in a semiprime ring, the center of a nonzero one-sided ideal is contained in the

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center of R; in particular, any commutative one-sided ideal is contained in the center of R;
(IV) in a prime ring, the centralizer of any nonzero one-sided ideal is equal to the center of R; in particular, if R has a nonzero central ideal, R must be commutative.

In addition, we require two deeper results, the first apparently due to Levitzki, and the second due to Martindale:

(V) [2, Lemma 1.1] Let n be a fixed positive integer. If a ring R contains a nonzero left ideal I such that $x^n = 0$ for all $x \in I$, then R contains a nonzero nilpotent ideal. In particular, a semiprime ring has no nonzero nil left ideals of bounded index.
(VI) [6] Let R be a prime ring. If a, b are elements of R with $axb = bxa$ for all $x \in R$, and if $a \neq 0$, then $b = \lambda a$ for some λ in the extended centroid of R.

Following established practice, we shall on occasion use the symbols x^F and S^F to denote the images of elements and subsets under the mapping F.

2. **Results on Centralizing Endomorphisms**. We being with two lemmas, both extending results of Mayne [8, 9].

LEMMA 1. *Let T be an endomorphism of the prime ring R, and let U be a nonzero left ideal of R. Then*

(i) if $u^T = u$ for all $u \in U$, T is the identity map on R;
(ii) if T is one-to-one on U, it is one-to-one on R.

PROOF. (i) For arbitrary $r \in R$ and $u \in U$, $ru = (ru)^T = r^T u^T = r^T u$; hence $(r - r^T)U = 0$ and therefore $r = r^T$.
(ii) Observe that $(\ker T)U \subseteq (\ker T) \cap U = \{0\}$; and since $U \neq \{0\}$, $\ker T = \{0\}$.

LEMMA 2. *Let $U \neq \{0\}$ be a left ideal of the semiprime ring R. If T is an endomorphism of R which is centralizing on U, then T is commuting on U.*

PROOF. Polarizing the condition that $[x, x^T] \in Z$ for all $x \in U$, we obtain

(2) $$[x, y^T] + [y, x^T] \in Z \text{ for all } x, y \in U.$$

Replacing y by x^2, we then get $2x^T[x, x^T] + 2x[x, x^T] \in Z$ for all $x \in U$; and since the second summand commutes with x, we have $2[x^T[x, x^T], x] = 0$, from which it follows that $2[x, x^T]^2 = 0$ for all $x \in U$. Since the center of a semiprime ring contains no nonzero nilpotent elements, we conclude that

(3) $$2[x, x^T] = 0 \text{ for all } x \in U,$$

and hence

(4) $$2([x, y^T] + [y, x^T]) = 0 \text{ for all } x, y \in U.$$

Making use of (2), we can show easily that for all $x, y \in U$,

$[xy + yx, x^T] + [x^2, y^T] = 2x([x, y^T] + [y, x^T]) + 2[x, x^T]y$; and applying (3) and

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(4), we get the crucial identity

$$(5) \qquad [xy + yx, x^T] + [x^2, y^T] = 0 \text{ for all } x, y \in U.$$

For $x \in U$, take $y = x^T x^2$ in (5), thereby obtaining

$$(6) \qquad (xx^T + x^T x)[x^2, x^T] + x^{TT}[x^2, x^T x^T] + [xx^T + x^T x, x^T]x^2 + [x^2, x^{TT}]x^T x^T = 0.$$

Noting that $[x^2, x^T] = x[x, x^T] + [x, x^T]x = 2x[x, x^T] = 0$, we now get

$$(7) \qquad [xx^T + x^T x, x^T]x^2 + [x^2, x^{TT}]x^T x^T = 0 \text{ for all } x \in U.$$

But $[xx^T + x^T x, x^T] = [[x, x^T] + 2x^T x, x^T] = 2[x^T x, x^T] = 2x^T[x, x^T] = 0$, so from (7) it follows that

$$(8) \qquad [x^2, x^{TT}]x^T x^T = 0 \text{ for all } x \in U.$$

On the other hand, taking $y = x^T x$ in (5) yields $[xx^T x + x^T xx, x^T] + [x^2, x^{TT} x^T] = 0$, hence $[([x, x^T] + 2x^T x)x, x^T] + x^{TT}[x^2, x^T] + [x^2, x^{TT}]x^T = 0$, and finally

$$(9) \qquad [x, x^T]^2 + [x^2, x^{TT}]x^T = 0 \text{ for all } x \in U.$$

From (9) it follows that $w = [x^2, x^{TT}]x^T$ is central, and from (8) that $w^2 = 0$. It is now apparent from (9) that $[x, x^T]^4 = 0$, and the absence of nonzero central nilpotent elements implies that $[x, x^T] = 0$ for all $x \in U$.

THEOREM 1. *Let R be a semiprime ring and U a nonzero left ideal of R. Suppose that R admits an endomorphism T which is one-to-one on U and centralizing on U; suppose also that the left ideal $Q = U \cap T^{-1}(U) \cap T^{-2}(U) \cap T^{-3}(U)$ is nonzero and that T is not the identity map on Q. Then R contains a non-zero central ideal.*

PROOF. In view of Lemma 2, we have $[x, x^T] = 0$ for all $x \in U$ – *a* condition which polarizes to

$$(10) \qquad [x, y^T] = [x^T, y] \text{ for all } x, y \in U.$$

Substituting xy for y and applying (10), we then get

$$(11) \qquad (x - x^T)[x^T, y] = 0 \text{ for all } x, y \in U;$$

and replacing y by wy for $w \in U$ yields $(x - x^T)U[x^T, y] = 0$ for all $x, y \in U$, so that

$$(12) \qquad (x - x^T)RU[x^T, y] = 0 \text{ for all } x, y \in U.$$

Now choose a family $\mathcal{P} = \{P_\alpha | \alpha \in \Lambda\}$ of prime ideals of R for which $\cap P_\alpha = \{0\}$; and let P denote a fixed one of the P_α. From (12) it follows that for each $x \in U$, either

(i) $x - x^T \in P$, or

(ii) $U[x^T, y] \subseteq P$ for all $y \in U$.

Define $U_{(i)}$ to be the set of all $x \in U$ for which (i) holds and $U_{(ii)}$ the set of $x \in U$ for which (ii) holds; note that both are additive subgroups of U and their union is equal to

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U. Thus, either $U_{(i)} = U$ or $U_{(ii)} = U$; hence P satisfies one of the following:

(i)′ $x - x^T \in P$ for all $x \in U$;

(ii)′ $U[x^T, y] \subseteq P$ for all $x, y \in U$.

Call a prime ideal in $\mathcal{P}$ a type-one prime if it satisfies (i)′; call all other members of $\mathcal{P}$ type-two primes. Define P_1 and P_2 respectively as the intersection of all type-one primes and the intersection of all type-two primes; and note that

$$P_1P_2 = P_2P_1 = P_1 \cap P_2 = \{0\}. \tag{13}$$

Define W to be $U \cap T^{-1}(U)$, so that for $x \in W$, both x^T and $x - x^T$ are in U. Thus, from (i)′ and (ii)′ we can conclude that $W^T[x^T - x^{TT}, y^T] \subseteq P_1 \cap P_2 = \{0\}$ for all x, $y \in W$; and since T was one-to-one on U, we have

$$W[x - x^T, y] = 0 \text{ for all } x, y \in W. \tag{14}$$

Recalling (ii)′, we now have

$$W[x, y] \subseteq P_\alpha \text{ for all } x, y \in W \text{ and all type-two primes } P_\alpha. \tag{15}$$

Now returning to (10) and replacing x by xy, we get $[x^T, y](y - y^T) = 0$ for all x, $y \in U$, so that $[x^T, y^T](y^T - y^{TT}) = 0$ for all $x \in U$ and $y \in W$. Invoking the one-to-one-ness of T on U shows that $[x, y](y - y^T) = 0$ for all $x \in U$ and $y \in W$; and replacing x by xu for $u \in U$, we see that $[x, y]U(y - y^T) = [x, y]RU(y - y^T) = 0$. Again considering a fixed P_α in $\mathcal{P}$, we see that for fixed $y \in W$, either $U(y - y^T) \subseteq P_\alpha$ or $[x, y] \in P_\alpha$ for all $x \in U$. Since the sets of $y \in W$ for which these two alternatives hold form two additive subgroups of W with union equal to W, we conclude that either $[x, y] \in P_\alpha$ for all $x \in U$ and $y \in W$, or $U(y - y^T) \subseteq P_\alpha$ for all $y \in W$. Thus,

$$[x, y]U(z - z^T) \subseteq \bigcap_{\alpha \in \Lambda} P_\alpha = \{0\} \text{ for all } x \in U \text{ and } y, z \in W;$$

in particular,

$$[x, y](z_1 - z_1^T)(z_2 - z_2^T) = 0 \text{ for all } x, y, z_1, z_2 \in W. \tag{16}$$

We can now identify a central ideal of R. Specifically, define V to be the left ideal generated by all elements of form $u(v - v^T)$ for $u, v \in W$. In fact, V is the set of all finite sums and differences of the generating elements; and to show that V is a commutative, hence central, ideal of R, it will suffice to show that

$$[u_1(v_1 - v_1^T), u_2(v_2 - v_2^T)] = 0 \text{ for all } u_1, u_2, v_1, v_2 \in W. \tag{17}$$

Accordingly, we note that

$$\begin{aligned}
[u_1(v_1 - v_1^T), u_2(v_2 - v_2^T)] &= u_1[v_1 - v_1^T, u_2(v_2 - v_2^T)] \\
&\quad + [u_1, u_2(v_2 - v_2^T)](v_1 - v_1^T) \\
&= u_1[v_1 - v_1^T, u_2v_2] - u_1u_2[v_1 - v_1^T, v_2^T] \\
&\quad - u_1[v_1 - v_1^T, u_2]v_2^T + u_2[u_1, v_2 - v_2^T](v_1 - v_1^T) \\
&\quad + [u_1, u_2](v_2 - v_2^T)(v_1 - v_1^T).
\end{aligned}$$

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It is immediate from (14) and (16) that in this last sum, every summand except possibly the second is 0; moreover, (i)′, (ii)′, and the definition of type-one and type-two primes show that the second summand belongs to $P_1 \cap P_2 = \{0\}$. Thus, (17) holds and V is a central ideal. Our theorem will be established once we show that $V \neq \{0\}$.

Suppose, then, that $V = \{0\}$. Then

$$W(y - y^T) = 0 \text{ for all } y \in W. \tag{18}$$

Define $W_1 = W \cap T^{-1}(W) = U \cap T^{-1}(U) \cap T^{-2}(U)$, and define $F = \{u \in W_1 | u^T = u\}$. From (10) and (18) it follows that

$$xy + yx \in F \text{ for all } x, y \in W_1. \tag{19}$$

Restricting x to F and applying (10) yields $x(y - y^T) = (y - y^T)x$ for all $x \in F$ and $y \in U$; hence (18) yields

$$(y - y^T)x = 0 \text{ for all } x \in F \text{ and } y \in W. \tag{20}$$

In particular, (19) now gives

$$(y - y^T)(xz + zx) = 0 \text{ for all } x, y, z \in W_1. \tag{21}$$

But by (15) we have $(y - y^T)(xz - zx) = 0$ for all $x, y, z \in W_1$; therefore, $2(y - y^T)W_1^2 = 0$ and hence

$$2(y - y^T)W_1 = 0 \text{ for all } y \in W_1. \tag{22}$$

Observe that (18) implies $(y - y^T)^2 = 0$ for all $y \in W_1$—a result which together with (22) yields $(y^T)^2 = (y - (y - y^T))^2 = y^2 - 2(y - y^T)y + (y - y^T)^2 = y^2$; thus, $y^2 \in F$ for all $y \in W_1$. In view of (20), it follows by taking x^2 for x and rx for z in (21) that

$$(y - y^T)Rx^3 = 0 \text{ for all } x, y \in W_1. \tag{23}$$

It is now clear that for each $P_\alpha \in \mathcal{P}$, either $y - y^T \in P_\alpha$ for all $y \in W_1$ or $x^3 \in P_\alpha$ for all $x \in W_1$. Call P_α a prime of type three if the first of these alternatives holds, otherwise call P_α a prime of type 4; let P_3 and P_4 be the intersections of all type-three and type-four primes respectively; note that $P_3 \cap P_4 = \{0\}$. Noting that the left ideal Q in the hypotheses of our theorem is $W_1 \cap T^{-1}(W_1)$, and recalling that T is not the identity on Q, we see that there exists $y \in Q$ such that $y - y^T \neq 0$; hence $\tilde{W}_1 = P_3 \cap W_1 \neq \{0\}$. Since $x^3 \in P_4$ for each $x \in \tilde{W}_1$, we see that $\tilde{W}_1$ is a nonzero left ideal with $x^3 = 0$ for each $x \in \tilde{W}_1$. But a semiprime ring cannot have such a left ideal, so we have contradicted our assumption that $V = \{0\}$, thereby completing the proof of Theorem 1.

COROLLARY 1. *Let R be semiprime and U a nonzero left ideal; and suppose that R admits an endomorphism T which is one-to-one on U, centralizing on U, and not the identity on U. If $U^T \subseteq U$, then R contains a nonzero central ideal.*

For prime rings, we shall require the following corollary, which depends on Lemma 1(i) as well as Theorem 1.

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COROLLARY 2. *Let R be a prime ring and U a nontrivial left ideal of R. If R admits a nonidentity endomorphism T which is one-to-one on U and centralizing on U, and if U contains a nonzero element which is fixed by T, then R is commutative.*

In the statement of Theorem 1, the hypothesis that T is not the identity on Q may appear somewhat technical or even artificial; however, some such hypothesis is required. Consider, for example, the semiprime ring $R = S \oplus S$, where S is a simple noncommutative ring; and define the endomorphism T by $T((x, y)) = (y, x)$. Then if U is the ideal consisting of all elements of form $(s, 0)$ with $s \in S$, the endomorphism T is clearly one-to-one on U, commuting on U, and not the identity on U; but R has no nontrivial central ideal.

If we could replace our hypothesis involving Q by a condition which is satisfied for all prime rings, then our next theorem would follows as an immediate consequence. However, we have been unable to find an appropriate hypothesis of this kind; hence, some additional work is required in the prime case.

THEOREM 2. *Let R be a prime ring and U a nontrivial left ideal of R. If R admits a nonidentity endomorphism T which is one-to-one on U and centralizing on U, then R is commutative.*

PROOF. Since (12) holds for R and since Lemma 1 guarantees that T is not the identity on U, we conclude that

$$U[x^T, y] = 0 \text{ for all } x, y \in U. \tag{24}$$

Moreover, recalling from the proof of Theorem 1 that $[x^T, y](y - y^T) = 0$ for all x, $y \in U$, we get $[x^T, y]RU(y - y^T) = 0$ for all $x, y \in U$; hence either $[x^T, y] = 0$ for all $x, y \in U$ or $U(y - y^T) = 0$ for all $y \in U$. If the first of these alternatives holds, then U^T is contained in the centralizer of U; hence by (IV), U^T is contained in the center of R. Applying the first isomorphism theorem to the ring homomorphism from U to U^T induced by T, we then get $U^T \cong U/(U \cap \ker T) \cong U$, so that U is commutative and hence R is commutative. Thus, we assume henceforth that

$$U(y - y^T) = 0 \text{ for all } y \in U, \tag{25}$$

which combines with (24) to give

$$U[x, y] = 0 \text{ for all } x, y \in U. \tag{26}$$

Note that if U contains a non-zero element which is fixed by T, then by Corollary 2, R has a nonzero central ideal and so must be commutative. Therefore we assume that U contains no fixed points.

Invoking (V), choose $a \in U$ such that $a^3 \neq 0$. Observe that by (25) we have

$$a^2 = aa^T = a^Ta. \tag{27}$$

From (26), we have $a[xa, ya] = 0$ for all $x, y \in R$—that is, $axaya = ayaxa$. Fixing x and applying (VI), we get an element $\lambda(x)$ in the extended centroid C of R such that

$axa = \lambda(x)a$. An immediate consequence is that $[axa, a] = 0$, or equivalently $axa^2 = a^2xa$, for all $x \in R$; hence another application of (VI) yields a nonzero $\lambda \in C$ for which $a^2 = \lambda a$. It follows that

$$a = \lambda^{-1}a^2 = \lambda^{-1}aa^T = \lambda^{-1}a^Ta. \tag{28}$$

Let us use the notation $x \perp y$ to mean that $xy = yx = 0$. Since $a \perp a^T - a$, we have

$$a^T \perp a^{TT} - a^T \text{ and } a^{TT} \perp a^{TTT} - a^{TT}; \tag{29}$$

the first of these conditions, together with (28), yields $a \perp a^{TT} - a^T$ and hence

$$a^T \perp a^{TTT} - a^{TT}. \tag{30}$$

Recalling (27) and (29), we now get

$$a^2 \perp a^{TTT} - a^{TT} \text{ and } a^{TT} - a^T \perp a^{TTT} - a^{TT}. \tag{31}$$

Now, for arbitrary $x \in R$ we have

$$[a^2 + (a^{TT} - a^T)xa, (a^T)^2 + (a^{TTT} - a^{TT})x^Ta^T] = 0;$$

and applying (29) and (31) gives

$$(a^{TT} - a^T)xa(a^T)^2 - (a^{TTT} - a^{TT})x^Ta^Ta^2 = 0.$$

Left-multiplying by $a^{TT} - a^T$ and noting (31), we get $(a^{TT} - a^T)^2xa(a^T)^2 = 0$, which by (27) is the same as $(a^{TT} - a^T)^2xa^3 = 0$. The primeness of R and the fact that $a^3 \neq 0$ show that $(a^{TT} - a^T)^2 = 0$; and since T is one-to-one on R by Lemma 1, we conclude that $(a^T - a)^2 = 0 = (a^2)^T - 2aa^T + a^2$. It now follows from (27) that a^2 is a nonzero element of U fixed by T. But this is a contradiction, so our proof is complete.

3. **Centralizing Derivations**. For derivations, we need analogues of Lemmas 1 and 2.

LEMMA 3. [9] *Let U be a nonzero left ideal of the prime ring R. If D is a nonzero derivation of R, then D is nonzero on U.*

PROOF. If $D(x) = D(rx) = 0$ for all $x \in U$ and $r \in R$, it follows that $D(r)x = 0$; hence $D(R)U = \{0\}$ and $D(R) = \{0\}$.

LEMMA 4. *Let R be a semiprime ring and U a nonzero left ideal. If D is a derivation of R which is centralizing on U, then D is commuting on U.*

PROOF. For arbitrary $x \in U$, we have $[x^2, D(x^2)] \in Z$—that is, $[x^2, xD(x) + D(x)x] = [x^2, 2xD(x) - [x, D(x)]] = 2[x^2, xD(x)] = 4x^2[x, D(x)] \in Z$. Thus, $4[x^2[x, D(x)], D(x)] = 0$, from which it follows that $8x[x, D(x)]^2 = 0$ and hence $8[x, D(x)]^3 = 0$. Again invoking (II), we get

$$2[x, D(x)] = 0 \text{ for all } x \in U; \tag{32}$$

and it follows at once that

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$$(33)\qquad [x^2, D(x)] = 0 \text{ for all } x \in U.$$

By polarizing both (32) and the original hypothesis that $[x, D(x)] \in Z$ for all $x \in U$, we see that $[x, D(y)] + [y, D(x)] \in Z$ and $2([x, D(y)] + [y, D(x)]) = 0$ for all $x, y \in U$; and by combining these results with (32), we can show that

$$(34)\qquad [xy + yx, D(x)] + [x^2, D(y)] = 0 \text{ for all } x, y \in U.$$

Replacing y by yx yields

$$(xy + yx)[x, D(x)] + ([xy + yx, D(x)] + [x^2, D(y)])x + y[x^2, D(x)] + [x^2, y]D(x) = 0 \text{ for all } x, y \in U.$$

Rewriting the first summand as $([x, y] + 2yx)[x, D(x)]$ and using (32), (33), and (34), we get

$$[x, y][x, D(x)] + [x^2, y]D(x) = 0 \text{ for all } x, y \in U;$$

taking $y = D(x)x$ and using (33), we thus conclude that $[x, D(x)]x[x, D(x)] = 0 = [x, D(x)]^3$ and hence $[x, D(x)] = 0$ for all $x \in U$.

THEOREM 3. *Let R be a semiprime ring and U a nonzero left ideal. If R admits a derivation D which is nonzero on U and centralizing on U, then R contains a nonzero central ideal.*

PROOF. Since D is commuting on U by Lemma 4, we have

$$(35)\qquad [u, D(v)] + [v, D(u)] = 0 \text{ for all } u, v \in U.$$

In particular, for $x, y \in U$, we have $[x, D(yx)] + [yx, D(x)] = 0$, which reduces to $([x, D(y)] + [y, D(x)])x + [x, y]D(x) = 0$; and (35) now gives $[x, y]D(x) = 0$ for all $x, y \in U$. By replacing y by wy for arbitrary $w \in U$, we get

$$(36)\qquad [x, w]RUD(x) = 0 \text{ for all } x, w \in U.$$

Now, as in the proof of Theorem 1, we let $\mathcal{P} = \{P_\alpha | \alpha \in \Lambda\}$ be a family of prime ideals with $\cap P_\alpha = \{0\}$. From (36) it follows that for each P_α, either

$$\text{(a)}\qquad [x, w] \in P_\alpha \text{ for all } x, w \in U$$

or

$$\text{(b)}\qquad UD(U) \subseteq P_\alpha.$$

Call P_α a type-one prime if it satisfies (a), a type-two prime otherwise; let P_1 and P_2 be respectively the intersections of all type-one and type-two primes; note that $P_1 \cap P_2 = \{0\}$.

We now investigate a typical type-two prime $P = P_\alpha$. From (b) and the fact that $[u, D(u)] = 0$ for all $u \in U$, we have $uD(u) \in P$ and $D(u)u \in P$ for all $u \in U$. Thus $(x + y)(D(x) + D(y)) \in P$ and $(D(x) + D(y))(x + y) \in P$ for all $x, y \in U$; consequently, $xD(y) + yD(x) \in P$ and $D(x)y + D(y)x \in P$. Direct calculation now yields

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(37) $$D(xy + yx) \in P \text{ for all } x, y \in U.$$

It follows that $D(z(xy + yx) + (xy + yx)z) \in P$ for all $x, y, z \in U$. Writing this as $D(z)(xy + yx) + zD(xy + yx) + D(xy + yx)z + (xy + yx)D(z) \in P$, and noting that the last three summands are in P by (b) and (37), we get

$$D(z)(xy + yx) \in P \text{ for all } x, y, z \in U.$$

Replacing x by zx and noting that $D(z)z \in P$, we see that $D(z)\ yzx \in P$ for all $x, y, z \in U$; hence

$$D(z)Ryzx \subseteq P \text{ for all } x, y, z \in U.$$

The fact that P is a prime ideal now shows that either $D(U) \subseteq P$ or $U^3 \subseteq P$. But if the latter holds, we get $U \subseteq P$ and hence (a) holds for P, contradicting our definition of type-two prime; therefore, $D(U) \subseteq P$. It now follows that for $r \in R$ and $u \in U$, $D(r)u = D(ru) - rD(u) \in P$, so that $RD(R)U \subseteq P$; and since $U \not\subseteq P$, we conclude that $RD(R) \subseteq P$. This being true for every type-two prime, we have

(38) $$RD(R) \subseteq P_2.$$

Consider now the left ideal V generated by the set $D(R)U$; we shall show that V is commutative, hence a two-sided central ideal. A typical element of V is a sum of elements of form $D(r)u$ and $sD(r)u$, where $r, s \in R$ and $u \in U$. Thus we need only show that commutators of the forms $[D(r_1)u_1, D(r_2)u_2]$, $[s_1D(r_1)u_1, D(r_2)u_2]$ and $[s_1D(r_1)u_1, s_2D(r_2)u_2]$ are all trivial. Clearly all three types are in P_1 by (a), and they are all in P_2 by (38); hence all belong to $P_1 \cap P_2 = \{0\}$.

If $V \neq \{0\}$, we are finished. Assume, therefore, that $V = \{0\}$, in which case $D(R)U = \{0\}$. The left ideal $UD(R)$ is therefore nilpotent, so $UD(R) = 0$. Thus, $uD(rs) = 0$ for all $u \in U$ and all $r, s \in R$, so that $uD(r)s + urD(s) = 0$ and therefore

(39) $$URD(R) = 0.$$

In particular, for each $u \in U$ and $x \in R$, $uxD(u) = 0$ and hence $D(uxD(u)) = uD(xD(u)) + D(u)xD(u) = 0$. Expanding the first term yields $uxD^2(u) + uD(x)D(u) + D(u)xD(u) = 0$; and since the first two summands are trivial by (39), $D(u)RD(u) = 0$ for all $u \in U$ and hence $D(U) = 0$. This contradicts our initial hypothesis, so the central ideal V must in fact be nonzero.

In view of (IV) and Lemma 3, our final theorem is immediate from Theorem 3.

THEOREM 4. *Let R be a prime ring and U a nonzero left ideal. If R admits a nonzero derivation which is centralizing on U, then R is commutative.*

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