EXTREME POINTS AND LINEAR ISOMETRIES OF THE BANACH SPACE OF LIPSCHITZ FUNCTIONS

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Introduction. Let X be a compact metric space with metric d. A complex-valued function f on X is said to satisfy a *Lipschitz condition* if, for all points x and y of X, there exists a constant K such that

$$|f(x) - f(y)| \leqq Kd(x, y).$$

The smallest constant for which the above inequality holds is called the *Lipschitz constant* for f and is denoted by $||f||_d$, that is,

$$||f||_d = \sup_{\substack{x,y \in X; \\ x \neq y}} \frac{|f(x) - f(y)|}{d(x, y)}.$$

The space of Lipschitz functions, denoted by $\mathrm{Lip}(X, d)$ as in (**7**), consists of all functions f for which $||f||_d$ is finite. It is clear that $\mathrm{Lip}(X, d)$ is a vector space over the complex numbers C and we make it a Banach space by defining

$$||f|| = \max(||f||_\infty, ||f||_d),$$

where

$$||f||_\infty = \sup_{x \in X} |f(x)|.$$

In §1, we show that if X is connected, then the isometries of $\mathrm{Lip}(X, d)$ are precisely those induced by the isometries of the metric space X. In §2, we prove that if X is the n-dimensional cube in R^n and d is the metric given by the l_1-norm of R^n, then $\mathrm{Lip}(X, d)$ is a conjugate space, that is, it is isometrically isomorphic to the conjugate space of another Banach space. Here the method of proof is a close imitation of one used by de Leeuw in (**4**).

The results of §2 and the Kreĭn-Milman theorem suggest that when X is the n-dimensional cube, the unit ball of $\mathrm{Lip}(X, d)$ has quite a few extreme points. In §3, we explicitly determine these extreme points when X is the unit interval on the real line and prove that the unit ball of $\mathrm{Lip}[0, 1]$ is the closed convex hull of these extreme points in the norm topology.

1. Linear isometries. By a linear isometry of $\mathrm{Lip}(X, d)$ we mean, of course, a norm-preserving linear transformation of $\mathrm{Lip}(X, d)$ onto itself. It is

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necessary to give some definitions and establish several lemmas before we can state and prove the main result of this section.

Let W be the complement of the diagonal in the Cartesian product space $X \times X$. Let $Y = X \cup \beta W$, where βW is the *Stone-Čech compactification* (**1**, p. 276) of W. Then Y is a compact Hausdorff space.

If $C(Y)$ denotes the space of continuous functions on Y, define $\tilde{f}$ on $C(Y)$ as follows:

$$\tilde{f}(x) = f(x), \qquad x \in X,$$

$$\tilde{f}(w) = (\beta f^*)(w), \qquad w \in \beta W,$$

where

$$f^*(x, y) = \frac{f(x) - f(y)}{d(x, y)}$$

is defined on W and βf^* is its norm-preserving extension to βW.

LEMMA 1.1. *The map $f \to \tilde{f}$ establishes a linear and norm-preserving correspondence between* $\operatorname{Lip}(X, d)$ *and the closed subspace S of $C(Y)$,*

$$S = \{\tilde{f}: f \in \operatorname{Lip}(X, d)\}.$$

Proof. This follows from the definition of the function $\tilde{f}$.

LEMMA 1.2. *If* $\operatorname{Lip}^*(X, d)$ *denotes the conjugate space of* $\operatorname{Lip}(X, d)$, *then all the extreme points of the unit ball of* $\operatorname{Lip}^*(X, d)$ *are contained in*

$$\{e^{i\theta}L_x: x \in X\} \cup \{e^{i\eta}L_w: w \in \beta W\},$$

where L_x, L_w are 'point evaluations' at x, w respectively, and θ, η range over $[0, 2\pi)$. *Moreover, every $e^{i\theta}L_x$ is an extreme point of the unit ball of* $\operatorname{Lip}^*(X, d)$.

Proof. It is well known (see, e.g., **1**, p. 441) that every extreme point L of the unit ball of S^* is of the form

$$L(\tilde{f}) = e^{i\alpha}\tilde{f}(y),$$

where $\alpha \in [0, 2\pi)$ and $y \in Y$. From the definition of $\tilde{f}$ it follows that each extreme point of the unit ball of $\operatorname{Lip}^*(X, d)$ is of the form described in the statement of the lemma.

The proof that every $e^{i\theta}L_x$ (or every L_x equivalently) is an extreme point will be based on the following fact which is a special case of a somewhat more general result proved by de Leeuw (**4**): Let X be any compact Hausdorff space and let B be a closed subspace of $C(X)$ and let $x \in X$. If there exists an $f \in B$ such that $f(x) = ||f||_\infty$ and $|f(y)| \leqq ||f||_\infty$, $y \in X$, $y \neq x$, with equality holding only for those $y \in X$ for which

$$g(y) = g(x)$$

or

$$g(y) = -g(x)$$

for all $g \in B$, then the linear functional $L_x(h) = h(x)$, $h \in B$, is an extreme point of the unit ball of B^*.

To use the above result, let x_0 be a point in our compact metric space X and define $f(x) = d(x, x_0)$. Then $f \in C(X)$, thus it attains its maximum at a point $\tilde{x}$,

$$||f||_\infty = d(\tilde{x}, x_0).$$

Let

$$g(x) = 2 + d(\tilde{x}, x_0) - d(x, x_0).$$

Then it is clear that $g \in \text{Lip}(X, d)$, $g(x) \geqq 2$ for all $x \in X$ and that $||g||_d = 1$. The function $\tilde{g} \in S$ fulfills the conditions of de Leeuw's theorem and this proves our lemma.

LEMMA 1.3. *Assume that in addition to being compact, X is connected with diameter at most 1. Let $B_1, \ldots, B_n$ be open, pairwise disjoint, and proper subsets of X. Define the functions*

$$\delta_i(x) = d(x, X \sim B_i), \qquad i = 1, \ldots, n.$$

Suppose that

(i) $\lambda_i \in C$, $|\lambda_i| = 1$ *and* $|1 + \lambda_i| \leqq 1$,

(ii) $\min_{i \neq j} d(B_i, B_j) \geqq 2 \sup_i ||\delta_i||_\infty$.

Then $1 + \sum_{i=1}^n \lambda_i \delta_i$ is an extreme point of U, where U is the unit ball of $\text{Lip}(X, d)$.

Proof. It is quite easy to check that

$$\left|\left(\sum_{i=1}^n \lambda_i \delta_i\right)(x) - \left(\sum_{i=1}^n \lambda_i \delta_i\right)(y)\right| \leqq d(x, y)$$

for all $x, y \in X$. This means that $||1 + \sum \lambda_i \delta_i||_d \leqq 1$. It is also clear from the assumptions made in the lemma that $||1 + \sum \lambda_i \delta_i||_\infty \leqq 1$. Hence, $1 + \sum \lambda_i \delta_i \in$ U, where U is the unit ball of $\text{Lip}(X, d)$.

Let

$$1 + \sum \lambda_i \delta_i = \tfrac{1}{2} g + \tfrac{1}{2} h,$$

where $g, h \in U$. If $x \notin \bigcup_i B_i$, we must have

$$1 = \tfrac{1}{2} g(x) + \tfrac{1}{2} h(x),$$

which implies that $g(x) = h(x) = 1$. If $x \in B_k$, choose $y \in X \sim B_k$ such that

$$\delta_k(x) = d(x, y).$$

Now we maintain that $y \notin \bigcup_{i=1}^n B_i$ for otherwise if $y \in B_j$ for some j, then

$$d(B_k, B_j) \leqq d(x, y) \leqq ||\delta_k||_\infty$$

which contradicts hypothesis (ii) of the lemma. This yields

$$\tfrac{1}{2} g(y) + \tfrac{1}{2} h(y) = 1$$

and therefore $g(y) = h(y) = 1$. Now,

$$|g(x) - 1| = |g(x) - g(y)| \leqq d(x, y) = |(\lambda_k \delta_k)(x)| = |(\sum \lambda_i \delta_i)(x)|$$

and similarly

$$|h(x) - 1| \leqq |(\sum \lambda_i \delta_i)(x)|.$$

From

$$\tfrac{1}{2}(g(x) - 1) + \tfrac{1}{2}(h(x) - 1) = (\sum \lambda_i \delta_i)(x)$$

we get

$$g(x) - 1 = h(x) - 1 = (\sum \lambda_i \delta_i)(x)$$

and therefore

$$g(x) = h(x).$$

Since we have proved that $g(x) = h(x)$ for all $x \in X$, we can conclude that $1 + \sum \lambda_i \delta_i$ is an extreme point of U.

We assume from now on that X is connected.

LEMMA 1.4. *If $f \in U$ and f is not a constant, then there exists a $g \in E$ such that $||e^{i\theta} f + g|| > 1$ for all $\theta \in [0, 2\pi)$, where E denotes the set of extreme points of U.*

Proof. Let $c_1, \ldots, c_{31}$ be distinct points of $f(X)$ (these exist because f is not identically equal to a constant and $f(X)$ is connected). Then

$$f^{-1}(c_i), \qquad i = 1, \ldots, 31,$$

are disjoint closed, hence compact, subsets of X. Let

$$\min_{i \neq j} d(f^{-1}(c_i), f^{-1}(c_j)) = 8\epsilon > 0.$$

$f^{-1}(c_1)$ is a non-empty, closed, and proper subset of X. From the connectedness of X, there exist $x_1 \in f^{-1}(c_1)$ and $y_1 \in B(x_1)$ such that $f(x_1) \neq f(y_1)$, where $B(x_1)$ is the open ball of radius ϵ with x_1 as centre. Choosing $(x_2, y_2), \ldots, (x_{31}, y_{31})$ in a similar manner we get 31 non-zero complex numbers

$$(f(x_k), f(y_k)), \qquad k = 1, \ldots, 31.$$

Divide $C \sim \{0\}$ into six equal wedges. One of these wedges (call it P) contains at least six of these complex numbers, say

$$(f(x_i), f(y_i)), \qquad i = 1, \ldots, 6.$$

Let $z_i \in \{x_i, y_i\}$ and let

$$B_{z_i} = \{y \in X : d(y, z_i) < d(x_i, y_i)\}, \qquad i = 1, \ldots, 6.$$

It is easy to check that $d(z_i, z_j) \geqq 6\epsilon$ $(i \neq j)$, from which it follows that $d(B_{z_i}, B_{z_j}) \geqq 4\epsilon$. If, as before, $\delta_{z_i}(x) = d(x, X \sim B_{z_i})$, then $2\epsilon \geqq ||\delta_{z_i}||_\infty$. We then have that

$$\inf_{i \neq j} d(B_{z_i}, B_{z_j}) \geqq 2 \sup_i ||\delta_{z_i}||_\infty,$$

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showing that condition (ii) of Lemma 1.3 is satisfied. Note that

$$\delta_{z_i}(z_i) = d(x_i, y_i)$$

and $\delta_{z_i}(\tilde{z}_i) = 0$, where $\tilde{z}_i = \{x_i, y_i\} \sim \{z_i\}$. Let

$$\lambda_{2k} = \lambda_{2k-1} = \exp(i(1 + k)\pi/3), \qquad k = 1, 2, 3.$$

Then $|\lambda_i| = 1$ and $|1 + \lambda_i| \leqq 1$, where $i = 1, \ldots, 6$. Let

$$g(x) = 1 + \left(\sum_{k=1}^{3} \lambda_{2k-1}\delta_{y_{2k-1}} + \lambda_{2k}\delta_{x_{2k}}\right)(x).$$

By Lemma 1.3, $g \in E$. Fix θ. Then $(-1)^\nu\lambda_\nu \in e^{i\theta}P$ for exactly one

$$\nu \in \{1, 2, \ldots, 6\}.$$

Now,

$$\left|\frac{(e^{i\theta}f + g)(x_\nu) - (e^{i\theta}f + g)(y_\nu)}{d(x_\nu, y_\nu)}\right| = \left|e^{i\theta}\frac{f(x_\nu) - f(y_\nu)}{d(x_\nu, y_\nu)} + (-1)^\nu\lambda_\nu\frac{d(x_\nu, y_\nu)}{d(x_\nu, y_\nu)}\right| > 1.$$

This means that $||e^{i\theta}f + g|| > 1$ for all $\theta \in [0, 2\pi)$.

LEMMA 1.5. *If T is a linear isometry of* Lip(X, d), *then $T(1)$ is the constant function $e^{i\theta_0}$, $\theta_0 \in [0, 2\pi)$.*

Proof. Since the constant function $1 \in E$ and a linear isometry carries extreme points onto themselves, we get that $T(1) \in E$.

Let $T(1) = f$ and suppose that f is not identically equal to a constant. By Lemma 1.4, there exists a $g \in E$ such that

$$||e^{i\theta}f + g|| > 1$$

for all $\theta \in [0, 2\pi)$. Since $T^{-1}g \in E$, it is quite easy to see that there exists a $\theta_1 \in [0, 2\pi)$ such that

$$||T^{-1}g + e^{i\theta_1}|| \leqq 1.$$

We then have that

$$||T(T^{-1}g + e^{i\theta})|| = ||g + e^{i\theta}f|| > 1$$

for all θ, but

$$||T(T^{-1}g + e^{i\theta_1})|| = ||T^{-1}g + e^{i\theta_1}|| \leqq 1,$$

which is a contradiction.

LEMMA 1.6. *T preserves 'supnorms', that is, $||Tf||_\infty = ||f||_\infty$ for all*

$$f \in \text{Lip}(X, d),$$

T as in Lemma 1.5.

Proof. Let T^* be the adjoint of T. T^* is also a linear isometry and it carries the extreme points of the unit ball of Lip$^*(X, d)$ onto themselves. We claim

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that $T^*L_x = e^{i\eta}L_{x'}$ for some $\eta \in [0, 2\pi)$ and some $x' \in X$. If this were not true, then by Lemma 1.2, $T^*L_x = e^{i\eta}L_w$, where $w \in \beta W$. Therefore,

$$(T^*L_x)(1) = (e^{i\eta}L_w)(1) = 0$$

or,

$$L_x(T(1)) = 0.$$

Since $T(1)$ is a non-zero constant function by Lemma 1.5, this yields a contradiction and proves the above claim.

Consider now an $f \in \mathrm{Lip}(X, d)$. If x_0 is a point at which Tf attains its maximum modulus, then we may write

$$(Tf)(x_0) = ||Tf||_\infty e^{i\theta_0}$$

for some $\theta_0 \in [0, 2\pi)$. By what was just proved,

$$T^*L_{x_0} = e^{i\eta}L_{x_1}$$

for some $\eta \in [0, 2\pi)$ and some $x_1 \in X$. Hence

$$e^{i\eta}f(x_1) = ||Tf||_\infty e^{i\theta_0},$$

showing that $||Tf||_\infty \leqq ||f||_\infty$. Since T^{-1} is also an isometry, the same inequality holds for it, which gives the reverse inequality for T. This proves the lemma.

We are now in a position to state and prove the main result of this section.

THEOREM 1.7. *Let (X, d) be a compact, connected metric space with diameter at most* 1. *Then*

$$T\colon \mathrm{Lip}(X, d) \to \mathrm{Lip}(X, d)$$

is a linear isometry if and only if

$$(1) \qquad (Tf)(x) = e^{i\theta}f(\tau x),$$

where $\tau\colon X \to X$ is an isometry of X onto itself and θ is a constant in $[0, 2\pi)$.

Proof. It is an easy matter to check that every isometry τ of X induces an isometry T of $\mathrm{Lip}(X, d)$ if T is defined as in (1). It remains to prove that every isometry T of $\mathrm{Lip}(X, d)$ is of the above type. Without loss of generality, we can assume that $T(1) = 1$ by Lemma 1.5. By Lemma 1.6, T preserves supnorms. Since $\mathrm{Lip}(X, d) \subset C(X)$ is a complex linear algebra (as the product of two Lipschitz functions is also a Lipschitz function), we can quote a well-known result to conclude that T is an algebra automorphism, that is,

$$T(fg) = T(f)T(g)$$

for all $f, g \in \mathrm{Lip}(X, d)$. The result that we have in mind is stated as follows: Let X be a compact Hausdorff space and let A be a complex linear subalgebra of $C(X)$. Suppose that T is a one-to-one linear map of A onto A which is isometric:

$$||Tf||_\infty = ||f||_\infty.$$

If $T(1) = 1$, then T is multiplicative. (For a proof, see **3**, p. 144.)

We have proved that $T(fg) = T(f)T(g)$. Now T is also a bounded operator on $\text{Lip}(X, d)$ with the norm $||f||_1 = ||f||_\infty + ||f||_d$ which is in fact a Banach algebra norm. Since T is a Banach algebra automorphism, we have, by (**6**), that

$$(Tf)(x) = f(\tau x), \qquad f \in \text{Lip}(X, d),\ x \in X,$$

where $\tau: X \to X$ is a homeomorphism such that

$$K_1 d(x, y) \leqq d(\tau x, \tau y) \leqq K_2 d(x, y)$$

for all $x, y \in X$ and for some constants K_1 and K_2.

Define, for fixed y, $f(x) = d(x, \tau y)$. T being an isometry, $||Tf||_d \leqq 1$. Now

$$|(Tf)(x) - (Tf)(y)| = |d(\tau x, \tau y) - d(\tau y, \tau y)| \leqq d(x, y).$$

We have thus proved that

$$d(\tau x, \tau y) \leqq d(x, y)$$

for all $x, y \in X$. The reverse inequality follows by applying this to T^{-1} and τ^{-1}, and we can conclude that d is an isometry of X onto itself. This completes the proof of our theorem.

2. Lipschitz functions as a dual space. Our aim in this section is to show that when (X, d) is the n-dimensional cube in R^n and d is the metric induced by the l_1-norm of R^n, $\text{Lip}(X, d)$ is a dual space. The theorem of Kreĭn-Milman (**1**, p. 440) then shows that the unit ball U of $\text{Lip}(X, d)$ not only has extreme points but sufficiently many to span U in a weak sense.

For $x, y \in X$ with $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$,

$$d(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

This ensures that for a Lipschitz function f defined on X,

$$||f||_d = \max_{1 \leqq i \leqq n} \left(\left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right),$$

it being well known that the partial derivatives of f exist and are bounded measurable functions. Let C^1 be the linear space of functions on X with continuous first-order partial derivatives normed by

$$||f|| = \max_{1 \leqq i \leqq n} \left(||f||_\infty, \left\| \frac{\partial f}{\partial x_i} \right\|_\infty \right).$$

C^1 is a closed (and therefore complete) subspace of $\text{Lip}(X, d)$.

Let S be the norm-closure, with the norm on $(C^1)^*$ induced by C^1, of the linear span of the point evaluations ϕ_x of C^1,

$$\phi_x(f) = f(x), \qquad f \in C^1,\ x \in X.$$

Then we have the following theorem.

THEOREM 2.1. *$\mathrm{Lip}(X, d)$ is isometrically isomorphic to S^*.*

Proof. As we remarked in the Introduction, the proof is almost identical with that given by de Leeuw in (**4**) and therefore the details need not be repeated here. The only fact that requires a little proof is the following: given a Lipschitz function h on X, there is a sequence of functions $\{h_m\}$ in C^1 such that

$$||h_m|| \leqq ||h||, \qquad m = 1, 2, \ldots,$$

and

$$\lim_m h_m = h$$

uniformly on X. We obtain the functions h_m by using the Fourier series for h. To this end, make the function h periodic in each variable with period 1 (this clearly will not affect anything). Let σ_m be the mth Cesàro (rectangular) sum of the Fourier series for h. Then (see **9**, p. 302)

$$\sigma_m(x) = (K_m * h)(x) = \int_X h(t)K_m(x - t)\, dt,$$

where dt is an n-dimensional Lebesgue measure and K_m is Fejer's kernel. Recall that K_m satisfies

(i) $K_m(x) \geqq 0$ for all $x \in X$,
(ii) $\int_X K_m(t)\, dt = 1$,
(iii) $K_m \in C^1$.

It follows (again see **9**, p. 304) that $\sigma_m \to h$ uniformly on X. Moreover, $\sigma_m \in C^1$ for all m and

$$|\sigma_m(x)| \leqq \int_X |h(x - t)|K_m(t)\, dt \leqq ||h||_\infty,$$

$$\begin{aligned} |\sigma_m(x) - \sigma_m(y)| &= \left| \int_0^1 [h(x - t) - h(y - t)]K_m(t)\, dt \right| \\ &\leqq ||h||_d ||x - y|| \end{aligned}$$

for all $x, y \in X$. Therefore, $||\sigma_m|| \leqq ||h||$.

3. Extreme points of $\mathrm{Lip}[0, 1]$**.** We now wish to describe all the extreme points of $\mathrm{Lip}[0, 1]$, the metric d being the standard Euclidean metric on $[0, 1]$. We have that

$$||f||_d = ||f'||_\infty.$$

As before, U denotes the unit ball of $\mathrm{Lip}[0, 1]$ and E the set of extreme points of U. The first thing to note is that if $f \in U$ and $|f(x)| = 1$ for all $x \in [0, 1]$, then $f \in E$. This is because f is already extreme in $C[0, 1]$. The following theorem completely describes the other members of E.

THEOREM 3.1. *If f is not a constant function and if f is not of modulus one everywhere, then a necessary and sufficient condition for f to belong to E is that $|f'| = 1$ a.e. on $[0, 1] \sim M_f$, where M_f is the maximum set for f, i.e.,*

$$M_f = \{x \in [0, 1]: |f(x)| = ||f||_\infty\}.$$

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Proof. First note that for f to belong to E, it is necessary that $\|f\|_\infty = 1$ for otherwise, just add and subtract a suitable constant to f to show that it is not extreme.

Let $|f'| < 1$ on a set F of positive measure, F being contained in $[0, 1] \sim M_f$. Since the latter set is open, we may assume that F is contained in some interval I. We may also assume that

(i) F is compact because Lebesgue measure is regular,

(ii) $\operatorname{ess\,sup}_{x \in F} |f'(x)| = \alpha < 1$,

(iii) $|f(x)| \leqq 1 - \epsilon$ for all x in I and for some $\epsilon > 0$.

The function

$$g(x) = \int_0^x C_F(\xi)\, d\xi, \qquad x \in [0, 1],$$

where $C_F(x)$ is the characteristic function of F, is a continuous function on $[0, 1]$ and hence there exists a point x_0 such that

$$\int_0^{x_0} C_F(\xi)\, d\xi = \frac{1}{2} \int_0^1 C_F(\xi)\, d\xi = \tfrac{1}{2} m(F),$$

m denoting the Lebesgue measure on $[0, 1]$. Define

$$f_0(x) = C_F(x) C_{[0,x_0]}(x) - C_F(x) C_{(x_0,1]}(x), \qquad x \in [0, 1],$$

and let

$$g_0(x) = \int_0^x f_0(\xi)\, d\xi.$$

Then $g_0 \in \operatorname{Lip}[0, 1]$, it vanishes off I, and $g_0' = 0$ off F. Thus, if δ is small enough, then $f \pm \delta g_0 \in U$ and

$$f = \tfrac{1}{2}(f + \delta g_0) + \tfrac{1}{2}(f - \delta g_0),$$

showing that f is not extreme. Therefore the condition is necessary.

Now suppose that $|f'| = 1$ a.e. on $[0, 1] \sim M_f$. Let

$$2f = g + h, \qquad g, h \in U.$$

Clearly,

$$g = h = f \quad \text{on } M_f.$$

Also,

$$2f' = g' + h' \quad \text{a.e.}$$

and hence

$$g' = h' = f' \quad \text{a.e. on } [0, 1] \sim M_f.$$

If $x \notin M_f$, let y be the closest point of M_f to the left (or the right as the case may be) of x. Then $g' = h'$ a.e. on (y, x) and we can integrate from y to x to deduce that

$$g(x) = h(x).$$

Thus the condition is also sufficient.

We propose to prove now that the unit ball U of Lip[0, 1] is the *closed convex hull*, denoted henceforth by $\overline{\text{co}}$, of its extreme points E. More precisely, given $\epsilon > 0$ and $f \in U$, there exist $f_i \in E$, $a_i \in R$, $a_i \geqq 0$ $(1 \leqq i \leqq n)$, $\sum_{i=1}^{n} a_i = 1$ such that

$$\left\| f - \sum_{i=1}^{n} a_i f_i \right\| < \epsilon.$$

Before we prove this theorem, we have to state and summarize some special results about Banach spaces. Suppose that X is a Banach space. Let

$$S^* = \{L \in X^*: ||L|| = 1\}$$

be the *unit sphere* in X^*. Let U be the unit ball of X and D a subset of U. Then a necessary and sufficient condition that

$$U = \overline{\text{co}}(\text{D})$$

is that

$$\sup \operatorname{Re} L(D) = 1$$

for all L in a norm-dense subset of S^*. This is easily proved by the 'separation' theorem for locally convex topological vector spaces. For a Banach space X, it is known (see **6**) that the set

$$P = \{L \in S^*: \text{there exists } x \in U \text{ such that } L(x) = 1 = ||L||\}$$

is dense in S^*. When X happens to be a closed subspace of some $C(Y)$, where Y is a compact Hausdorff space, then the elements of P can be represented by measures on Y. This is done as follows (again see **6**): Suppose that

$$L(f) = ||L|| = 1$$

for some $f \in U \subset X \subset C(Y)$. By the Hahn-Banach theorem, we can extend L with preservation of norm to the whole of $C(Y)$. Then by the Riesz representation theorem (**1**, p. 265) and the so-called 'polar decomposition' of complex measures, it is easy to see that

$$L(g) = \int_Y g\bar{f}\, d\mu, \qquad g \in C(Y),$$

where μ is a positive, regular Borel measure on Y of total mass 1 and $|f| = 1$ on the closed support of μ.

We are now in a position to prove the following theorem.

THEOREM 3.2. *The unit ball of* Lip[0, 1] *is the closed convex hull of its extreme points.*

Proof. Consider

$$C[0, 1] \oplus L^\infty[0, 1]$$

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with the norm

$$||(f, g)|| = \max(||f||_\infty, ||g||_\infty).$$

Define the map:

$$\mathrm{Lip}[0, 1] \to C[0, 1] \oplus L^\infty [0, 1] \quad \text{given by} \quad f \to (f, f').$$

This is a linear and isometric map and the image of Lip[0, 1] under it is a closed (and hence complete) subspace of $C[0, 1] \oplus L^\infty [0, 1]$. Let A denote this image and, as before, let U be its unit ball and E the set of extreme point of U.

In the following discussion, we shall identify $L^\infty [0, 1]$ with the space of continuous functions $C(M)$ on its maximal ideal space M. (For results concerning M, see **3**.)

By what was stated above, a necessary and sufficient condition for

$$U = \overline{\mathrm{co}}(E)$$

is that

$$\sup \operatorname{Re} L(E) = 1$$

for all $L \in P$, P being the set we defined earlier. By what was discussed on page 1159, L has the form

$$\text{(1)} \qquad L(f, f') = \int_0^1 (f\bar{f}_0)\, d\mu + \int_M (f'\bar{f}_0{}')^\wedge\, d\mu,$$

where μ is a positive, regular Borel measure on $[0, 1] \cup M$ of total mass 1 and f_0 belongs to the unit ball of Lip[0, 1] and has the following properties:

$|f_0| = 1$ on the closed support of μ in [0, 1] and

$|\hat{f}_0{}'| = 1$ on the closed support of μ in M.

Here $\hat{f}_0{}'$ denotes the Gelfand transform of $f_0{}'$.

In order to prove our theorem, we have to show that

$$\sup \operatorname{Re} L(f, f') = 1, \qquad (f, f') \in E$$

for all $L \in P$. We shall, in fact, show this sup is actually attained for some $(f, f') \in E$.

We have now to establish some preliminary lemmas.

LEMMA 3.3. *Let ϕ, ψ belong to the unit ball of $L^\infty [0, 1]$. Then a necessary and sufficient condition for*

$$\hat{\phi} = \hat{\psi} \quad on\ [|\hat{\phi}| = 1]$$

is that, given $\epsilon > 0$, there exists $\delta(\epsilon)$ such that

$$|\phi - \psi| < \epsilon \quad a.e.\ on\ [|\phi| > 1 - \delta].$$

Proof. The condition is sufficient: Let $p \in [|\hat{\phi}| = 1]$ and suppose, if possible, that $|\hat{\phi}(p) - \hat{\psi}(p)| \neq 0$. Choose $\epsilon < |\hat{\phi}(p) - \hat{\psi}(p)|$. Let δ be the positive number given by the stated condition. Then $[|\hat{\phi}| > 1 - \delta]$ is an open set

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containing $[|\hat{\phi}| = 1]$. Since the sets

$$[\hat{C}_E = 1], \qquad E \text{ Lebesgue measurable, } E \subset [0, 1],$$

form a basis for the topology of the maximal ideal space M of L^∞, we can find an E such that

$$p \in [\hat{C}_E = 1] \subset [|\hat{\phi}| > 1 - \delta].$$

This means that barring a set of measure zero,

$$E \subset [|\phi| > 1 - \delta].$$

Then $|\phi - \psi| < \epsilon$ a.e. on E by the given condition which implies that $|\hat{\phi} - \hat{\psi}| < \epsilon$ on $[\hat{C}_E = 1]$. But as $p \in [\hat{C}_E = 1]$, this yields a contradiction.

The condition is necessary. Suppose that for a certain $\epsilon > 0$ no such δ exist. Choose a monotone increasing sequence of δ_n's such that

$$\lim_n \delta_n = 1.$$

If

$$E_n = [|\phi - \psi| > \epsilon] \cap [|\phi| > \delta_n = 1 - (1 - \delta_n)],$$

then, by our assumption, $m(E_n) > 0$ and $E_n \downarrow$. Let

$$F_n = [\hat{C}_{E_n} = 1] \subset M,$$

then the F_n's are compact and decreasing and therefore have non-empty intersection. But this intersection is contained in $[|\hat{\phi}| = 1]$, which shows that $\hat{\phi} \neq \hat{\psi}$ on $[|\hat{\phi}| = 1]$. This completes the proof of Lemma 3.3.

Let us examine (1). If we can find a g in E which coincides with f_0 on its maximum set and which is also such that $\widehat{g'} = \hat{f}_0'$ on $[|\hat{f}_0'| = 1]$, then our proof will be complete. By Lemma 3.3, g has to have the property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f_0' - g'| < \epsilon \quad \text{a.e. on } [|f_0'| > 1 - \delta].$$

If we can construct such a g on each of the disjoint intervals comprising the complement of M_{f_0}, then the proof will be complete. We should note that $g = f_0$ on M_{f_0} ensures that $g' = f_0'$ a.e. on M_{f_0}. This is because M_{f_0} can be written as the union of a perfect set X_1, and a countable set X_2: as g and f_0 are Lipschitz functions, their derivatives will exist and agree a.e. on X_1, and therefore a.e. on M_{f_0} since $m(X_2) = 0$. Therefore the problem is reduced to proving the following lemma.

LEMMA 3.4. *Given an interval* $[a, b] \subset [0, 1]$ *and a (complex-valued) Lipschitz function* f_0 *defined on it which is such that* $|f_0(a)| = |f_0(b)| = 1$, $|f_0(x)| < 1$ *for all* x *in* (a, b), $|f_0'| \leqq 1$ *a.e. on* $[a, b]$, *we can construct another Lipschitz function* g *with the following properties*:

(i) $g(a) = f_0(a)$, $g(b) = f_0(b)$,

(ii) $|g| \leqq 1$ *on* $[a, b]$,

(iii) $|g'| = 1$ *a.e. on* $[a, b]$, *and*
(iv) *for every* $\epsilon > 0$, *there exists a* δ *such that*

$$|f_0' - g'| < \epsilon \quad \textit{a.e. on } [|f_0'| > 1 - \delta].$$

Proof. Take any subinterval $[a_1, b_1] \subset [a, b]$, $a \neq a_1$, $b \neq b_1$. The image of this subinterval under f_0 is a compact subset of the unit disc which does not meet the boundary of the disc. Therefore, this compact set has positive distance, say r, from the boundary of the disc. Subdivide $[a_1, b_1]$ into intervals of length smaller than r. Let $[\alpha, \beta]$ be a typical interval. Define the function $\theta(x)$ on $[\alpha, \beta]$ as follows:

When $f_0'(x) \neq 0$ there is a unique chord of the unit circle perpendicular to (and having as midpoint) $f_0'(x)$. Let $\theta(x)$ be one of the vectors defining half this chord, so that

$$\overline{\theta(x)} \cdot f_0'(x) = 0 \quad \text{and} \quad |f_0'(x) \pm \theta(x)| = 1$$

(hence $|\theta|^2 + |f_0'|^2 = 1$). When $f_0'(x) = 0$, let $\theta(x) = 1$. Now define

$$g' = f_0' + h\theta \quad \text{on } [\alpha, \beta],$$

where h is a real-valued, measurable function assuming only the values ± 1 and satisfying

$$\int_\alpha^\beta h\theta \, dx = 0.$$

(We shall presently indicate how h can be constructed.) It is clear that $|g'| = 1$ a.e. on $[\alpha, \beta]$ and that it is a measurable function. Let

$$g(x) = f_0(\alpha) + \int_\alpha^x g'(\xi)\, d\xi, \qquad x \in [\alpha, \beta].$$

Then

$$g(\beta) = f_0(\alpha) + \int_\alpha^\beta f_0'(x)\, dx = f_0(\beta).$$

Moreover,

$$|g(x) - f_0(\alpha)| \leqq |x - \alpha| < r$$

which means that $|g(x)| \leqq 1$ for all x in $[\alpha, \beta]$. Also,

$$|g' - f_0'| = |\theta| = \sqrt{(1 - |f_0'|)}\sqrt{(1 + |f_0'|)} \leqq 2\sqrt{(1 - |f_0'|)},$$

whence

$$|g' - f_0'| < \epsilon \text{ a.e. on } [|f_0'| > 1 - \delta]$$

if $\delta \leqq \epsilon^2/4$. Performing a similar construction on the other intervals of $[a_1, b_1]$, we can define a g on $[a_1, b_1]$ which has all the properties (i)–(iv) of Lemma 3.4. Now take an increasing sequence of intervals $[a_n, b_n]$, $a < a_n$ and $b_n < b$,

$$\lim_n a_n = a, \quad \lim_n b_n = b \qquad (n = 1, 2, 3, \ldots)$$

and by the method explained above, define functions g_n on $[a_n, b_n]$ which have all the properties listed in Lemma 3.4. Evidently,

$$g_{n+1} = g_n \quad \text{on } [a_n, b_n].$$

Define g on $[a, b]$ by

$$g(x) = g_n(x), \qquad x \in [a_n, b_n],$$
$$g(a) = \lim_n g_n(a_n) = \lim_n f(a_n) = f(a),$$
$$g(b) = \lim_n g_n(b_n) = \lim_n f(b_n) = f(b).$$

It is clear that g has all the required properties. Thus the lemma is proved.

We now show how h can be chosen so that

$$\int_\alpha^\beta h\theta\, dx = 0.$$

For this, we appeal to a theorem due to Liapunov (see **5**) in which it is stated that the range of a finite non-atomic, countably additive vector-valued set function (assuming values in R^n), is a closed convex set. Let

$$\nu(E) = \int_E \theta\, dm,$$

where m denotes Lebesgue measure and E runs through all the Lebesgue measurable sets in $[\alpha, \beta]$. Clearly, ν is a non-atomic complex measure. Since the range of ν contains the complex numbers 0 and $\int_\alpha^\beta \theta\, dm$, it also contains $\frac{1}{2}\int_\alpha^\beta \theta\, dm$ by Liapounoff's theorem. Therefore there is a measurable set $E \subset [\alpha, \beta]$ such that

$$\int_E \theta\, dm = \frac{1}{2}\int_\alpha^\beta \theta\, dm.$$

Define

$$h = 1 \quad \text{on } E,$$
$$= -1 \quad \text{on } [\alpha, \beta] \sim E,$$

Then

$$\int_\alpha^\beta h\theta\, dm = 0.$$

Finally, the referee has kindly pointed out that a much more general result than Theorem 2.1 is valid:

Lip(X, d) *is a dual space for any compact* X *and* d

Proof. By Ascoli's theorem, the unit ball B of Lip(X, d) is uniformly, hence pointwise, compact. If E denotes Lip(X, d) in the pointwise topology, F its dual, then B is $w(E, F)$ compact. By Smulian's theorem (**3**, p. 142) the polar $B^0 \subset F$ is radial at 0, and if L is a linear functional on F which is bounded on B^0, then there is an $f \in E$ such that $L(u) = \langle u, f\rangle$, $u \in F$. If G denotes the space F with the norm defined by B^0, then it may be readily verified that $\mathrm{Lip}(X, d)$ is isometric with G^*. [Define $T\colon \mathrm{Lip}(X, d) \to G^*$ by $(Tf)(u) = \langle f, u\rangle$, $u \in F$; T is an isometry.]

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