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AN UPPER BOUND FOR THE NUMBER OF DIOPHANTINE QUINTUPLES

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Abstract

We improve the known upper bound for the number of Diophantine $D(4)$-quintuples by using the most recent methods that were developed in the $D(1)$ case. More precisely, we prove that there are at most 6.8587×10^{29} $D(4)$-quintuples.

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1. Introduction

Definition 1.1. Let $n \neq 0$ be an integer. We call a set of m distinct positive integers a Diophantine $D(n)$-m-tuple if the product of any two distinct elements of the set increased by n is a perfect square.

Research on $D(n)$-m-tuples has been quite active recently, especially in the case $n = 1$. The cases $n = -1$ and $n = 4$ have also been actively studied. Details of problems concerning $D(n)$-m-tuples, together with the history and recent references, can be found on the webpage [7].

In this paper, we will consider only Diophantine $D(4)$-quintuples $\{a, b, c, d, e\}$, ordered so that $a < b < c < d < e$. It is conjectured (see [9, Conjecture 1]) that all $D(4)$-quadruples $a < b < c < d$ are regular: that is

$$d = d_+ = a + b + c + \tfrac{1}{2}(abc + rst),$$

where r, s and t are positive integers satisfying $ab + 4 = r^2$, $ac + 4 = s^2$ and $bc + 4 = t^2$. This conjecture obviously implies that there does not exist a $D(4)$-quintuple.

The second author, in [11], has proved that an irregular $D(4)$-quadruple cannot be extended to a quintuple with a larger element. This is important because it implies that if $\{a, b, c, d, e\}$ is a $D(4)$-quintuple with $a < b < c < d < e$, then d is uniquely given by a, b and c. Moreover, the second author also proved, in [12], that there are at most four ways to extend a $D(4)$-quadruple to a quintuple with a larger element.

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The best published bound on the number of $D(4)$-quintuples is 7×10^{36}; this was found by Baćić and the second author in [3]. By using the most recent methods, mostly from [5], we prove the following theorem.

THEOREM 1.2. *There are at most* 6.8587×10^{29} *Diophantine* $D(4)$*-quintuples.*

2. The lower bound on b

In this section, we will firstly improve some of the results from [2] and [3].

LEMMA 2.1. *Let* $\{a, b, c, d, e\}$ *be a* $D(4)$*-quintuple with* $a < b < c < d < e$. *Then* $\{a, b, c, d\}$ *is a regular* $D(4)$*-quadruple and at least one of the following is true:*

(i) $b > 4a$ *and* $d > b^2$;

(ii) $b \leq 4a$, $c = a + b + 2r$ *and* $d > c^2$;

(iii) $b \leq 4a$, $c = c_- = (ab + 2)(a + b - 2r) + 2(a + b)$ *and* $c^{5/3} < d < c^2$;

(iv) $b \leq 4a$, $c = c_+ = (ab + 2)(a + b + 2r) + 2(a + b)$ *and* $c^{4/3} < d < c^{5/3}$.

PROOF. The statement follows from [3, Propositions 2.2, and 2.3]. □

The next lemma gives an improvement of [2, Lemma 3] for the lower bound on b in a $D(4)$-quintuple.

LEMMA 2.2. *Let* $\{a, b, c, d, e\}$ *be a* $D(4)$*-quintuple such that* $a < b < c < d < e$. *Then* $b > 10^5$.

PROOF. We used Baker–Davenport reduction, as described in [2, Lemma 3]. It took around 80 hours in the Mathematica 10 package with the processor Intel(R) Core(TM) i7-4510U CPU @2.00–3.10 GHz. □

The next lemma shows that cases (iii) and (iv) from Lemma 2.1 are not possible.

LEMMA 2.3. *If* $b < 4a$ *in a* $D(4)$*-quintuple* $\{a, b, c, d, e\}$ *with* $a < b < c < d < e$, *then the only possibility for* c *is* $c = a + b + 2r$.

PROOF. Suppose $c = c_\pm = (ab + 2)(a + b \pm 2r) + 2(a + b)$. The second author proved, in [13], that $b > a + 57\sqrt{a}$. Then, for $b > 10^5$, using a short computer search, it can be proved that $a + b - 2r > 700$, which yields $c_\pm > ab(a + b - 2r) > 700ab > 7 \times 10^7 a$ and $d = d_+ > abc > 700a^2b^2$.

To use the version of Rickert's theorem from [2] and [2, Lemmas 6 and 7] for the $D(4)$-quadruple $\{a, b, d, e\}$, we must have $d > 308.07\, a'b(b - a)^2/a$, where $a' = \max\{4a, 4(b - a)\}$. But, since

$$4a \leq a' < 4(4a - a) = 12a$$

and

$$57\sqrt{a} < b - a < 3a,$$

$$ac > \frac{7 \times 10^7}{12 \cdot 9} \frac{a'(b - a)^2}{a} > 308.07 \frac{a'(b - a)^2}{a},$$

and the inequality we need is satisfied, since $d = d_+ > abc$.

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Now

$$32.02\,aa'b^4d^2 < 32.02\,a\cdot 12a\cdot (4a)^4d^2 = 98365.44\,a^6d^2,$$
$$0.026\,ab(b-a)^{-2}d^2 < 0.0264\cdot 4a\cdot \frac{1}{(57\sqrt{a})^2}d^2 < 0.000033\,ad^2,$$
$$bd > ad$$

and, finally,

$$0.00325\,a(a')^{-1}b^{-1}(b-a)^{-2}d > 0.00325\,a\frac{1}{12a\cdot 4a\cdot (3a)^2}d > 7\times 10^{-6}a^{-3}d.$$

Let us also recall that when we consider the extension of a triple to a quadruple, we are actually solving equations of the form $v_m = w_n$, where (v_m) and (w_n) are binary recurrence sequences. Now, from [2, Lemmas 6 and 7] and using the fact that we only have to solve the equation $v_m = w_n$ for even indices (see [12]), when we have the extension of a triple $\{a, b, d\}$ to a quintuple, we see that $v_{2m} = w_{2n}$ implies that

$$n < \frac{\log(98365.44\,a^6d^2)\log(0.000033\,ad^2)}{\log(ad)\log(7\times 10^{-6}a^{-3}d)}.$$

The right-hand side of the inequality is decreasing in d for $d > 700a^2b^2 > 7\times 10^7a^3$, which yields

$$n < \frac{12\log(52.916\,a)\cdot 7\log(39.925\,a)}{4\log(91.469\,a)\log(490)} < 3.391\frac{\log(52.916\,a)\log(39.925\,a)}{\log(91.469\,a)}.$$

On the other hand, from the proof of [3, Proposition 2.3], $v_{2m} = w_{2n}$ implies that

$$n > 0.5\cdot 0.495\,b^{-0.5}d^{0.5} > 0.2475\cdot (4a)^{-0.5}a^2 > 0.12375\,a^{1.5}.$$

By solving the inequality

$$a^{1.5} < 27.41\frac{\log(52.916a)\log(39.925a)}{\log(91.469a)},$$

we get $a \le 32$. But $4a > b > 10^5$, so $a > 25000$, which leads to a contradiction. □

The authors in [8, Lemma 1] show that $c = a + b + 2r$ or $c > ab$ in a $D(4)$-triple $\{a, b, c\}$ with $a < b < c$. As in [3], to get the better bound on the number on quintuples, we will also consider the subcases $ab < c \le a^2b^2$ and $c > a^2b^2$.

Lemma 2.4. *Let $\{a, b, c, d, e\}$ be a $D(4)$-quintuple such that $a < b < c < d < e$. Then $\{a, b, c, d\}$ is a regular quadruple and one of the following is true:*

(i) *$b > 4a$, $c > a^2b^2$ and $d > b^3$;*
(ii) *$b > 4a$, $a^2b^2 \ge c > ab$ and $d > b^2$;*
(iii) *$b > 4a$, $c = a + b + 2r$ and $d > b^2$; or*
(iv) *$b \le 4a$, $c = a + b + 2r$ and $d > 6250c^2$.*

Proof. The statement follows from [3] and the previous considerations. In the last case, we have a better constant in the lower bound on d. More precisely, since $4a < c < 4b$ and $a > \frac{1}{4}\times 10^5 = 25000$, $c < \frac{4}{25000}ab$ which gives us $d > abc > 6250c^2$. □

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3. The lower bound on m

As we said earlier, elements of a $D(4)$-quadruple are defined as solutions of three simultaneous Pellian equations (see, for example, [11]). The solutions are obtained as a common term of two second-order linear recurrence sequences v_m and w_n such that $v_m = w_n$ for some positive integers m and n. The next proposition gives us a connection between those integers and the elements of a quadruple.

Proposition 3.1. *Let $\{A, B, C, D\}$ be a $D(4)$-quadruple with $A < B < C < D$ for which $v_{2m} = w_{2n}$ has a solution with $2n \geq m > n \geq 2$, $m \geq 3$. Suppose that $A \geq A_0$, $B \geq B_0$, $C \geq C_0$, $B \geq \rho A$ for some positive integers A_0, B_0, C_0 and a real number $\rho > 1$. Then*

$$m > \alpha B^{-1/2} C^{1/2},$$

where α is any real number satisfying the two inequalities

$$\alpha^2 + (1 + 2B_0^{-1} C_0^{-1})\alpha \leq 1 \tag{3.1}$$

$$3\alpha^2 + \alpha(B_0(\lambda + \rho^{-1/2}) + 2C_0^{-1}(\lambda + \rho^{1/2})) \leq B_0 \tag{3.2}$$

with $\lambda = (A_0 + 4)^{1/2}(\rho A_0 + 4)^{-1/2}$. Moreover, if $C^\tau \geq \beta B$ for some positive real numbers β and τ, then

$$m > \alpha \beta^{1/2} C^{(1-\tau)/2}. \tag{3.3}$$

Proof. The proof is similar to the proof of [5, Proposition 3.1] using the results from [11] and [12]. □

Since the conditions of Proposition 3.1 are satisfied for $D(4)$-quintuples (see [12]), we can use it to obtain the lower bound on m in terms of d. From now on, we will assume that $\{a, b, c, d = d_+\}$ is a regular quadruple, since this follows from [11].

Lemma 3.2. *If $\{a, b, c, d, e\}$ is a $D(4)$-quintuple with $a < b < c < d < e$, then we have the following bounds on m depending on the respective cases from Lemma 2.4:*

(i) $m > 0.618034 d^{1/3}$;
(ii) $m > 0.618034 d^{1/4}$;
(iii) $m > 0.618034 d^{1/4}$; *and*
(iv) $m > 48.85 d^{1/4}$.

Proof. We prove this by using Proposition 3.1 for $\{A, B, C, D\} = \{a, B, d, e\}$, where $B \in \{b, c\}$.

In case (i), since $B = b > 4a = 4A$, we can take $\rho = 4$. From $C = d > abc > a^3 b^3$ and $d = d_+$, we have $\tau = \frac{1}{3}$ and $\beta = A_0$. From previous considerations, $A_0 = 1$, $B_0 = 10^5$, $C_0 = 10^{15}$ and, after a short computer search, using inequalities (3.1) and (3.2), we get $\alpha = 0.618034$.

In cases (ii) and (iii), $B = b > 4a = 4A$ and, again, $\rho = 4$. From $d > b^2$, $\tau = \frac{1}{2}$, $\beta = 1$ and we get $\alpha = 0.618034$, by using $A_0 = 1$, $B_0 = 10^5$ and $C_0 = 10^{10}$.

In the last case, $B = c = a + b + 2r = a + b + 2\sqrt{ab + 4} > 4a = 4A$, which again implies that $\rho = 4$. Since $d > 6250c^2$, $\tau = \frac{1}{2}$, $\beta = 1$ and, with the lower bounds $A_0 = 2500$, $B_0 = 10^5$, $C_0 = 6250 \times 10^{10}$, we get $\alpha = 0.618034$ again.

Inserting these values in the inequality (3.3) concludes the proof. □

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REMARK 3.3. Notice that the inequality (3.1) tends to $\alpha^2 + \alpha \le 1$ when B_0 and C_0 tend to infinity. The maximal solution of that inequality is $\frac{1}{2}(-1 + \sqrt{5}) \approx 0.618034$, which means that we have the optimal value of α and we cannot get any better results by using Proposition 3.1 and increasing the lower bounds for A, B and C.

4. The upper bound on d

First, we state the theorem that we will use, as the authors have done in [5], to get better results on the upper bound on d by using the results from Lemma 3.2. This theorem gives slightly better results than the Baker–Wüstholz theorem, which was used in previous papers on this topic.

THEOREM 4.1 Aleksentsev [1]. *Let Λ be a linear form in the logarithms of n multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_n$, with rational coefficients $b_1, \dots, b_n$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j for $1 \le j \le n$. Let d be the degree of the number field $\mathcal{K} = \mathbb{Q}(\alpha_1, \dots, \alpha_n)$ and let $A_j = \max(dh(\alpha_j), |\log \alpha_j|, 1)$. Finally, let*

$$E = \max\Big(\max_{1\le i,j\le n}\Big\{\frac{|b_i|}{A_j} + \frac{|b_j|}{A_i}\Big\}, 3\Big).$$

Then

$$\log|\Lambda| \ge -5.3n^{(1-2n)/2}(n+1)^{n+1}(n+8)^2(n+5)31.44^n d^2(\log E)A_1\cdots A_n \log(3nd).$$

As in [5], we apply the previous theorem to the algebraic numbers

$$\alpha_1 = \frac{S+\sqrt{AC}}{2}, \quad \alpha_2 = \frac{T+\sqrt{BC}}{2}, \quad \alpha_3 = \frac{\sqrt{B}(\sqrt{C}\pm\sqrt{A})}{\sqrt{A}(\sqrt{C}\pm\sqrt{B})},$$

where the signs in α_3 coincide depending on whether $z_0 = z_1 = 2$ or $z_0 = z_1 = -2$. Also $S = \sqrt{AC+4}$ and $T = \sqrt{BC+4}$. The linear form is

$$\Lambda = j\log\alpha_1 - k\log\alpha_2 + \log\alpha_3,$$

where $j = 2m$, $k = 2n$ and it is easy to see that $n = 3$ and $d = 4$.

In order to determine E, we have to find estimates for A_j. The proof of these estimates is only slightly different from the one presented in [5] for $D(1)$-quintuples, so we will state the results without going into details.

In the following, C_1 denotes an integer such that $C_1 \ge C$.

First, we consider A_1. Since the minimal polynomial of α_1 is $p(X) = X^2 - SX + 1$, $h(\alpha_1) = \frac{1}{2}\log\alpha_1$, so $A_1 = 2\log\alpha_1$. We get

$$\log Cg_2(A_0, C_1) < A_1 < \log Cg_1(\beta, \rho, \tau, C_1),$$

where

$$g_1(\beta, \rho, \tau, C_1) = 1 + \tau - \frac{\log(\beta\rho)}{\log C_1} \quad \text{and} \quad g_2(A_0, C_1) = 1 + \frac{\log A_0}{\log C_1}.$$

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Similarly, $A_2 = 2\log\alpha_2$ and

$$\log C g_4(B_0, C_1) < A_2 < \log C g_3(\beta, \tau, C_0),$$

where

$$g_3(\beta, \tau, C_0) = 1 + \tau + \frac{\log(\beta^{-1} + 2C_0^{-1-\tau})}{\log C_0} \quad \text{and} \quad g_4(B_0, C_1) = 1 + \frac{\log B_0}{\log C_1}.$$

Since $A_3 = 4h(\alpha_3) = B^2(C-A)^2$ and since the same conditions hold as in [5],

$$\log C g_6(\beta, \rho, \tau, A_0, C_1) < A_3 < \log C g_5(\beta, \tau, C_1),$$

where

$$g_6(\beta, \rho, \tau, A_0, C_1) = 1 - \tau + \frac{\log(\beta\rho^2/4) + 2\log(1 - A_0/C_1) - \log(1 - 4/C_1)}{\log C_1}.$$

Using the fact that $C_1 > 10^{10} = C_0$ and the other parameters we have, it is easy to show that $g_6 < g_2 < g_4$ in all of our cases. For simplicity, from now on, we denote the value of $g_6(\beta, 4, \tau, 1, C_1)$ by g_6 and we will use g_i similarly for the other bounds. Since

$$\frac{j}{g_6 \log C} > \frac{j}{A_1} > \frac{k}{A_1} > \frac{1}{A_1}, \quad \frac{j}{g_6 \log C} > \frac{j}{A_2} > \frac{k}{A_2} > \frac{1}{A_2}$$

and

$$\frac{j}{g_6 \log C} > \frac{j}{A_3},$$

it follows that

$$\max_{1 \le i,j \le 3} \left\{ \frac{|b_i|}{A_j} + \frac{|b_j|}{A_i} \right\} \le \frac{2j}{g_6 \log C}.$$

From $C_1 > C_0 = 10^{10}$, $g_6 < 0.561$. Also, since $d > 10^{10}$, the worst case from Lemma 3.2 is $m > 0.618034 d^{1/4}$, which gives us $m \ge 196$. If we assume that $2j/(g_6 \log C_0) < 3$, from [3], we know that $d < 10^{89}$, so

$$2j < 3 g_6 \log C_0 < 3 \cdot 0.561 \log(10^{89}) < 345,$$

which yields $m \le 86$, which is a contradiction. We conclude that $2j/(g_6 \log C_0) \ge 3$ and take

$$E \le \frac{2j}{g_6 \log C_0}.$$

In [10], it is proved that $\Lambda > 0$. Now we can use Theorem 4.1 to get

$$\begin{aligned} -\log\Lambda &\le 1.5013 \times 10^{11} A_1 A_2 A_3 \log E \\ &\le 1.5013 \times 10^{11} \cdot 2\log\alpha_1 \cdot g_3 \cdot g_5 \cdot \log^2 C \log \frac{2j}{g_6 \log C_0}. \end{aligned}$$

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Also, from [10],

$$\Lambda < 2AC\alpha_1^{-2j} \implies -\log\Lambda < -\log(2AC) + 2j\log\alpha_1,$$

which gives

$$2j\log\alpha_1 < 1.5013\times 10^{11}\cdot 2\log\alpha_1\cdot g_3\cdot g_5\cdot \log^2 C\log\frac{2j}{g_6\log C_0} + \log(2AC)$$

and, since $\log 2x/2\log\frac{1}{2}(\sqrt{x+4}+\sqrt{x}) < 1$,

$$j-1 < 1.5013\times 10^{11}\cdot g_3\cdot g_5\cdot \log^2 C\log\frac{2j}{g_6\log C_0}.$$

Finally, we can use $j = 2m$ and $C = d$ to get the inequality

$$\frac{2m-1}{\log(4m/g_6\log C_0)} < 1.5013\times 10^{11}\cdot g_3\cdot g_5\log^2 d. \tag{4.1}$$

The function on the left-hand side of inequality (4.1) is increasing in m for $m > 0$, so we can use the upper bound on m from Lemma 3.1 to get the upper bound on d in each case of Lemma 2.4. Inserting appropriate parameters for case (i), yields $d < 1.294\times 10^{52}$ and we can use that value as the new value for C_1 and calculate again the upper bound on d, but the result is not much better than the previous one. We repeat this procedure in all cases, which gives us the next Lemma.

LEMMA 4.2. *For a $D(4)$-quintuple $\{a,b,c,d,e\}$ with $a<b<c<d<e$, in the respective cases from Lemma 2.4:*

(i) $d < 1.294\times 10^{52}$;
(ii) $d < 1.096\times 10^{71}$;
(iii) $d < 1.096\times 10^{71}$; *and*
(iv) $d < 5.452\times 10^{62}$.

5. Some arithmetical sums used for bounding the number of quintuples

By combining methods from [4], [5] and [6], we can improve the bounds for some number-theoretic sums used in [3]. As in [14], we use notation $f(x) = \vartheta(g(x))$ to mean $|f(x)| \le g(x)$ for all x under consideration.

LEMMA 5.1 [14, Lemma 13]. *For all $t > 0$,*

$$\sum_{n\le t}\frac{d(n)}{n} = \frac{1}{2}\log^2 t + 2\gamma\log t + \gamma^2 - 2\gamma_1 + \vartheta(1.16t^{-1/3}),$$

where γ is Euler's constant and γ_1 is the second Stieltjes constant, which satisfies $-0.07282 < \gamma_1 < -0.07281$.

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LEMMA 5.2 [14, Lemma 14]. *Let $\{g_n\}_{n\geq 1}$, $\{h_n\}_{n\geq 1}$ and $\{k_n\}_{n\geq 1}$ be three sequences of complex numbers satisfying $g = h * k$, that is, g is the Dirichlet convolution of h and k. Let $H(s) = \sum_{n\geq 1} h_n n^{-s}$ and $H^*(s) = \sum_{n\geq 1} |h_n| n^{-s}$, where $H^*(s)$ converges for $Re(s) \geq -\frac{1}{3}$. If there are four constants A, B, C and D satisfying*

$$\sum_{n\leq t} k_n = A\log^2 t + B\log t + C + \vartheta(Dt^{-1/3}) \quad (t > 0),$$

then

$$\sum_{n\leq t} g_n = u\log^2 t + v\log t + w + \vartheta(Dt^{-1/3}H^*(-1/3))$$

and

$$\sum_{n\leq t} ng_n = Ut\log t + Vt + W + \vartheta(2.5Dt^{2/3}H^*(-1/3)),$$

where

$$u = AH(0), \quad v = 2AH'(0) + BH(0), \quad w = AH''(0) + BH'(0) + CH(0),$$
$$U = 2AH(0), \quad V = -2AH(0) + 2AH'(0) + BH(0),$$
$$W = A(H''(0) - 2H'(0) + 2H(0)) + B(H'(0) - H(0)) + CH(0).$$

Let $g(d)$ denote the number of solutions $n \in \mathbb{Z}_d$ to the congruence $n^2 \equiv 4 \pmod{d}$. It is easy to see, from [15], that, for $d = 2^a q$, $g(d) = 2^{\omega(q)+s(a)}$, where

$$s(a) = \begin{cases} 0 & \text{if } a = 0, 1, \\ 1 & \text{if } a = 2, 3, \\ 2 & \text{if } a = 4, \\ 3 & \text{if } a \geq 5. \end{cases}$$

Since $g(d)$ is a multiplicative function, we can easily determine its values by using the values in prime powers: for $e_1 \geq 1, e_2 \geq 5$ and p odd,

$$g(2) = 1, \quad g(4) = g(8) = g(p^{e_1}) = 2, \quad g(16) = 4, \quad g(2^{e_1}) = 8.$$

To determine the upper bound on the number of $D(4)$-quintuples, we will need an upper bound on the sum $\sum_{d\leq N} g(d)/d$.

LEMMA 5.3. *Let $g(d)$ denote the number of solutions of $n^2 \equiv 4 \pmod{d}$ with $0 \neq n < d$ and let $N \in \mathbb{N}$. Then*

$$\sum_{d\leq N} \frac{g(d)}{d} \leq \frac{3}{\pi^2}\log^2 N + 1.078763\log N + 0.160201 + 7.07945N^{-1/3}$$

and

$$\sum_{d\leq N} g(d) \leq \frac{6}{\pi^2} N\log N + 0.470835N - 0.310634 + 17.6986N^{2/3}.$$

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Proof. For the Dirichlet series $F(s) = \sum_{d=1}^{\infty} g(d)/d^{s+1}$, using the values at prime factors of $g(d)$, we get the Euler product

$$F(s) = \left(1 + \frac{1}{2^{s+1}} + \frac{2}{2^{2(s+1)}} + \frac{2}{2^{3(s+1)}} + \frac{4}{2^{4(s+1)}} + 8\left(\frac{1}{2^{5(s+1)}} + \frac{1}{2^{6(s+1)}} + \cdots\right)\right) \times \prod_{p, p\neq 2}\left(1 + \frac{2}{p^{s+1}} + \frac{2}{p^{2(s+1)}} + \cdots\right).$$

Dudek, in [6], showed that

$$\frac{\zeta^2(s+1)}{\zeta(2(s+1))} = \prod_p \frac{1 + p^{-(s+1)}}{1 - p^{-(s+1)}} = \prod_p \left(1 + \frac{2}{p^{s+1}} + \frac{2}{p^{2(s+1)}} + \cdots\right),$$

where $\zeta(s)$ is the Riemann zeta function. To use Lemma 5.2, we must first find $H(s) = \sum_{n\geq 1} h_n n^{-(s+1)}$, such that $F(s) = H(s)\cdot K(s) = \zeta^2(s+1)H(s)$, where $K(s) = \sum_{n=1}^{\infty} d(n)n^{-(s+1)} = \zeta^2(s+1)$. By comparing the coefficients of appropriate Euler products,

$$\begin{aligned} &h(1) = 1, \quad h(p^2) = -1, \quad h(p^{e_1}) = 0 \quad \text{for } p \neq 2 \text{ and } e_1 \in \mathbb{N}\setminus\{2\};\\ &h(2) = h(8) = -1, \quad h(4) = 1, \quad h(16) = h(32) = 2, \quad h(64) = -4,\\ &h(2^{e_2}) = 0, \quad \text{for } e_2 \geq 7. \end{aligned}$$

This gives

$$H(s) = \left(1 - \frac{1}{2^{s+1}} + \frac{1}{2^{2(s+1)}} - \frac{1}{2^{3(s+1)}} + \frac{2}{2^{4(s+1)}} + \frac{2}{2^{5(s+1)}} - \frac{4}{2^{6(s+1)}}\right)\prod_{p>2}\left(1 - \frac{1}{p^{2(s+1)}}\right).$$

Now $H^*(s) = \sum_{n\geq 1} |h_n| n^{-(s+1)}$ converges for all $s > -1$ and, in its Euler product, the product over the primes is equal to $\zeta(s+1)/\zeta(2(s+1))$, so $H^*(-\frac{1}{3}) \leq 6.103$. Similarly, since $\zeta(s)^{-1} = \prod_p (1 - p^{-s})$, we easily find that $H(0) = 6/\pi^2$, $H'(0) \leq 0.377$ and $H''(0) \leq -1.1321$. We can now use Lemma 5.2 to get the upper bounds in the statement of the lemma. □

From the previous Lemma and considerations from [6], we obtain the next result.

Lemma 5.4. *Let $d(n)$ denote the number of divisors of $n \in \mathbb{N}$. Then*

$$\begin{aligned} E &= \sum_{n=3}^{N} d(n^2 - 4)\\ &\leq N\left(\frac{6}{\pi^2}\log^2 N + 2.15752\log N + 0.320402 + 14.159N^{-1/3}\right). \end{aligned}$$

Proof. This follows from $\sum_{n=2}^{N} d(n^2 - 4) \leq 2N\sum_{d\leq H} g(d)/d$. □

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6. Counting the number of quintuples

This section completes the proof of Theorem 1.2. Lemma 5.4 can be used when we know N such that $r < N$, where $r = \sqrt{ab+4}$. Then we can conclude that the total number of $D(4)$-pairs $\{a, b\}$, such that $a < b$, is less than $E/2$. We will now determine the upper bound on the number of $D(4)$-quintuples for each case in Lemma 2.4.

Case (i). Here, $b > 4a$, $d > b^3$ and $d < 1.294 \times 10^{52}$. Since $c > a^2b^2$, $d > abc > a^3b^3 > 0.99r^6$, so

$$r < \left(\frac{d}{0.99}\right)^{1/6} < 4.8535 \times 10^8,$$

and $d > b^3$ yields $b < 2.3478 \times 10^{17}$. Using the method described before, we see that the number of pairs $\{a, b\}$ is less than 6.9567×10^{10}. For a fixed pair $\{a, b\}$, the number of elements c which extend it to triple $\{a, b, c\}$ depends on the binary recurrence sequences described in [2], and the number of those sequences is less than $8 \cdot 2^{\omega(b)}$. In every sequence, $\sqrt{bc_\nu + 4} > 2(r-1)^{\nu-1}$. Since $b > 10^5$ and $d > abc > 10^5 c_\nu$, $c_\nu < 2.85 \times 10^{47}$, which gives $\nu \leq 13$: that is, each sequence has at most 13 elements. The product of the first 15 primes is greater than $6.14 \times 10^{17} > b$, which means that the number of sequences is less than

$$8 \cdot 2^{\omega(b)} < 8 \cdot 2^{14} = 131072.$$

As we said before, in every $D(4)$-quintuple $d = d_+$ is unique and, from [12], we know there are at most four ways to extend a regular $D(4)$-quadruple to a quintuple, so we conclude that, in this case, the number of $D(4)$-quintuples is less than

$$6.9567 \times 10^{10} \cdot 131072 \cdot 13 \cdot 4 < 4.74151 \times 10^{17}.$$

Case (ii). Here $d < 1.096 \times 10^{71}$. From $ab < c \leq a^2b^2$, we get $d > abc > a^2b^2 > 0.99r^4$, that is, $r < 5.76825 \times 10^{17}$. Since $d > b^2$, it is easy to get $b < 3.31059 \times 10^{35}$. The number of pairs $\{a, b\}$ is less than 3.18788×10^{20}. As in case (i), we see that the product of the first 25 primes is the first product greater than the upper bound on b, so

$$8 \cdot 2^{\omega(b)} < 8 \cdot 2^{24} = 1.3422 \times 10^8.$$

From $c \leq a^2b^2$, we get $\nu \leq 4$ and conclude that the number of quintuples is less than

$$3.18788 \times 10^{20} \cdot 1.3422 \times 10^8 \cdot 4 \cdot 4 < 6.84604 \times 10^{29}.$$

Case (iii). In this case, $c = a + b + 2r > 3r + 1$ and $d > abc > (r^2 - 4)(3r + 1)$. Since the upper bounds on b and d are the same as in case (ii), $r < 3.32 \times 10^{23}$ and the upper bound on the number of pairs $\{a, b\}$ is less than 3.1547×10^{26} . We conclude that the number of quintuples is less than

$$3.1547 \times 10^{26} \cdot 4 < 1.2583 \times 10^{27}.$$

Case (iv). Here $c = a + b + 2r > 3r + 1$, $b \leq 4a$ and $d > 6250c^2 > 6250\frac{81}{16}b^2$. From $d < 5.452 \times 10^{62}$, we get $b < 1.3127 \times 10^{29}$ and $r < 5.6643 \times 10^{20}$. The number of pairs $\{a, b\}$ is less than 4.475×10^{30}, so the number of quintuples is less than

$$5.6643 \times 10^{20} \cdot 4 < 1.69 \times 10^{24}.$$

If we sum up everything, we have proved the main result: that is, the number of $D(4)$-quintuples is less than

$$4.74151 \times 10^{14} + 6.84604 \times 10^{29} + 1.2583 \times 10^{27} + 1.69 \times 10^{24} < 6.8587 \times 10^{29}.$$

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