BULL. AUSTRAL. MATH. SOC. 16A72
VOL. 36 (1987) 121-125

SEMI-HOMOMORPHISMS OF RINGS

S.A. Huq

In this note, following Kaplanasky's study of semi-automorphisms of rings and Herstein's study of semi-homomorphisms of groups, we present a general study of semi-homomorphisms of rings.

0. Introduction

Perhaps the notion of a semi-homomorphism was conceived as a common generalization of both the notion of a homomorphism and an anti-homomorphism. The study was begun, for semi-automorphisms only, by Anchochea [1] and Kaplanasky [4]. Anchochea, studied semi-automorphisms of quaternion algebras and division algebras and proved that if A is a simple algebra of characteristic $\neq 2$, then a semi-automorphism of A is either an automorphism or an anti-automorphism. This was extended by Kaplanasky [4] to simple algebras over any field. Later on Hua [3] proved that a semi-automorphism of any skewfield is either an automorphism or an anti-automorphism. Herstein [2] proceeded later on to study semi-homomorphisms of groups. Here, following Herstein and Kaplanasky, we present the study of semi-homomorphisms of rings in general.

1. Main Results

DEFINITION 1. Let R be a ring and S a subset of R; We call S a *semi-subring*, if for all $x,y \in S$

Received 18 August 1986. The author is grateful to ICTP, Trieste for a two weeks grant in the summer of 1985 where the idea of this paper was conceived and to Monash University for a visiting grant in 1986, when this paper was completed.

Copyright Clearance Centre, Inc. Serial-fee code: 0004-9729/87 $A2.00 + 0.00

(i) $x + y + x \in S$.

(ii) $x y x \in S$.

Every subring is a semi-subring but the converse need not be true.

DEFINITION 2. A mapping $\phi : R \to S$ between two rings is called a *semi-homomorphism*, if for $a, b \in R$

(i) $\phi(a + b + a) = \phi(a) + \phi(b) + \phi(a)$ and

(ii) $\phi(a b a) = \phi(a) \phi(b) \phi(a)$

hold. Clearly any homomorphism or antihomomorphism is a semi-homomorphism but the converse need not be true; for example the constant function from Z_2 to Z_2 having value 1 , is a semi-endomorphism of Z_2 but not an endomorphism.

The set $K = \{x | \phi(x) = 0\}$ if it exists is called the kernel of the semi-homomorphism ϕ . One notices that K and $\phi(K)$ are then semi-subrings of R and S respectively.

PROPOSITION 3. *If* $\phi : R \to R'$ *is a semi-homomorphism of rings, then for* $a \in R$

$$\phi(-a) = -\phi(a) .$$

Proof. $\phi(a) = \phi[a + (-a) + a] = \phi(a) + \phi(-a) + \phi(a)$ from which our result follows.

LEMMA 4. *A semi-homomorphism* $\phi : R \to R'$ *of rings will be a homomorphism of the underlying additive groups if the characteristic of the codomain* $R' \neq 2$,

Proof. For $a, b \in R$

$$\begin{aligned}\phi(a + b) &= \phi[(a + b) + [-(a + b)] + a + b] \\ &= \phi[a + b + [-(a + b)] + b + a]. \\ &= \phi(a) + \phi[b + [-(a + b)] + b] + \phi(a) . \\ &= \phi(a) + \phi(b) + \phi[-(a + b)] + \phi(b) + \phi(a) . \\ &= 2\phi(a) + 2\phi(b) - \phi(a + b) \quad \text{by proposition 1.3}\end{aligned}$$

That is $2[\phi(a + b) - \phi(a) - \phi(b)] = 0$.

Thus $\phi(a + b) = \phi(a) + \phi(b)$ since char $R' \neq 2$.

A consequence of Lemma 4 is the corollary:

COROLLARY 5. *For a semi-homomorphism* $\phi : R \to R'$ *with char* $R' \neq 2$. *we have* $\phi(-na) = -n \phi(a)$ *for any integer* n .

https://doi.org/10.1017/S0004972700026356 Published online by Cambridge University Press

PROPOSITION 6. *If* $\phi : R \to R'$ *is a semi-homomorphism of rings with identities* $1, 1'$ *respectively and if* $1' \in \phi(R)$, *then* $[\phi(1)]^2 = 1'$. *If further* R' *is a nontrivial ring without zero divisors, then* $\phi(1) = 1'$ *or* $-1'$ *(which are distinct when* $\text{char } R' \neq 2$*).*

Proof. If $1' = \phi(r)$ for $r \in R$ then $\phi(r) = \phi(1)\ \phi(r)\ \phi(1)$ so $\phi(r) = [\phi(1)]^2$. Hence $[\phi(1)]^2 = 1'$.

Now $[\phi(1)]^2 - 1' = 0 \Rightarrow [\phi(1) + 1'][\phi(1) - 1'] = 0$ from which the second part follows.

PROPOSITION 7. *If* $\phi : F \to F'$ *is a semi-homomorphism of fields, then the kernel of* ϕ *is either zero or a regular semi-subring, provided* $\text{char } F' \neq 2$.

Proof. One notices that if the char $F' = 2$, the kernel may fail to exist. For the definition of reqularity, we refer to [3]. Thus if $K \neq 0$, let $a \neq 0 \in K$, then $a^{-1} = a^{-1}aa^{-1} \in K$. Thus $a = aa^{-1}a$ for $a^{-1} \in K$.

PROPOSITION 8. *For fields* F, F' *and a semi-homomorphism* $\phi : F \to F'$, *if* $a \neq 0 \in F$, *does not belong to the kernel of* ϕ , *then*

$$\phi(a^{-1}) = [\phi(a)]^{-1} .$$

Proof. $$\phi(a) = \phi(aa^{-1}a) = \phi(a)\ \phi(a^{-1})\ \phi(a)$$

Therefore $$\phi(a^{-1}) = [\phi(a)]^{-1} .$$

PROPOSITION 9. *If* $\phi : F \to F'$ *is not a null semi-homomorphism of fields, then*

$$\phi(1) \in \text{Centre of } \phi(F) .$$

Proof. Since ϕ is not null $1 \notin \text{Ker } \phi$, if $\text{Ker } \phi$ exists. Therefore as $\phi(1) = [\phi(1)]^{-1}$ by proposition 8; that is $[\phi(1)]^2 = 1'$

so $$\phi(r) = \phi(1)\phi(r)\phi(1) ,$$

and $$\phi(1)\phi(r) = \phi(r)\phi(1) .$$

Now we ask when is a semi-homomorphism a homomorphism?

We present a sufficient condition in a special situation.

https://doi.org/10.1017/S0004972700026356 Published online by Cambridge University Press

THEOREM 10. *A semi-monomorphism* $\phi : R \to R'$ *of rings will be a monomorphism, if*

(i) *char* $R' \neq 2$,

(ii) $\phi(R)$ *is a skew subfield of* R' *and*

(iii) $\phi(2y + yz) - 2\,\phi(y) = \phi(yzy)\,[\phi(y)]^{-1}$.

Proof. (i) By Lemma 4, (i) guarantees ϕ to be a homomorphism for the underlying additive groups. If either of $y,z = 0$, then $\phi(y)$ and $\phi(z) = 0$, or $\phi(z) = 0$ and $\phi(yz) = 0$, so in either case $\phi(yz) = \phi(y)\ (z)$.

When both $y,z \neq 0$, then $\phi(y)$ and $\phi(z)$ both $\neq 0$

so $$\phi(yzy) = \phi(y)\ \phi(z)\ \phi(y)$$

implies $$\phi(y)\ \phi(z) = [\phi(yzy)]\ [\phi(y)]^{-1} .$$

Also $$\phi(2y + yz) = \phi(y + yz + y) = \phi(y) + \phi(yz) + \phi(y)$$

that is $$\phi(2y + yz) - 2\phi(y) = \phi(yz) .$$

Hence by (iii), $$\phi(yz) = \phi(y)\ \phi(z) .$$

One notices that (iii) in Theorem 10 can be equivalently replaced by

(iii)' $$\phi(zy) = [\phi(y)]^{-1}\ \phi(yzy).$$

The open problem here is to find the necessary and sufficient condition for a semi-homomorphism to be a homomorphism.

THEOREM 11. *For commutative ring* R *and* R' *with identities, if* $\phi : R \to R'$ *is an identity-preserving semi-homomorphism, and* *char* $R' \neq 2$, *then* ϕ *is a homomorphism or an antihomomorphism.*

Proof. We know by Lemma 4, ϕ is an additive homomorphism.

Now $$\phi[(x + y).1(x + y)] = \phi(x + y).1'.\phi(x + y)$$

that is $$\phi[x.x + x.y + y.x + y.y] = \phi(x)\phi(x) + \phi(x)\phi(y) + \phi(y)\phi(x) + \phi(y.y) .$$

Then using $\phi(x.1.x) = \phi(x).1'.\phi(x)$ and since ϕ is additive we have,

$$\phi(xy) + \phi(yx) = \phi(x)\phi(y) + \phi(y)\phi(x)$$

that is $$2\phi(xy) = 2\phi(x)\phi(y)$$

that is $$\phi(xy) = \phi(x)\phi(y) .$$

A slight weakening of Hua's theorem now reads as:

THEOREM 12. *If* $\phi : K \to K'$ *is an identity preserving semi-monomorphism of skew fields, then* ϕ *is either a monomorphism or an* *anti-monomorphism, provided char* $K' \neq 2$.

Proof. Applying Lemma 4, one proceeds to prove this result exactly as in Hua [3].

Remarks. (A) The set of semi-endomorphisms of an abelian group A form a ring, the *ring of semi-endomorphisms* of the group A which contains the ring of endomorphism as a subring, under the usual pointwise addition and composition of functions. Thus any ring $(R, +, .)$ can be embedded into the ring of semi-endomorphism of $(R,+)$.

(B) The concept of a semi-homomorphism becomes more significant when both the additive and the multiplicative structures are not commutative, for example, near-ring [4]. This case if left for future study.

References

[1] G. Anchochea, "On semi-automorphisms of division algebras", *Ann. of Math.*, 48 (1947), 147-153.

[2] I.N. Herstein, "Semi-homomorphisms of groups", *Canad. J. Math.* 20 (1968), 384-388.

[3] L.K. Hua, "On the automorphisms of a field", *Proc. Nat. Acad. Sci. U.S.A.*, 35 (1949), 386-389.

[4] I. Kaplanasky, "Semi-automorphisms of Rings", *Duke Math. J.*, 14 (1947), 521-525.

[5] N.H. CcCoy, *Theory of rings*, (Collier-Macmillan, 1968).

[6] G. Pilz, *Near-rings*, (North Holland, 1977).

Department of Mathematics
Monash University, Clayton
Victoria 3168, Australia.

and

Faculty of Science
Sana'a University
YEMEM ARAB REPUBLIC.

https://doi.org/10.1017/S0004972700026356 Published online by Cambridge University Press