Can. J. Math., Vol. XXXIX, No. 5, 1987, pp. 1174-1209

THE MULTIPLE Q-CONSTRUCTION

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Introduction. Products, and closely associated questions of infinite loop space structure, have always been a source of trouble in higher algebraic K-theory. From the first description of the product in terms of the plus construction, up to the current tendency to let the infinite loop space machines do it, the constructions have never been completely explicit, and many mistakes have resulted.

Since Waldhausen introduced the double Q-construction [**16**], there has been the tantalizing prospect of an infinite loop space structure for the nerve $BQ\mathscr{A}$ of the Q-construction $Q\mathscr{A}$ of an exact category $\mathscr{A}$, which would be understandable to the man on the street, and which also would be well-behaved with respect to products induced by biexact pairings. Gillet [**3**] showed that most of these conditions could be met with his introduction of the multiple Q-construction $Q^k\mathscr{A}$. Shimakawa [**14**] filled in some of the details later.

This version of the multiple Q-construction is, however, based on multiple categories and their associated multi-simplicial nerves. I freely admit that I still have trouble seeing into such things. It is a rather painful exercise, for example, to get an explicit description of a multi-simplex in the nerve for $Q^k\mathscr{A}$. Such a description is necessary if one wants to describe, let alone analyze homotopically, the action of the symmetric group on $BQ^k\mathscr{A}$.

The central point of this paper is that, by perturbing the description of the Q-construction itself, it is possible to explicitly write down the multiple Q-constructions, and avoid multiple categories in the process.

In the notation of this paper, $Q^1\mathscr{A}$ is a simplicial set which is homotopically equivalent to the nerve $BQ\mathscr{A}$ of Quillen's category $Q\mathscr{A}$. Essentially, an n-simplex of $Q^1\mathscr{A}$ is what you get if you write down a representing diagram for an n-simplex of $BQ\mathscr{A}$. The point is that, homotopically, the equivalence relation that describes the morphisms of $Q\mathscr{A}$ is unnecessary. $Q\mathscr{A}$ can also be recovered from $Q^1\mathscr{A}$; it is the associated path category.

This description of $Q^1\mathscr{A}$ appears in the first section of this paper. There is a certain amount of sport in the fact that $Q^1\mathscr{A}$ is an axiomatically defined subcomplex of a generalized nerve $B_{\langle\Delta\rangle}\mathscr{A}$ for $\mathscr{A}$. Explicit homotopies are constructed within that context.

Received June 25, 1986. This research was supported by NSERC.

https://doi.org/10.4153/CJM-1987-060-0 Published online by Cambridge University Press

$Q^1\mathscr{A}$ is the set of objects of a simplicial exact category, and so the Q^1 construction may be iterated, giving k-fold simplicial sets

$$Q^k\mathscr{A} = Q^1Q^1 \dots Q^1\mathscr{A} \quad \text{for } k \geqq 1.$$

The multisimplices of $Q^k\mathscr{A}$ may also be explicitly described. $Q^k\mathscr{A}$ is the k-fold simplicial set of objects of a k-fold simplicial exact category. These results appear in the second section of the paper.

It turns out that Waldhausen's construction $L\mathscr{A}$, when made precise, is contractible via explicit homotopies. This, together with the "fibre homotopy sequence"

$$Q^k_*\mathscr{A} \to LQ^k_*\mathscr{A} \to Q^{k+1}_*\mathscr{A},$$

implies that there is an isomorphism

$$Q^k_*\mathscr{A} \simeq \Omega Q^{k+1}_*\mathscr{A}$$

in the pointed homotopy category. $Q^k_*\mathscr{A}$ is obtained from $Q^k\mathscr{A}$ by collapsing the subcomplex of 0-objects. This gives an infinite loop space structure for $Q^1\mathscr{A} \simeq Q^1_*\mathscr{A}$. These results appear in the third section of the paper. I also show there that the canonical Σ_k-action on $Q^k_*\mathscr{A}$ induces multiplication by sign in the homotopy category.

From the explicit description of $Q^k\mathscr{A}$, it is easily seen that any biexact pairing

$$\otimes : \mathscr{A} \times \mathscr{B} \to \mathscr{C}$$

induces pairings of the form

$$\otimes : Q^k_*\mathscr{A} \wedge Q^l_*\mathscr{B} \to Q^{k+l}_*\mathscr{C}$$

which fit together, up to homotopy, in such a way that they induce K-theory pairings

$$\otimes : K_i\mathscr{A} \times K_j\mathscr{B} \to K_{i+j}\mathscr{C}.$$

The tensor product $\otimes : \mathbf{P}(R) \times \mathbf{P}(R) \to \mathbf{P}(R)$ on the category $\mathbf{P}(R)$ of finitely generated projective modules on a ring R is the standard example of a biexact pairing. A graded commutative ring structure on the K-theory $K_*(R)$ of R is now easily derived by analyzing this biexact pairing. This ring structure coincides with the original one given by Loday [**8**]. These results appear in the fourth section. The existence of the projection formula can be proved in the same way.

The last section of the paper has mostly to do with my ulterior motive for getting involved with this construction. I show how to construct on the K-theory simplicial presheaves $[Q^k\mathbf{P}]_*$ on the étale site $\mathrm{et}|_S$ of a scheme S. The technical difficulty is that $Q^k\mathbf{P}$ arises from the categories $\mathbf{P}(U)$ of vector bundles on the total spaces of the étale maps $U \to S$, and these do not give an honest contravariant functor on $\mathrm{et}|_S$ with values in exact categories. **P** is only a pseudo-functor; the deviation from functoriality is given by coherent natural isomorphism. Such things can, however, be straightened out up to homotopy equivalence, as in [**9**]. In fact, to construct $[Q^k\mathbf{P}]_*$, it is better to rectify the simplicial lax functor Iso $Q^k\mathbf{P}$ of isomorphisms in the simplicial object $Q^k\mathbf{P}$. All of the other constructions of the previous sections may also be run through the same process, giving a presheaf of infinite loop spaces, and pairings of simplicial presheaves induced by tensor product. This procedure works for any diagram of schemes, in fact.

There is one striking defect in this game, however, in that it is not at all clear that the naive pairings

$$\otimes: Q^k_*\mathscr{A} \wedge Q^l_*\mathscr{B} \to Q^{k+l}_*\mathscr{C}$$

fit together to give a smash product pairing of spectra. One is unlikely, for example, to be able to construct the presheaf of Bott-periodic K-theory spectra on the étale site without such a pairing on the presheaf level (see [**15**]). The problem, I believe, is that the infinite loop space structure of $Q^1_*\mathscr{A}$ that I construct here bears no resemblance to any of the existing infinite loop space machines. $Q^2_*\mathscr{A}$ and the direct sum nerve $B_\oplus Q^1_*\mathscr{A}$ seem to be very different objects.

It still appears that, in order to get the complete story on K-theory products, one is obliged to use Γ-spaces, via the correct mixture of [**1**] and [**9**]. Ultimately, this may even be more aesthetically appealing. To quote a classic phrase, the theory described here is "elementary, but not short".

1. The rigid Q-construction. Let **n** be the finite ordinal number

$$0 \to 1 \to 2 \to \dots \to n,$$

thought of as a partially ordered set. Associated to **n** is a poset $\langle \mathbf{n} \rangle$ whose objects are the pairs (i, j) with $0 \leqq i \leqq j \leqq n$. Write $(i, j) \leqq (i', j')$ if $i \leqq i'$ and $j \geqq j'$. Observe, more generally, that a poset $\langle P \rangle$ may be associated to an arbitrary poset P in an analogous fashion, and that this process is functorial. It follows that $\langle n \rangle$ is part of cosimplicial poset $\langle \Delta \rangle$, where Δ is the category of finite ordinal numbers. There is a cosimplicial poset map $\tau:\langle \Delta \rangle \to \Delta$, where $\tau_n:\langle \mathbf{n} \rangle \to \mathbf{n}$ is the map defined by

$$\tau_n(i, j) = i.$$

Let $\mathscr{A}$ be an exact category, as in [**11**]. Define a simplicial set $Q^1\mathscr{A}$ by requiring its n-simplices to be those functors $Q:\langle \mathbf{n} \rangle \to \mathscr{A}$ which satisfy the following requirements:

(1.1) each induced map $Q(i, j) \to Q(i', j)$ is an admissible monic,

(1.2) each map $Q(i, j) \to Q(i, j')$ is an admissible epi,

(1.3) if $(i, j) \leqq (i', j')$ in $\langle \mathbf{n} \rangle$, then the diagram

$$\begin{tikzcd} Q(i,j) \arrow[r, tail] \arrow[d, two heads] & Q(i',j) \arrow[d, two heads] \\ Q(i,j') \arrow[r, tail] & Q(i',j') \end{tikzcd}$$

is bicartesian.

These requirements are stable under precomposition with all maps

$$\langle\theta\rangle:\langle\mathbf{m}\rangle \to \langle\mathbf{n}\rangle$$

induced by ordinal number morphisms $\theta:\mathbf{m} \to \mathbf{n}$, and so $Q^!\mathscr{A}$ is a simplicial set.

One sees, for example, that a 2-simplex of $Q^!\mathscr{A}$ is a commutative diagram

$$\begin{tikzcd} Q(0,2) \arrow[r, tail] \arrow[d, two heads] & Q(1,2) \arrow[r, tail] \arrow[d, two heads] & Q(2,2) \\ Q(0,1) \arrow[r, tail] \arrow[d, two heads] & Q(1,1) & \\ Q(0,0) & & \end{tikzcd}$$

in which the square is bicartesian. In other words, a 2-simplex of $Q^!\mathscr{A}$ consists of representations of a composeable pair of arrows in Quillen's category $Q\mathscr{A}$, together with a representative of their composite.

$Q^!\mathscr{A}$ is an axiomatically defined subcomplex of the simplicial set $B_{\langle\Delta\rangle}\mathscr{A}$, whose n-simplices consist of all functors $\langle\mathbf{n}\rangle \to \mathscr{A}$. $B_{\langle\Delta\rangle}\mathscr{A}$ is a generalized nerve for $\mathscr{A}$. There is a corresponding construction B_PC for any cosimplicial poset P and any small category C. In this notation, the ordinary nerve BC is $B_\Delta C$.

Now let $\langle\mathbf{n}\rangle^*$ be the poset whose objects are those of $\langle\mathbf{n}\rangle$, but with $(i, j) \leqq (i', j')$ if $i \leqq i'$ and $j \leqq j'$. If $Q:\langle\mathbf{n}\rangle \to \mathscr{A}$ is an n-simplex of $Q^!\mathscr{A}$, and $(i, j) \leqq (i', j')$ in $\langle\mathbf{n}\rangle^*$, then Q determines a diagram

$$Q(i,j) \overset{p}{\twoheadleftarrow} Q(i,j') \overset{m}{\rightarrowtail} Q(i',j')$$

in $\mathscr{A}$, and hence an arrow

$$m_!p^!:Q(i,j) \to Q(i',j')$$

in $Q\mathscr{A}$. In fact, $m_!p^!$ is the image of $(i, j) \leqq (i', j')$ under a functor

$$Q^*:\langle\mathbf{n}\rangle^* \to Q\mathscr{A}$$

which is associated to Q. The assignment $Q \mapsto Q^*$ determines a simplicial set map

$$\eta_{\mathscr{A}}:Q^!\mathscr{A} \to B_{\langle\Delta\rangle *}Q\mathscr{A}.$$

On the other hand, there is a cosimplicial poset map $\gamma:\Delta \to \langle\Delta\rangle^*$, where $\gamma_n:\mathbf{n} \to \langle\mathbf{n}\rangle^*$ is the map defined by $\gamma_n(i) = (i, i)$. Clearly, γ induces a simplicial set map

$$\gamma^*_{\mathscr{A}}:B_{\langle\Delta\rangle *}Q\mathscr{A} \to BQ\mathscr{A}.$$

Let $\pi_{\mathscr{A}}:Q^!\mathscr{A} \to BQ\mathscr{A}$ be the composite of $\gamma^*_{\mathscr{A}}$ with $\eta_{\mathscr{A}}$.

PROPOSITION 1.4. *The simplicial set map*

$$\pi_{\mathscr{A}}:Q^!\mathscr{A} \to BQ\mathscr{A}$$

is a homotopy equivalence, which is natural with respect to exact functors.

Proof. The naturality of $\pi_{\mathscr{A}}$ is clear. We construct a non-natural homotopy inverse

$$\iota_{\mathscr{A}}:BQ\mathscr{A} \to Q^!\mathscr{A}$$

for $\pi_{\mathscr{A}}$.

First of all, choose a fixed representative

$$\cdot \xleftarrow{p_v} \cdot \xrightarrow{m_v} \cdot$$

for each arrow v of the category $Q\mathscr{A}$. One may suppose that m_v and p_v are identities of $\mathscr{A}$ if v is an identity of $Q\mathscr{A}$. Then, given an n-simplex $R:n \to Q\mathscr{A}$ of $BQ\mathscr{A}$, there is a unique n-simplex $R^*:\langle\mathbf{n}\rangle \to \mathscr{A}$ of $Q^!\mathscr{A}$ such that

(1.5) $\pi_{\mathscr{A}}R^* = R$, and

(1.6) for each object (i, j) of $\langle\mathbf{n}\rangle$, the diagram

$$R(i) = R^*(i, i) \twoheadleftarrow R^*(i, j) \rightarrowtail R^*(j, j) = R(j)$$

is the chosen representative of the map $R(i) \to R(j)$ in $Q\mathscr{A}$ which is induced by $i \leqq j$.

It follows easily that $R \mapsto R^*$ defines a simplicial map

$$\iota_{\mathscr{A}}:BQ\mathscr{A} \to Q^!\mathscr{A}$$

such that

$$\pi_{\mathscr{A}}\iota_{\mathscr{A}} = 1_{BQ\mathscr{A}}.$$

If Q and Q' are n-simplices of $Q^!\mathscr{A}$ such that $\pi_{\mathscr{A}}Q = \pi_{\mathscr{A}}Q'$, then there is a unique natural isomorphism

$$\theta:Q \xrightarrow{\cong} Q'$$

which restricts to the identity on $\pi_{\mathscr{A}}Q$.

It follows that there is a unique natural isomorphism

$$\theta_Q: Q \cong \iota_{\mathscr{A}}\pi_{\mathscr{A}}Q$$

for each n-simplex Q of $Q^1\mathscr{A}$. Each of these determines a homotopy of functors

$$\begin{tikzcd}
\langle n \rangle \arrow[d] \arrow[dr, "Q"] & \\
\langle n \rangle \times 1 \arrow[r] & \mathscr{A} \\
\langle n \rangle \arrow[u] \arrow[ur, "\iota_{\mathscr{A}}\pi_{\mathscr{A}}Q"'] &
\end{tikzcd}$$

in the standard way. The uniqueness of the θ_Q implies that this homotopy is natural in Q in the sense that, if $\gamma:\mathbf{m} \to \mathbf{n}$ is an ordinal number map, then the diagram

$$\begin{tikzcd}
\langle m \rangle \times 1 \arrow[d, "\langle \gamma \rangle \times 1"] \arrow[dr, "h_{Q\langle \gamma \rangle}"] & \\
\langle n \rangle \times 1 \arrow[r, "h_Q"'] & \mathscr{A}
\end{tikzcd}$$

commutes. This implies that $j_{\mathscr{A}}\iota_{\mathscr{A}}\pi_{\mathscr{A}}$ is homotopic to the inclusion

$$j_{\mathscr{A}}: Q^1\mathscr{A} \subset B_{\langle \Delta \rangle}\mathscr{A}.$$

In effect, the h_Q induce to composites

$$\begin{tikzcd}
\Delta^n \times \Delta^1 \arrow[r, "j_n \times \tau^*"] & B_{\langle \Delta \rangle}\langle n \rangle \times B_{\langle \Delta \rangle}(1) \cong B_{\langle \Delta \rangle}(\langle \mathbf{n} \rangle \times \mathbf{1}) \arrow[d, "B_{\langle \Delta \rangle}h_Q"] \\
& B_{\langle \Delta \rangle}\mathscr{A},
\end{tikzcd} \tag{1.7}$$

where j_n classifies the identity functor on $\langle \mathbf{n} \rangle$, and $\tau^*(\mathbf{m} \to \mathbf{1})$ is the composite

$$\langle \mathbf{m} \rangle \xrightarrow{\tau} \mathbf{m} \to \mathbf{1}.$$

These composites are natural in the simplices of $Q^1\mathscr{A}$, and hence induce the desired homotopy

$$Q^1\mathscr{A} \times \Delta^1 \to B_{\langle \Delta \rangle}\mathscr{A}.$$

Finally, to show that this homotopy takes values in $Q^1\mathscr{A}$, it suffices to show that each composite (1.7) factors through $Q^1\mathscr{A}$ by checking it on the maximal non-degenerate simplices of $\Delta^1 \times \Delta^1$. But this follows from

LEMMA 1.8. *If* $Q:\langle n\rangle \to \mathscr{A}$ *is an* n*-simplex of* $Q^1\mathscr{A}$, *and*

$$h:\langle \mathbf{n}\rangle \times \mathbf{1} \to \mathscr{A}$$

is a natural isomorphism of Q *with a functor* $Q':\langle \mathbf{n}\rangle \to \mathscr{A}$, *then the composite functors*

$$\langle \mathbf{n} + \mathbf{1}\rangle \xrightarrow{\langle\sigma_i\rangle} \langle \mathbf{n} \times \mathbf{1}\rangle \cong \langle \mathbf{n}\rangle \times \langle \mathbf{1}\rangle \xrightarrow{1 \times \tau} \langle \mathbf{n}\rangle \times \mathbf{1} \xrightarrow{h} \mathscr{A}$$

are $(n + 1)$*-simplices of* $Q^1\mathscr{A}$ *for* $0 \leqq i \leqq n$.

Proof. The simplex $\sigma_i:\mathbf{n} + \mathbf{1} \to \mathbf{n} \times \mathbf{1}$ is defined by

$$\sigma_i(j) = \begin{cases} (j, 0) & 0 \leqq j \leqq i \\ (j - 1, 1) & i + 1 \leqq j \leqq n + 1. \end{cases}$$

The composite is of the form

$$\langle \mathbf{n} + \mathbf{1}\rangle \to \mathscr{A}$$
$$(i, j) \mapsto h((\omega i, \omega j), \gamma i),$$

where $\omega:\mathbf{n} + \mathbf{1} \to \mathbf{n}$ and $\gamma:\mathbf{n} + \mathbf{1} \to \mathbf{1}$ are ordinal number morphisms. $Q':\langle \mathbf{n}\rangle \to \mathscr{A}$ is necessarily an n-simplex of $Q^1\mathscr{A}$, and so

(1) $(i, j) \leqq (i, j')$ induces an admissible epi

$$h((\sigma i, \sigma j), \gamma i) \twoheadrightarrow h((\sigma i, \sigma j'), \gamma i),$$

(2) $(i, j) \leqq (i', j)$ induces an admissible monic

$$h((\sigma i, \sigma j), \gamma i) \rightarrowtail h((\sigma i', \sigma j), \gamma i) \cong h((\sigma i', \sigma j), \gamma i')$$

(3) if $(i, j) \leqq (i', j')$, then the induced outer square in the diagram

$$\begin{tikzcd} h((\sigma i, \sigma j), \gamma i) \arrow[r, tail] \arrow[d] & h((\sigma i', \sigma j), \gamma i) \arrow[r, phantom, "\cong"] \arrow[d] & h((\sigma i', \sigma j), \gamma i') \arrow[d] \\ h((\sigma i, \sigma j'), \gamma i) \arrow[r, tail] & h((\sigma i', \sigma j'), \gamma i) \arrow[r, phantom, "\cong"] & h((\sigma i', \sigma j'), \gamma i') \end{tikzcd}$$

is bicartesian.

The path category PX of a simplicial set X is the category whose objects are the vertices of X and whose morphisms are the 1-simplices of X, subject to the relation that the diagram

$$\begin{tikzcd} v_0 \arrow[rr, "d_2\sigma"] \arrow[dr, "d_1\sigma"'] & & v_1 \arrow[dl, "d_0\sigma"] \\ & v_2 & \end{tikzcd}$$

commutes in PX for each 2-simplex σ of X (see [**2**]). There is an obvious natural simplicial map $\eta:X \to BPX$ which is universal for all simplicial set maps of the form $X \to BC$, where C is a category. It follows that there is a unique functor

$$(\pi_{\mathscr{A}})_*:PQ^1\mathscr{A} \to Q\mathscr{A}$$

such that the following diagram of simplicial set maps commutes:

$$\begin{tikzcd} Q^1\mathscr{A} \arrow[r, "\eta"] \arrow[dr, "\pi"'] & BPQ^1\mathscr{A} \arrow[d, "B(\pi_{\mathscr{A}})_*"] \\ & BQ\mathscr{A} \end{tikzcd}$$

PROPOSITION 1.9. *$(\pi_{\mathscr{A}})_*$ is an isomorphism of categories.*

Proof. Use the presentation of $Q\mathscr{A}$ by generators and relations given in [**11**] to construct the inverse $Q\mathscr{A} \to PQ^1\mathscr{A}$ for $(\pi_{\mathscr{A}})_*$.

2. Iterating Q^1. $Q^1\mathscr{A}$ is the simplicial set of objects of a simplicial exact category, which will also be denoted by $Q^1\mathscr{A}$. It is clear that the construction may be iterated as often as one likes, yielding multisimplicial exact categories

$$Q^1(Q^1(\dots(Q^1\mathscr{A})\dots)).$$

The objects of such will give deloopings for the simplicial set $Q^1\mathscr{A}$. But, in order to define multiplication arising from biexact pairings properly, it is necessary to write down explicit descriptions of these multisimplicial objects which see all of the symmetries that can occur. This will be somewhat messy.

A *k-fold exact sequence* in $\mathscr{A}$ is a functor $E:\mathbf{2}^k \to \mathscr{A}$, such that each sequence

$$0 \to E(j_1, \dots, \overset{i}{0}, \dots, j_k) \to E(j_1, \dots, \overset{i}{1}, \dots, j_k) \to E(j_1, \dots, \overset{i}{2}, \dots, j_k) \to 0$$

is exact in $\mathscr{A}$. The k-fold exact sequences are the objects of an exact category $\mathrm{Ex}^k\,\mathscr{A}$. There is a canonical isomorphism

$$\mathrm{Ex}(\mathrm{Ex}^k\,\mathscr{A}) \cong \mathrm{Ex}^{k+1}\,\mathscr{A},$$

which is given by the exponential law.

Recall that the ordinal number morphisms d^0 and d^2 from $\mathbf{1}$ to $\mathbf{2}$ are defined, respectively, by

$$d^0(0 \to 1) = 1 \to 2, \quad \text{and}$$
$$d^2(0 \to 1) = 0 \to 1.$$

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A *k-fold epi-monic* in $\mathscr{A}$ is defined to be a functor $\varphi:\mathbf{1}^k \to \mathscr{A}$ which extends to a k-fold exact sequence of $\mathscr{A}$ in the sense that there is a sequence of numbers $(\epsilon_1, \dots, \epsilon_k)$, such that ϵ_i is 0 or 2, and a commutative diagram

$$\begin{tikzcd} 1^k \arrow[r, "\varphi"] \arrow[d, "d^{\epsilon_1} \times \dots \times d^{\epsilon_k}"'] & \mathscr{A} \\ 2^k \arrow[ur, "\varphi_*"'] & \end{tikzcd}$$

where φ_* is a k-fold exact sequence. Observe that φ_* is determined up to isomorphism by φ and the sequence $(\epsilon_1, \dots, \epsilon_k)$. It follows that the k-fold epi-monics corresponding to the fixed sequence $(\epsilon_1, \dots, \epsilon_k)$ are the objects of an exact category

$$EM(\epsilon_1, \dots, \epsilon_k)(\mathscr{A})$$

which is additively equivalent to $\mathrm{Ex}^k\,\mathscr{A}$. Also, an admissible monic or epi of this category is $(k + 1)$-fold epi-monic.

A *non-degenerate generator* in the poset $\langle \mathbf{n} \rangle$ is a relation having either of the following two forms:

(2.1) $$R_2:(i_1, j) \leqq (i_2, j), \quad i_1 \neq i_2,$$
$$R_0:(i, j_1) \leqq (i, j_2), \quad j_1 \neq j_2.$$

Set $\epsilon(R_2) = 2$ for all generators of the form R_2, and set $\epsilon(R_0) = 0$ for all generators of the form R_0.

$Q^k(n_1, \dots, n_k)$ is inductively defined to be the set of all functors

$$\varphi:\langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \to \mathscr{A}$$

such that the following conditions are satisfied:

(2.2) for each $x_i \in \langle n_i \rangle$, $i = 1, \dots, k$, the composite

$$\begin{tikzcd} \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_{i-1} \rangle \times \langle \mathbf{n}_{i+1} \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[d, "j_{x_i}"'] & \\ \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[r, "\varphi"'] & \mathscr{A} \end{tikzcd}$$

is in $Q^{k-1}(n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_k)$, where j_{x_i} is defined on relations by

$$j_{x_i}(R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_k) = (R_1, \dots, 1^i_{x_i}, \dots, R_k),$$

(2.3) for each k-tuple $(R_1, \dots, R_k)$ of non-degenerate generators, with R_i in $\langle n_i \rangle$, the composite functor

$$\mathbf{1}^k \xrightarrow{i_{R_1} \times \dots \times i_{R_k}} \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \xrightarrow{\varphi} \mathscr{A}$$

is a k-fold epi-monic for the sequence $(\epsilon(R_1), \dots, \epsilon(R_k))$.

The functor $i_{R_1}:\mathbf{1} \to \langle \mathbf{n}_1 \rangle$ classifies the map R_1.

PROPOSITION 2.4. *The functors in $Q^k\mathscr{A}(n_1, \dots, n_k)$ are the objects of a k-fold simplicial exact category $Q^k\mathscr{A}$.*

Proof. The morphisms of the category $Q^k\mathscr{A}(n_1, \dots, n_k)$ are natural transformations. The exact sequences are pointwise exact sequences. To show that $Q^k\mathscr{A}(n_1, \dots, n_k)$ is exact, one assumes inductively that all $Q^{k-1}\mathscr{A}(m_1, \dots, m_{k-1})$ are exact, and then shows that an admissible monic (respectively epi) of $Q^k\mathscr{A}(n_1, \dots, n_k)$ is a pointwise monic (respectively epi) which restricts to admissible monics (respectively epis) along each functor j_{x_1} and along each $i_{R_1} \times \dots \times i_{R_k}$.

It is also necessary to show that precomposition of functors

$$\varphi:\langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \to \mathscr{A} \quad \text{in } Q^k\mathscr{A}(n_1, \dots, n_k)$$

with a multisimplicial structure functor

$$\langle \mathbf{m}_1 \rangle \times \dots \times \langle \mathbf{m}_k \rangle \xrightarrow{\langle \theta_1 \rangle \times \dots \times \langle \theta_k \rangle} \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle$$

produces functors in $Q^k\mathscr{A}(m_1, \dots, m_k)$. This is done by induction on k again. In effect, the diagram of functors

$$\begin{tikzcd}
\langle \mathbf{m}_1 \rangle \times \dots \times \langle \mathbf{m}_k \rangle \arrow[rr, "\langle \theta_1 \rangle \times \dots \times \langle \theta_k \rangle"] & & \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[r, "\varphi"] & \mathscr{A} \\
\langle \mathbf{m}_1 \rangle \times \dots \times \langle \mathbf{m}_{i-1} \rangle \times \langle \mathbf{m}_{i+1} \rangle \times \dots \times \langle \mathbf{m}_k \rangle \arrow[u, "j_{x_i}"] \arrow[rr, "\langle \theta_1 \rangle \times \dots \times \langle \theta_{i-1} \rangle \times \langle \theta_{i+1} \rangle \times \dots \times \langle \theta_k \rangle"'] & & \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_{i-1} \rangle \times \langle \mathbf{n}_{i+1} \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[u, "j_{\theta_i(x_i)}"'] &
\end{tikzcd}$$

commutes for each $x_i \in \langle \mathbf{m}_i \rangle$, and so the first axiom is satisfied. The second axiom is proved by showing that every sequence of generators $(R_1, \dots, R_k)$, where R_i in $\langle \mathbf{n}_i \rangle$ is possibly degenerate in the sense that $i_1 = i_2$ or $j_1 = j_2$ in (1.10), determines a k-fold epi-monic

$$\mathbf{1}^k \xrightarrow{i_{R_1} \times \dots \times i_{R_k}} \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \xrightarrow{\varphi} \mathscr{A}$$

for the sequence $(\epsilon(R_1), \dots, \epsilon(R_k))$, where $\epsilon(R_k)$ can be either 0 or 2 in the degenerate places. In effect, if $R_1 = 1_{x_1}$, then there is a diagram

$$\begin{tikzcd}
\mathbf{1}^k \arrow[r, "i_{R_1} \times \dots \times i_{R_k}"] \arrow[d, "pr"] & \langle \mathbf{n}_1 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[r, "\varphi"] & \mathscr{A} \\
\mathbf{1}^{k-1} \arrow[r, "i_{R_2} \times \dots \times i_{R_k}"] & \langle \mathbf{n}_2 \rangle \times \dots \times \langle \mathbf{n}_k \rangle \arrow[u, "i_{x_1}"'] &
\end{tikzcd}$$

Then the composite

$$\varphi i_{x_1}(i_{R_2} \times \ldots \times i_{R_k})$$

is a $(k - 1)$-fold epi-monic for the sequence $(\epsilon(R_2), \ldots, \epsilon(R_k))$, by induction. It follows, by inserting a 0 in exact sequences in the right places, that

$$\varphi(i_{R_1} \times \ldots \times i_{R_k})$$

is an epi-monic for the sequence $(\epsilon(R_1), \ldots, \epsilon(R_k))$.

PROPOSITION 2.5. *There is an isomorphism*

$$Q^1 Q^k \mathscr{A} \cong Q^{k+1}\mathscr{A}$$

of $(k + 1)$*-fold simplicial exact categories.*

Proof. The isomorphism is given by the exponential law. The category

$$Q^1(Q^k\mathscr{A}(n_2, \ldots, n_{k+1}))(n_1)$$

is identified with the category of functors

$$\varphi:\langle \mathbf{n}_1 \rangle \times \langle \mathbf{n}_2 \rangle \times \ldots \times \langle \mathbf{n}_{k+1} \rangle \rightarrow \mathscr{A}$$

such that

(2.6) each composite

$$\langle \mathbf{n}_2 \rangle \times \ldots \times \langle \mathbf{n}_{k+1} \rangle \xrightarrow{j_{x_1}} \langle \mathbf{n}_1 \rangle \times \ldots \times \langle \mathbf{n}_{k+1} \rangle \xrightarrow{\varphi} \mathscr{A}$$

is in $Q^k\mathscr{A}(n_2, \ldots, n_{k+1})$,

(2.7) for each generator $(i, j) \leqq (i', j)$ in $\langle n_1 \rangle$, the induced transformation

$$\varphi i_{(i,j)} \rightarrow \varphi i_{(i',j)}$$

is an admissible monic of $Q^k\mathscr{A}(n_2, \ldots, n_{k+1})$,

(2.8) for each generator $(i, j) \leqq (i, j')$ of $\langle \mathbf{n}_1 \rangle$, the induced transformation

$$\varphi i_{(i,j)} \rightarrow \varphi i_{(i,j')}$$

is an admissible epi of $Q^k\mathscr{A}(n_2, \ldots, n_{k+1})$, and

(2.9) for each relation $(i, j) \leqq (i', j')$ the induced diagram

$$\begin{array}{ccc}
\varphi i_{(i,j)} & \rightarrowtail & \varphi i_{(i',j)} \\
\downarrow & & \downarrow \\
\varphi i_{(i,j')} & \rightarrowtail & \varphi i_{(i',j')}
\end{array}$$

is pointwise bicartesian.

But such functors are precisely the objects of

$$Q^{k+1}\mathscr{A}(n_1, \ldots, n_{k+1}),$$

by induction on k, using the fact that an admissible epi or monic of k-fold epi-monics is a $(k + 1)$-fold epi-monic.

3. Delooping Q^k. Let $\mathbf{n}^{\mathrm{op}}$ be the poset

$$n^0 \rightarrow n - 1^0 \rightarrow \ldots \rightarrow 0^0.$$

Then the poset join $\mathbf{n}^{\mathrm{op}} * \mathbf{n}$ is the string of arrows

$$\begin{tikzcd} n^0 \arrow[r] & n - 1^0 \arrow[r] & \ldots \arrow[r] & 0^0 \arrow[d] \\ n & n - 1 \arrow[l] & \ldots \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

Define $L\mathscr{A}_n$ to be the set of all commutative diagrams of functors

$$\begin{tikzcd} \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n}\rangle^* \arrow[r, "\mathrm{pr}_L"] \arrow[d, "i_0"] & \mathbf{n}^{\mathrm{op}} * \mathbf{n} \arrow[d, "B"] \\ \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n}\rangle^* \times 2 \arrow[r, "E"] & \mathscr{A} \\ \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n}\rangle^* \arrow[r, "\mathrm{pr}_R"] \arrow[u, "i_1"] & \mathbf{n}^{\mathrm{op}} * \mathbf{n} \arrow[u, "B"] \end{tikzcd}$$

such that

(3.1) for each object x of $\langle \mathbf{n}^{\mathrm{op}} * \mathbf{n}\rangle^*$, the sequence

$$0 \rightarrow E(x, 0) \rightarrow E(x, 1) \rightarrow E(x, 2) \rightarrow 0$$

is exact in $\mathscr{A}$, and

(3.2) for each object i of $\mathbf{n}^{\mathrm{op}} * \mathbf{n}$, $E((i, i), 2)$ is a zero object of $\mathscr{A}$.

Here, i_ϵ, $\epsilon = 0, 1$, is induced by the functor $\epsilon: \mathbf{0} \rightarrow \mathbf{2}$ which picks out the object ϵ in the poset $\mathbf{2}$. Also,

$$\mathrm{pr}_L(i, j) = i \quad \text{and} \quad \mathrm{pr}_R(i, j) = j.$$

It follows from the definition that, for each $i \leqq j$ in $\mathbf{n}^{\mathrm{op}} * \mathbf{n}$, the induced map $Bi \rightarrow Bj$ is isomorphic to the admissible monic

$$E((i, j), 0) \rightarrow E((i, j), 1).$$

LEMMA 3.3. *$L\mathscr{A}_n$ is the set of n-simplices of a contractible simplicial set $L\mathscr{A}$.*

Proof. The simplicial structure of $L\mathscr{A}$ is induced by the functors

$$\theta^{\mathrm{op}} * \theta: \mathbf{m}^{\mathrm{op}} * \mathbf{m} \rightarrow \mathbf{n}^{\mathrm{op}} * \mathbf{n}.$$

Let 0 be a fixed zero object of $\mathscr{A}$, and let (E, B) be an n-simplex of $L\mathscr{A}$. The functor

$$B:\mathbf{n}^{\mathrm{op}} * \mathbf{n} \rightarrow \mathscr{A}$$

extends to a diagram

$$\begin{tikzcd}
& 0 \arrow[rr] \arrow[dd] & & 0 \arrow[r] \arrow[dd] & \cdots \arrow[r] & 0 \arrow[dl] \arrow[dd] \\
0 \arrow[dd] & & 0 \arrow[ll] \arrow[dd] & \cdots \arrow[l] & 0 \arrow[l] \arrow[dd] & \\
& Bn^{o} \arrow[rr] & & (n-1)^{o} \arrow[r] & \cdots \arrow[r] & B0^{o} \arrow[dl] \\
Bn & & B(n-1) \arrow[ll] & \cdots \arrow[l] & B0 \arrow[l] &
\end{tikzcd}$$

and hence determines a functor

$$\widetilde{B}:(\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times 1 \rightarrow \mathscr{A}.$$

The functor

$$E:\langle \mathbf{n}^{\mathrm{op}} * \mathbf{n}\rangle^{*} \times \mathbf{2} \rightarrow \mathscr{A}$$

extends to a diagram of exact sequences

$$\widetilde{E}:\langle\, (\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times \mathbf{1}\rangle^{*} \times \mathbf{2} \rightarrow \mathscr{A},$$

where $\widetilde{E}(\,(\,(i, 0), (j, 1)\,)\,), \,)$ is the exact sequence

$$0 \rightarrow 0 \rightarrow Bj \xrightarrow{1} Bj \rightarrow 0.$$

This construction is natural in the simplices of $L\mathscr{A}$ in the sense that

$$(\widetilde{\theta}^{*}E, \widetilde{\theta}^{*}B) = (\widetilde{E}\ \langle\, (\theta^{\mathrm{op}} * \theta) \times 1\rangle^{*}, \widetilde{B}(\,(\theta^{\mathrm{op}} * \theta) \times 1)\,)$$

if $\theta:\mathbf{m} \rightarrow \mathbf{n}$ is an ordinal number morphism.

There are natural poset maps

$$f:(n \times 1)^{\mathrm{op}} * (\mathbf{n} \times \mathbf{1}) \rightarrow (\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times 1 \quad \text{and}$$

$$b:(\mathbf{n} \times \mathbf{1})^{\mathrm{op}} * (\mathbf{n} \times \mathbf{1}) \rightarrow (\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times \mathbf{1},$$

which are determined, respectively, by the diagrams

$$f:\quad \begin{array}{ccc}
(n^{o}, 0) \rightarrow \cdots \rightarrow (0^{o}, 0) \\
\downarrow 1 \qquad\qquad\qquad \downarrow 1 \\
(n^{o}, 0) \rightarrow \cdots \rightarrow (0^{o}, 0) \\
\qquad\qquad\qquad\qquad \downarrow \\
(n, 0) \leftarrow \cdots \leftarrow (0, 0) \\
\downarrow \qquad\qquad\qquad \downarrow \\
(n, 1) \leftarrow \cdots \leftarrow (0, 1)
\end{array}$$

$$b:\quad \begin{array}{ccc} (n^o, 0) \to \cdots \to (0^o, 0) \\ \downarrow \qquad\qquad \downarrow \\ (n^o, 1) \to \cdots \to (0^o, 1) \\ \qquad\qquad\quad \downarrow \\ (n, 1) \leftarrow \cdots \leftarrow (0, 1) \\ \downarrow 1 \qquad\qquad \downarrow 1 \\ (n, 1) \leftarrow \cdots \leftarrow (0, 1) \end{array}$$

in $(\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times 1$. There is a poset map

$$\tau:(\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \to (\mathbf{n}^{\mathrm{op}} * \mathbf{n}) \times \mathbf{1}$$

which is determined by the string

$$\begin{array}{c} (n^0, 0) \to \cdots \to (0^0, 0) \\ \qquad\qquad\quad \downarrow \\ (n, 1) \leftarrow \cdots \leftarrow (0, 1) \end{array}$$

The poset maps f, b and τ are natural in $\mathbf{n}$. It follows that the assignment

$$(E, B) \mapsto (\tilde{E}(\langle \tau \rangle^* \times 1), \tilde{B}\tau)$$

defines a simplicial set map

$$\tau: L\mathscr{A} \to L\mathscr{A}.$$

The composites

$$(\tilde{E}(\langle f \rangle^* \times 1), \tilde{B}f)$$

determine a homotopy from τ to $1_{L\mathscr{A}}$, and the composites

$$(\tilde{E}(\langle b \rangle^* \times 1), \tilde{B}b)$$

determine a homotopy from the composite map

$$L\mathscr{A} \to \Delta^0 \xrightarrow{0} L\mathscr{A}$$

to τ. In effect, the composites

$$\begin{array}{c} \langle (\mathbf{n} + \mathbf{1})^{\mathrm{op}} * (\mathbf{n} + \mathbf{1}) \rangle \times \mathbf{2} \to \langle (\mathbf{n} \times \mathbf{1})^{\mathrm{op}} * (\mathbf{n} \times \mathbf{1}) \rangle^* \times \mathbf{2} \xrightarrow{\langle f \rangle^* \times 1} \langle (\mathbf{n} * \mathbf{n}^{\mathrm{op}}) \times \mathbf{1} \rangle^* \times \mathbf{2} \\ \hfill \downarrow \tilde{E} \\ \hfill E\mathscr{A} \end{array}$$

which are induced by the maximal non-degenerate simplices $\mathbf{n} + \mathbf{1} \to \mathbf{n} \times \mathbf{1}$ of $\Delta^n \times \Delta^1$ satisfy the relations for a simplicial homotopy $\tau \simeq 1_{L\mathscr{A}}$. The homotopy $\tau \simeq 0$ is similar.

https://doi.org/10.4153/CJM-1987-060-0 Published online by Cambridge University Press

$L\mathscr{A}$ is a construction of Waldhausen's [**16**], made precise. There is a cosimplicial poset map

$$\gamma:\langle \mathbf{n} \rangle \to \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n} \rangle^*,$$

which is defined by $\gamma(i, j) = (j^0, i)$. If (E, B) is an n-simplex of $L\mathscr{A}$, then the composite

$$\langle \mathbf{n} \rangle \xrightarrow{\gamma} \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n} \rangle^* \xrightarrow{i_2} \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n} \rangle^* \times \mathbf{2} \xrightarrow{E} \mathscr{A}$$

is an n-simplex of $Q^1\mathscr{A}$. One sees this by checking the axioms directly. It follows that γ induces a simplicial set map

$$\gamma: L\mathscr{A} \to Q^1\mathscr{A}$$

which is natural in $\mathscr{A}$.

$L\mathscr{A}$ is the simplicial set of objects of a simplicial exact category, which will also be denoted by $L\mathscr{A}$. The morphisms of $L\mathscr{A}_n$ are natural transformations; the exact sequences are the sequences of natural transformations which are pointwise exact in $\mathscr{A}$. In particular, a natural transformation f in $L\mathscr{A}_n$ is an admissible monic (respectively epi) if and only if f is a pointwise admissible monic (respectively epi). This is a consequence of the fact that objects of $L\mathscr{A}_n$ are defined by exact sequences. It follows that there is a simplicial exact functor $\gamma: L\mathscr{A} \to Q^1\mathscr{A}$ which extends the definition given above.

Let $\mathbf{O} \subset \mathscr{A}$ be the groupoid of all zero objects of $\mathscr{A}$. $\mathbf{O}$ is a full exact subcategory of $\mathscr{A}$, and $Q^1\mathbf{O}$ is contractible as a simplicial set, by Lemma 1.8 or otherwise. Define $F\mathscr{A}$ as a simplicial set (respectively simplicial exact category) by the cartesian square

$$\begin{tikzcd} F\mathscr{A} \arrow[r] \arrow[d] & L\mathscr{A} \arrow[d, "\gamma"] \\ Q^1\mathbf{O} \arrow[r] & Q^1\mathscr{A} \end{tikzcd}$$

The set of objects of $F\mathscr{A}_n$ therefore consists of those n-simplices (E, B) of $L\mathscr{A}_n$ such that $E(x, 2)$ is a zero object of $\mathscr{A}$ for each x in $\langle \mathbf{n}^{\mathrm{op}} * \mathbf{n} \rangle^*$.

PROPOSITION 3.4. (a) *The constant simplicial exact category* $\mathscr{A}$ *is canonically additively equivalent to* $F\mathscr{A}$.

(b) *The cartesian square of simplicial sets*

$$\begin{tikzcd} dQ^kF\mathscr{A} \arrow[r, "i"] \arrow[d] & dQ^kL\mathscr{A} \arrow[d, "\gamma"] \\ dQ^kQ^1\mathbf{O} \arrow[r] & dQ^kQ^1\mathscr{A} \end{tikzcd}$$

is homotopy cartesian for each $k \geqq 1$, *where, for example,* $dQ^kF\mathscr{A}$ *is the diagonal of the* $(k + 1)$*-fold simplicial set* $Q^kF\mathscr{A}$.

Proof. (a) There are simplicial exact functors

(3.5) $\quad \mathscr{A} \to \mathrm{Is}\,\mathscr{A} \leftarrow F\mathscr{A},$

where the set of objects of $\mathrm{Is}\,\mathscr{A}_n$ is the set of functors

$$B: \mathbf{n}^{\mathrm{op}} * \mathbf{n} \to \mathscr{A}$$

such that, for each $i \leqq j$ in $\mathbf{n}^{\mathrm{op}} * \mathbf{n}$, the induced map $B(i) \to B(j)$ is an isomorphism of $\mathscr{A}$. The functor

$$F\mathscr{A}_n \to \mathrm{Is}\,\mathscr{A}$$

sends (E, B) to B. The functor

$$\mathscr{A} \to \mathrm{Is}\mathscr{A}_n$$

sends an object A of $\mathscr{A}$ to the functor B_A, which is defined by

$$B_A(i) = A \quad \text{and} \quad B_A(i \leqq j) = 1_A.$$

Both functors are equivalences of categories.

(b) Observe that an exact equivalence $\mathscr{A} \to \mathscr{B}$ induces a multisimplicial exact equivalence

$$Q^k\mathscr{A} \to Q^k\mathscr{B}.$$

The functor Q^1 takes exact equivalences to weak equivalences of simplicial sets, by Lemma 1.8. Thus, by induction on k and Proposition 2.5, the functor dQ^k takes exact equivalences to weak equivalences of simplicial sets. It follows, in particular, that $dQ^kQ^1\mathbf{O}$ is contractible.

Let 0 be a specific choice of zero object for $\mathscr{A}$. Then 0 determines a diagram

$$\begin{tikzcd}
 & dQ^kF\mathscr{A} \arrow[rr] \arrow[dd] & & dQ^kL\mathscr{A} \arrow[dd, "="] \\
dQ^k\mathscr{A} \arrow[ur, "dQ^k\omega_0"] \arrow[rr] \arrow[dd] & & dQ^kL\mathscr{A} \arrow[ur] \arrow[dd] & \\
 & dQ^kQ^1\mathbf{O} \arrow[rr] & & dQ^kQ^1\mathscr{A} \\
* \arrow[ur] \arrow[rr] & & dQ^kQ^1\mathscr{A} \arrow[ur, "="'] &
\end{tikzcd}$$

where the simplicial exact functor $\omega_0: \mathscr{A} \to F\mathscr{A}$ is defined on n-simplices by sending the object A of $\mathscr{A}$ to the pair (E_A, B_A). B_A is defined as above, and for each $x \in \langle \mathbf{n}^{\mathrm{op}} * \mathbf{n} \rangle^*$, $E_A(x, \quad)$ is the exact sequence

$$0 \to A \xrightarrow{1} A \to 0 \to 0.$$

ω_0 is a simplicial exact equivalence by (a), and so it suffices to show that the sequence

$$(3.6) \qquad dQ^k\mathscr{A} \rightarrow dQ^kL\mathscr{A} \rightarrow dQ^kQ^1\mathscr{A}$$

is a homotopy fibre sequence.

In each of the sequences

$$(3.7) \qquad dQ^k\mathscr{A} \rightarrow dQ^k(L\mathscr{A}_n) \rightarrow dQ^k(Q^1\mathscr{A}_n),$$

the base and total space are connected; one sees this by induction on k, using the fact that $Q^1\mathscr{A}$ is connected for all $\mathscr{A}$. The bisimplicial sets of objects underlying (3.6) therefore satisfy the π_*-Kan condition of [**1**], and so it suffices to show that each sequence (3.7) is a homotopy fibre sequence. But this is just additivity; there is an exact equivalence of categories

$$L\mathscr{A}_n \rightarrow \mathscr{A} \times Q^1\mathscr{A}_n$$
$$(E, B) \mapsto (B_n, \gamma(E, B)),$$

which makes the sequence (3.7) equivalent to a projection (see also [**16**]).

COROLLARY 3.8. *There is a homotopy equivalence*

$$dQ^k\mathscr{A} \simeq \Omega dQ^{k+1}\mathscr{A}$$

for each $k \geqq 1$.

Proof. Since $L\mathscr{A}$ is defined by exact sequences, there is an isomorphism

$$Q^kL\mathscr{A} \cong LQ^k\mathscr{A}$$

of multisimplicial exact categories which is induced by the exponential law. Now use Lemma 3.3.

Henceforth, $Q^k\mathscr{A}$ will stand for the simplicial set $dQ^k\mathscr{A}$, as well as its underlying multisimplicial exact category and the corresponding multisimplicial set of objects.

Observe that the elements of $Q^2\mathscr{A}(1, 1)$ are Waldhausen's diagrams

$$\begin{tikzcd}
* & * \arrow[l, two heads] \arrow[r, tail] & * \\
* \arrow[u, tail] \arrow[d, two heads] & * \arrow[l, two heads] \arrow[r, tail] \arrow[u, dash] \arrow[d, dash] & * \arrow[u, tail] \arrow[d, two heads] \\
* & * \arrow[l, two heads] \arrow[r, tail] & *
\end{tikzcd}$$

We have shown that there is no need, in fact, to introduce an equivalence relation into this model to make it work homotopically. The symmetries in $Q^n\mathscr{A}$ are also apparent; the symmetric group Σ_n acts on the diagonal simplicial exact category $dQ^n\mathscr{A}$ by permuting variables.

At the risk of complicating the story by introducing an extra simplicial dimension, I shall now produce a model for $Q^k\mathscr{A}$ which is better for the applications which appear in subsequent sections of this paper. Let Iso $\mathscr{A}$ be the groupoid of isomorphisms of $\mathscr{A}$. Then the ordinary nerve B Iso $\mathscr{A}$ is the simplicial set of objects of a simplicial exact category having the same name. By the same argument as was used for Is $\mathscr{A}$, there is a simplicial exact equivalence $\mathscr{A} \to B$ Iso $\mathscr{A}$. The exponential law induces an isomorphism

(3.9) $\quad Q^kB \text{ Iso } \mathscr{A} \cong B \text{ Iso } Q^k\mathscr{A}$

of bisimplicial exact categories and of their associated bisimplicial sets of objects. It follows that the simplicial set dB Iso $Q^k\mathscr{A}$ is naturally weakly equivalent to $Q^k\mathscr{A}$. There are similar natural weak equivalences

$$dB \text{ Iso } Q^k \text{ Is } \mathscr{A} \cong dQ^k \text{ Is } B \text{ Iso } \mathscr{A} \leftarrow dQ^k \text{ Is } \mathscr{A},$$

$$dB \text{ Iso } Q^kF\mathscr{A} \cong dQ^kFB \text{ Iso } \mathscr{A} \leftarrow dQ^kF\mathscr{A},$$

$$dB \text{ Iso } Q^kL\mathscr{A} \cong dQ^kLB \text{ Iso } \mathscr{A} \leftarrow dQ^kL\mathscr{A},$$

and the analogue of Proposition 3.4 (b) holds for these models.

All constructions may now be pointed in a functorial way. In effect, define

$$Q^k_*\mathscr{A} := dB \text{ Iso } Q^k\mathscr{A}/dB \text{ Iso } Q^k\mathbf{O}.$$

The pointed simplicial sets $Q^kL_*\mathscr{A}$, $Q^kF_*\mathscr{A}$ and Q^k $\text{Is}_*\mathscr{A}$ are defined in a similar manner by collapsing, respectively, the contractible subcomplexes

$$dB \text{ Iso } Q^kL\mathbf{O}, \; dB \text{ Iso } Q^kF\mathbf{O} \quad \text{and} \quad dB \text{ Iso } Q^k \text{ Is}\mathbf{O}.$$

Observe that the sequence

(3.10) $\quad Q^kF_*\mathscr{A} \to Q^kL_*\mathscr{A} \to Q^{k-1}_*\mathscr{A}$

is now a fibre homotopy sequence, and that there are induced weak equivalences

(3.11) $\quad Q^k_*\mathscr{A} \to Q^k \text{ Is}_* \mathscr{A} \leftarrow Q^kF_*\mathscr{A}.$

There is also an induced action of Σ_k on $Q^k_*\mathscr{A}$.

LEMMA 3.12. *Σ_k acts on $Q^k_*\mathscr{A}$, in the homotopy category, by multiplication by sign.*

Proof. The commutative diagram

$$(3.13)\quad \begin{tikzcd} Q^1FF_*\mathscr{A} \arrow[r] \arrow[d] & Q^1LF_*\mathscr{A} \arrow[r] \arrow[d] & Q^1Q^1F_*\mathscr{A} \arrow[d] \\ Q^1FL_*\mathscr{A} \arrow[r] \arrow[d] & Q^1LL_*\mathscr{A} \arrow[r] \arrow[d] & Q^1Q^1L_*\mathscr{A} \arrow[d] \\ Q^1FQ^1_*\mathscr{A} \arrow[r] & Q^1LQ^1_*\mathscr{A} \arrow[r] & Q^1Q^1Q^1_*\mathscr{A} \end{tikzcd}$$

may be replaced, up to weak equivalence, by a 3×3 diagram such that all objects are fibrant, all rows and columns are fibration sequences, and all total spaces are contractible.

Any pointed fibration sequence

$$F \to E \xrightarrow{\pi} B$$

of fibrant objects with E contractible determines a canonical map $c:F \to \Omega B$ in the homotopy category. c is constructed by pulling back the path fibration for B along π. Let

$$\tau^*:\Omega^2 Q^1Q^1Q^1_*\mathscr{A} \to \Omega^2 Q^1Q^1Q^1_*\mathscr{A}$$

be the map which interchanges the loop factors. Then (3.13) induces a commutative diagram of the form

$$\begin{tikzcd} Q^1FF_*\mathscr{A} \arrow[r, "c"] \arrow[dr, "c"'] & \Omega Q^1Q^1F_*\mathscr{A} \arrow[r, "c"] & \Omega^2 Q^1Q^1Q^1_*\mathscr{A} \arrow[d, "\tau^*"] \\ & \Omega Q^1FQ^1_*\mathscr{A} \arrow[r, "c"] & \Omega^2 Q^1Q^1Q^1_*\mathscr{A} \end{tikzcd}$$

in the homotopy category.

Let

$$\tau_*:Q^1Q^1Q^1_*\mathscr{A} \to Q^1Q^1Q^1_*\mathscr{A}$$

be the map which interchanges the second two Q^1 factors. There is a corresponding map τ_* for each of the objects in the diagram (3.13), and τ_* is natural with respect to all maps in the diagram. Let

$$\xi:Q^1\mathscr{A} \to Q^1F_*\mathscr{A}$$

be the composite of the weak equivalences

$$Q^1_*\mathscr{A} \to Q^1_*\mathscr{A} \leftarrow Q^1F_*\mathscr{A}.$$

Then there is a commutative diagram

$$\begin{tikzcd}
Q^1_*\mathscr{A} \arrow[r, "f"] \arrow[dr, "f"'] & Q^1 FF_*\mathscr{A} \arrow[r, "c"] \arrow[d, "\tau_*"] & \Omega Q^1 FQ^1_*\mathscr{A} \arrow[r, "c"] \arrow[d, "\tau_*"] & \Omega^2 Q^1 Q^1 Q^1_*\mathscr{A} \arrow[d, "\tau_*"] \\
 & Q^1 FF_*\mathscr{A} \arrow[r, "c"] \arrow[dr, "c"'] & \Omega Q^1 Q^1 F_*\mathscr{A} \arrow[r, "c"] & \Omega^2 Q^1 Q^1 Q^1_*\mathscr{A} \arrow[d, "\tau^*"] \\
 & & \Omega Q^1 FQ^1_*\mathscr{A} \arrow[r, "c"] & \Omega^2 Q^1 Q^1 Q^1_*\mathscr{A}
\end{tikzcd}$$

in the homotopy category, where

$$f = \xi_F \circ \xi = F\xi \circ \xi.$$

It follows that the self map on $\Omega^2 Q^3_*\mathscr{A}$ which interchanges any two of the Q^1 factors is multiplication by -1. Σ_n therefore acts by multiplication by sign on $\Omega^2 Q^{n+2}_*\mathscr{A}$, and hence by multiplication by sign on $Q^n_*\mathscr{A}$.

4. Products. Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be exact categories. A *biexact functor* is a functor $\otimes:\mathscr{A} \times \mathscr{B} \to \mathscr{C}$ such that the restrictions $\otimes(x,\ \ )$ and $\otimes(\ \ ,y)$ are exact functors for each object of x of $\mathscr{A}$ and for each object y of $\mathscr{B}$. Each such $\otimes$ induces a canonical multi-simplicial biexact pairing

$$\otimes: Q^k\mathscr{A} \times Q^l\mathscr{B} \to Q^{k+l}\mathscr{C}.$$

In effect, the biexact pairing

$$\otimes: Q^k\mathscr{A}(n_1, \dots, n_k) \times Q^l\mathscr{B}(n_{k+1}, \dots, n_{k+l}) \\ \to Q^{k+l}\mathscr{C}(n_1, \dots, n_{k+l})$$

is defined by sending the pair

$$(\boldsymbol{\varphi}:\langle \mathbf{n}_1\rangle \times \dots \times \langle \mathbf{n}_k\rangle \to X,\ \boldsymbol{\varphi}:\langle \mathbf{n}_{k+1}\rangle \times \dots \times \langle \mathbf{n}_{k+l}\rangle \to \mathscr{A})$$

to the composite

$$\langle \mathbf{n}_1\rangle \times \dots \times \langle \mathbf{n}_k\rangle \times \langle \mathbf{n}_{k+1}\rangle \times \dots \times \langle \mathbf{n}_{k+l}\rangle \\ \xrightarrow{\boldsymbol{\varphi} \times \boldsymbol{\varphi}} \mathscr{A} \times \mathscr{B} \overset{\otimes}{\to} \mathscr{C}.$$

This composite is easily seen to be a multi-simplex of $Q^{k+l}\mathscr{A}$; use the explicit description given in Section 2.

The pairing $\otimes$ also induces the following diagrams of simplicial biexact pairings:

$$\begin{tikzcd}
\mathscr{A} \times F\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "1 \times i"] & F\mathscr{C} \arrow[d, "i"] \\
\mathscr{A} \times L\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "1 \times \gamma"] & L\mathscr{C} \arrow[d, "\gamma"] \\
\mathscr{A} \times Q^1\mathscr{B} \arrow[r, "\otimes"] & Q^1\mathscr{C}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{A} \times \mathscr{B} \arrow[r] \arrow[d, "\otimes"] & \mathscr{A} \times \mathrm{Is}\,\mathscr{B} \arrow[d, "\otimes"] & \mathscr{A} \times F\mathscr{B} \arrow[l] \arrow[d, "\otimes"] \\
\mathscr{C} \arrow[r] & \mathrm{Is}\,\mathscr{C} & F\mathscr{C} \arrow[l]
\end{tikzcd}$$

These diagrams induce diagrams of simplicial set maps

$$(4.1)\qquad \begin{tikzcd}
Q^k\mathscr{A} \times Q^lF\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "1 \times i"] & Q^{k+l}F\mathscr{C} \arrow[d, "i"] \\
Q^k\mathscr{A} \times Q^lL\mathscr{A} \arrow[r, "\otimes"] \arrow[d, "1 \times \gamma"] & Q^{k+l}L\mathscr{C} \arrow[d, "\gamma"] \\
Q^k\mathscr{A} \times Q^{l+1}\mathscr{A} \arrow[r, "\otimes"] & Q^{k+l+1}\mathscr{C}
\end{tikzcd}$$

$$(4.2)\qquad \begin{tikzcd}
Q^k\mathscr{A} \times Q^l\mathscr{B} \arrow[r] \arrow[d, "\otimes"] & Q^k\mathscr{A} \times Q^l\,\mathrm{Is}\,\mathscr{B} \arrow[d, "\otimes"] & Q^k\mathscr{A} \times Q^lF\mathscr{B} \arrow[l] \arrow[d, "\otimes"] \\
Q^{k+1}\mathscr{C} \arrow[r] & Q^{k+1}\,\mathrm{Is}\,\mathscr{C} & Q^{k+1}F\mathscr{C} \arrow[l]
\end{tikzcd}$$

Observe that any biexact pairing $\otimes:\mathscr{A} \times B \to \mathscr{C}$ induces a functor

$$\otimes:\mathrm{Iso}\,\mathscr{A} \times \mathrm{Iso}\,\mathscr{B} \to \mathrm{Iso}\,\mathscr{C},$$

and a simplicial biexact functor

$$\otimes:B\,\mathrm{Iso}\,\mathscr{A} \times B\,\mathrm{Iso}\,\mathscr{B} \to B\,\mathrm{Iso}\,\mathscr{C}.$$

It follows that there is an induced pairing of simplicial sets

$$\otimes:dB\,\mathrm{Iso}\,Q^k\mathscr{A} \times dB\,\mathrm{Iso}\,Q^l\mathscr{B} \to DB\,\mathrm{Iso}\,Q^{k+l}\mathscr{C}$$

which is weakly equivalent to the previous construction. Similar statements hold for the other pairings in (4.1) and (4.2).

The map

$$\otimes : dB \text{ Iso } Q^k\mathscr{A} \times dB \text{ Iso } Q^l\mathscr{B} \to dB \text{ Iso } Q^{k+l}\mathscr{C}$$

has the property that

$$\begin{aligned}\otimes(\,(dB \text{ Iso } Q^k\mathscr{A} \times dB \text{ Iso } Q^l\mathbf{O}) \cup \\ (dB \text{ Iso } Q^k\mathbf{O} \times dB \text{ Iso } Q^l\mathscr{B})\,) \\ \subset dB \text{ Iso } Q^{k+l}\mathbf{O},\end{aligned}$$

since the original functor $\otimes$ is biexact. It follows that $\otimes$ induces pointed simplicial set maps

$$\otimes : Q^k_*\mathscr{A} \wedge Q^l_*\mathscr{B} \to Q^{k+l}_*\mathscr{C}.$$

Furthermore, the diagrams (4.1) and (4.2) induce diagrams of pointed maps

$$(4.3)\quad \begin{tikzcd} Q^k_*\mathscr{A} \wedge Q^lF_*\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "1 \wedge i"] & Q^{k+l}F_*\mathscr{C} \arrow[d, "i"] \\ Q^k_*\mathscr{A} \wedge Q^lL_*\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "1 \wedge \gamma"] & Q^{k+l}L_*\mathscr{C} \arrow[d, "\gamma"] \\ Q^k_*\mathscr{A} \wedge Q^{l+1}_*\mathscr{B} \arrow[r] & Q^{k+l+1}_*\mathscr{C} \end{tikzcd}$$

$$(4.4)\quad \begin{tikzcd} Q^k_*\mathscr{A} \wedge Q^l_*\mathscr{B} \arrow[r] \arrow[d, "\otimes"] & Q^k_*\mathscr{A} \wedge Q^l \text{ Is}_* \mathscr{B} \arrow[d, "\otimes"] & Q^k_*\mathscr{A} \wedge Q^lF_*\mathscr{B} \arrow[l] \arrow[d, "\otimes"] \\ Q^{k+l}_*\mathscr{C} \arrow[r] & Q^{k+l} \text{ Is}_* \mathscr{C} & Q^{k+l}F_*\mathscr{C} \arrow[l] \end{tikzcd}$$

by a similar argument.

Let

$$\sigma : \Omega^n Q^k_*\mathscr{A} \to \Omega^{n+1}Q^{k+1}_*\mathscr{A}$$

be the map in the homotopy category which is defined to be the composite of weak equivalences

$$\Omega^n Q^k_*\mathscr{A} \to \Omega^n Q^k \text{ Is}_* \mathscr{A} \leftarrow \Omega^n Q^k F_*\mathscr{A} \xrightarrow{c} \Omega^{n+1}Q^{k+1}_*\mathscr{A},$$

where c is the canonical map associated to the fibre sequence

$$\Omega^n Q^k F_*\mathscr{A} \to \Omega^n Q^k L_*\mathscr{A} \to \Omega^n Q^{k+1}_*\mathscr{A}.$$

I define the i^{th} K-group $K_i(\mathscr{A})$ to be the filtered colimit of the following system of abelian group homomorphisms:

$$\dots \xrightarrow{\sigma_*} [*, \Omega^{k+i}Q_*^k\mathscr{A}] \xrightarrow{\sigma_*} [*, \Omega^{k+i+1}Q_*^{k+1}\mathscr{A}] \xrightarrow{\sigma_*} \dots$$

where the homotopy classes of maps are unpointed. This definition may be eccentric, but it generalizes nicely to simplicial sheaves.

Diagrams (4.3) and (4.4), together with Lemma 3.12, imply that the following diagram commutes in the homotopy category, for any biexact pairing $\otimes:\mathscr{A} \times \mathscr{B} \to \mathscr{C}$:

(4.5)
$$\begin{tikzcd}
\Omega^n Q_*^k\mathscr{A} \wedge \Omega^s Q_*^r\mathscr{B} \arrow[r, "\otimes"] \arrow[d, "\sigma \wedge \sigma"] & \Omega^{n+s}Q_*^{k+r}\mathscr{C} \arrow[d, "\sigma^2(-1)^{n+r+1}"] \\
\Omega^{n+1}Q_*^{k+1}\mathscr{A} \wedge \Omega^{s+1}Q_*^{r+1}\mathscr{B} \arrow[r, "\otimes"] & \Omega^{n+s+2}Q_*^{k+r+2}\mathscr{C}
\end{tikzcd}$$

It follows that $\otimes \circ (\sigma \wedge \sigma)^2 = \sigma^4 \circ \otimes$, which essentially proves

PROPOSITION 4.6. *Any biexact pairing $\otimes:\mathscr{A} \times \mathscr{B} \to \mathscr{C}$ induces a well-defined bilinear map*

$$\otimes:K_i(\mathscr{A}) \times K_j(\mathscr{B}) \to K_{i+j}(\mathscr{C}).$$

A *k-fold exact functor* is a functor

$$\otimes:\mathscr{A}_1 \times \dots \times \mathscr{A}_k \to \mathscr{B},$$

where $\mathscr{A}_1, \dots, \mathscr{A}_k$ and $\mathscr{B}$ are exact categories, such that for each $(k - 1)$-tuple $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k)$ of objects with x_i in $\mathscr{A}_i$, the composite functor

$$\mathscr{A}_i \to \mathscr{A}_1 \times \dots \times \mathscr{A}_k \xrightarrow{\otimes} \mathscr{B}$$

$$y \mapsto (x_1, \dots, x_{i-1}, y, x_{i+1}, \dots, x_k)$$

is exact. $\otimes$ induces a map of pointed simplicial sets

$$\otimes_*:Q_*^{l_1}\mathscr{A}_1 \wedge \dots \wedge Q_*^{l_k}\mathscr{A}_k \to Q_*^{\Sigma l_i}\mathscr{B}$$

by analogy with the two-fold or biexact case. Any natural isomorphism $\otimes \cong \tilde{\otimes}$ of k-fold exact functors induces a multisimplicial natural isomorphism of the induced functors

$$\otimes, \tilde{\otimes}:\text{Iso } Q^{l_1}\mathscr{A}_1 \times \dots \times \text{Iso } Q^{l_k}\mathscr{A}_k \to \text{Iso } Q^{\Sigma l_i}\mathscr{B},$$

and hence a pointed homotopy $\otimes_* \simeq \tilde{\otimes}_*$.

Let R be a commutative ring with unit. The category $\mathbf{P}(R)$ of finitely generated projective modules on R is everybody's favourite exact category. The higher K-groups $K_i(R)$, $i \geqq 0$, may be identified with the groups

$$K_i(\mathbf{P}(R)) \cong \pi_0(\Omega^{i+1}Q^1_*(\mathbf{P}(R))),$$

as defined above. The tensor product functor

$$\otimes : \mathbf{P}(R) \times \mathbf{P}(R) \to \mathbf{P}(R)$$

is the canonical example of a biexact functor. It is associative, symmetric and has two-sided unit, up to natural isomorphism.

It is now a simple matter to show

PROPOSITION 4.7. *The tensor product induces the structure of a graded commutative ring on* $K_*(R)$.

Proof. The associativity is a manipulation of 3-fold exact functors. The graded commutativity follows from Lemma 3.12. The existence of the multiplicative unit is trivial.

The H-bilinear pairing

$$(4.8) \quad \Omega^1 Q^1_* \mathbf{P}(R) \wedge \Omega^1 Q^1_* \mathbf{P}(R) \xrightarrow{\otimes} \Omega^2 Q^2_* \mathbf{P}(R)$$

induces the product in $K_*(R)$ of Proposition 4.7. There are other candidates, of course, for K-theory pairings. In particular, there is the Loday pairing

$$(4.9) \quad BGl(R)^+ \wedge BGl(R)^+ \xrightarrow{\otimes} BGl(R)^+.$$

It is defined [**8**, p. 332], up to weak homotopy, by the requirement that the following diagrams commute:

$$\begin{tikzcd} BGl_n(R) \times BGl_m(R) \arrow[r, "\hat{\otimes}"] \arrow[d, "i_n \wedge i_m"'] & BGl(R)^+ \\ BGl(R)^+ \wedge BGl(R)^+ \arrow[ur, "\otimes"'] & \end{tikzcd}$$

Here,

$$\hat{\otimes}(A, B) = i_{nm}(A \otimes B) - i_{nm}(A \otimes I_m) - i_{nm}(I_n \otimes B)$$

in terms of the H-group structure of $BGl(R)^+$, and

$$i_r : BGl_r(R) \to BGl(R)^+$$

is the canonical map.

The pairings (4.8) and (4.9) induce the same pairing in homotopy groups, in view of

PROPOSITION 4.10. *The Loday pairing*

$$\otimes : BGl(R)^+ \wedge BGl(R)^+ \to BGl(R)^+$$

is the restriction to 0-*components of the pairing*

$$\Omega^1 Q^1_* \mathbf{P}(R) \wedge \Omega^1 Q^1_* \mathbf{P}(R) \rightarrow \Omega^2 Q^2_* \mathbf{P}(R),$$

up to weak homotopy.

Proof. For notational convenience, let $\mathbf{P}$ denote the category $\mathbf{P}(R)$.
The sequence

$$F\mathbf{P} \rightarrow L\mathbf{P} \rightarrow Q^1\mathbf{P}$$

induces a map

$$\gamma: F_*\mathbf{P} \rightarrow \Omega Q^1_*\mathbf{P},$$

since $L_*\mathbf{P}$ is contractible by Lemma 3.3. The simplicial equivalence of categories

$$\mathrm{can}: F\mathbf{P} \rightarrow B \text{ Iso } \mathbf{P}$$

defined by

$$(E, B) \mapsto (B_n \leftarrow B_{n-1} \leftarrow \ldots \leftarrow B_0)$$

induces a homotopy equivalence

$$\mathrm{can}: F_*\mathbf{P} \rightarrow B \text{ Iso}_* \mathbf{P}.$$

Let

$$c: B \text{ Iso}_* \mathbf{P} \rightarrow \Omega Q^1_*\mathbf{P}$$

be the composite $\gamma \circ \mathrm{can}^{-1}$ in the homotopy category.
The diagrams of pairings

$$\begin{tikzcd}
Q^1\mathbf{P} \times F\mathbf{P} \arrow[r, "\otimes"] \arrow[d] & Q^1F\mathbf{P} \arrow[d] \\
Q^1\mathbf{P} \times L\mathbf{P} \arrow[r, "\otimes"] \arrow[d] & Q^1L\mathbf{P} \arrow[d] \\
Q^1\mathbf{P} \times Q^1\mathbf{P} \arrow[r, "\otimes"] & Q^2\mathbf{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
F\mathbf{P} \times F\mathbf{P} \arrow[r, "\otimes"] \arrow[d] & FF\mathbf{P} \arrow[d] \\
L\mathbf{P} \times F\mathbf{P} \arrow[r, "\otimes"] \arrow[d] & LF\mathbf{P} \arrow[d] \\
Q^1\mathbf{P} \times F\mathbf{P} \arrow[r, "\otimes"] & Q^1F\mathbf{P}
\end{tikzcd}$$

induce a diagram of the form

$$\begin{tikzcd}
B \text{ Iso}_* \mathbf{P} \wedge B \text{ Iso}_* \mathbf{P} \arrow[r, "\otimes"] \arrow[d, "c \wedge c"] & B \text{ Iso}_* \mathbf{P} \arrow[r, "c"] & \Omega Q^1_*\mathbf{P} \arrow[d, "\Omega c"] \\
\Omega Q^1_*\mathbf{P} \wedge \Omega Q^1_*\mathbf{P} \arrow[rr, "\otimes"] & & \Omega^2 Q^2_*\mathbf{P}
\end{tikzcd}$$

in the homotopy category.

https://doi.org/10.4153/CJM-1987-060-0 Published online by Cambridge University Press

Recall [**4**] that there is a category $\tilde{L}\mathbf{P}$ whose objects are the exact sequences of $\mathbf{P}$, and whose morphisms are the commutative diagrams of the form

$$\begin{tikzcd}
0 \arrow[r] & E_1' \arrow[r, tail] & E_1 \arrow[r, two heads] \arrow[d, tail] & E_1'' \arrow[r] & 0 \\
0 \arrow[r] & E_2' \arrow[r, tail] \arrow[u, tail] & E_2 \arrow[r, two heads] & E_2'' \arrow[r] & 0
\end{tikzcd}$$

There is a functor $\tilde{L}\mathbf{P} \rightarrow Q\mathbf{P}$ which sends this map to the map of $Q\mathbf{P}$ which is represented by

$$E_1'' \twoheadleftarrow E_1''/E_2' \rightarrowtail E_2''.$$

On the other hand, there is a simplicial map

$$\pi_L : L\mathbf{P} \rightarrow B\tilde{L}\mathbf{P}$$

which takes an n-simplex (E, B) to the simplex

$$\begin{tikzcd}
B0^0 \arrow[d, tail] & B1^0 \arrow[l, two heads] \arrow[d, tail] & \cdots \arrow[l, two heads] & Bn^0 \arrow[l, two heads] \arrow[d, tail] \\
B0 \arrow[r, tail] \arrow[d] & B1 \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & Bn \arrow[d] \\
E((0^0, 0), 2) & E((1^0, 1), 2) & & E((n^0, n), 2)
\end{tikzcd}$$

of $B\tilde{L}\mathbf{P}$. The diagram of simplicial set maps

$$\begin{tikzcd}
L\mathbf{P} \arrow[r, "\pi_L"] \arrow[d] & \tilde{L}\mathbf{P} \arrow[d] \\
Q^1\mathbf{P} \arrow[r, "\pi"] & BQ\mathbf{P}
\end{tikzcd}$$

commutes, where π is the weak equivalence of Proposition 1.4.

Let 0 be a specific choice of zero object in $\mathbf{P}$. Then 0 determines a commutative diagram of simplicial sets

$$\begin{tikzcd}
B \operatorname{Iso} \mathbf{P} \arrow[r, "j_0"'] \arrow[d, "i_0"'] \arrow[rr, bend left, "1"] & F\mathbf{P} \arrow[r, "\text{can}"'] \arrow[d, tail] & B \operatorname{Iso} \mathbf{P} \\
B\tilde{L}\mathbf{P} & L\mathbf{P} \arrow[l] &
\end{tikzcd}$$

where

$$j_0\left(B_0 \overset{\theta_1}{\underset{\cong}{\to}} B_1 \overset{\theta_2}{\underset{\cong}{\to}} \dots \overset{\theta_n}{\underset{\cong}{\to}} B_n\right)$$

is the n-simplex of $L\mathbf{P}$ which is determined by the diagram of isomorphisms

$$\begin{tikzcd}
B_n \arrow[r, "\theta_n^{-1}"] \arrow[d, "1"] & B_{n-1} \arrow[r, "\theta_{n-1}^{-1}"] \arrow[d, "1"] & \cdots \arrow[r, "\theta_1^{-1}"] & B_0 \arrow[d, "1"] \\
B_n & B_{n-1} \arrow[l, "\theta_n"'] & \cdots \arrow[l, "\theta_{n-1}"'] & B_0 \arrow[l, "\theta_1"']
\end{tikzcd}$$

and the assignment of 0 for all possible cokernels. Recall that, if S denotes Iso $\mathbf{P}$, then localizing at the S-action as in [**4**] gives a commutative diagram

$$\begin{tikzcd}
B \operatorname{Iso} \mathbf{P} \arrow[r, "\tau_0"] \arrow[d, "i_0"] & BS^{-1}S \arrow[d, "S^{-1}i_0"] \\
B\tilde{L}\mathbf{P} \arrow[r, "\tau_0"] \arrow[dr] & BS^{-1}\tilde{L}\mathbf{P} \arrow[d] \\
 & BQ\mathbf{P}
\end{tikzcd}$$

where the sequence on the right is the homotopy fibre sequence of the $Q = +$ Theorem, and τ_0 is the canonical map. It follows that there is a homotopy commutative diagram

$$\begin{tikzcd}
B \operatorname{Iso}_* \mathbf{P} \arrow[d] & B \operatorname{Iso} \mathbf{P} \arrow[l] \arrow[r, "\tau_0"] & BS^{-1}S \arrow[d, "\simeq"] \\
\Omega Q^1_*\mathbf{P} & \Omega_0 Q^1\mathbf{P} \arrow[l, "\simeq"'] \arrow[r, "\simeq"'] & \Omega_0 BQ\mathbf{P}
\end{tikzcd}$$

One can show that the composite

$$\omega_0 : BS^{-1}S \to \Omega Q^1_*\mathbf{P}$$

of the indicated string of homotopy equivalences in the homotopy category is an H-map.

We therefore have a homotopy commutative diagram

$$\begin{tikzcd}
B\,\mathrm{Iso}\,\mathbf{P} \times B\,\mathrm{Iso}\,\mathbf{P} \arrow[rr, "\otimes"] \arrow[d, "\tau_0 \times \tau_0"] & & B\,\mathrm{Iso}\,\mathbf{P} \arrow[d, "\tau_0"] \\
BS^{-1}S \times BS^{-1}S \arrow[d] & & BS^{-1}S \arrow[d, "\omega_0", "\simeq"'] \\
BS^{-1}S \wedge BS^{-1}S \arrow[urr, "\tilde{\otimes}"] \arrow[d, "\omega_0 \wedge \omega_0"] & & \Omega^1 Q^1_* \mathbf{P} \arrow[d, "\Omega c", "\simeq"'] \\
\Omega Q^1_* \mathbf{P} \wedge \Omega Q^1_* \mathbf{P} \arrow[rr, "\otimes"'] & & \Omega^2 Q^2_* \mathbf{P}
\end{tikzcd}$$

in which the induced map $\tilde{\otimes}$ is H-bilinear.

The proof of the $Q = +$ Theorem implies that there is a homotopy equivalence

$$BS^{-1}S_0 \simeq BG1(R)^+$$

of the 0-component of $BS^{-1}S$ with $BG1(R)^+$, where the composite j_n, defined by

$$BG1_n(R) \xrightarrow{i_n} BGL(R)^+ \simeq BS^{-1}S_0 \subset BS^{-1}S,$$

coincides with the composite

$$BG1_n(R) \subset B\,\mathrm{Iso}\,\mathbf{P} \xrightarrow{\tau_0} BS^{-1}S \xrightarrow[x \mapsto x - [I_n]]{} BS^{-1}S$$

up to homotopy. But then the bilinearity of $\tilde{\otimes}$ implies that

$$j_n \tilde{\otimes} j_m(x, y) = j_{nm}(x \otimes y) - j_{nm}(I_n \otimes y) - j_{nm}(x \otimes I_m)$$

up to homotopy. It follows that Loday's pairing coincides with the map

$$BS^{-1}S_0 \wedge BS^{-1}S_0 \xrightarrow{\tilde{\otimes}} BS^{-1}S_0$$

on 0-components, up to weak homotopy.

5. Naturality. Diagrams of exact functors almost never commute in practice. Let $\mathbf{P}(X)$ denote the exact category of vector bundles on a scheme X, and let the functor $D:I \to \mathbf{Sch}$ be a diagram in the scheme category. Suppose that

$$i \xrightarrow{\alpha} j \xrightarrow{\beta} k$$

is a composeable pair of morphisms in the index category I. Then α and β induce exact inverse image functors

$$\mathbf{P}(D(k)) \xrightarrow{\beta^*} \mathbf{P}(D(j)) \xrightarrow{\alpha^*} \mathbf{P}(D(i)), \quad \text{and}$$

$$\mathbf{P}(D(k)) \xrightarrow{(\beta\alpha)^*} \mathbf{P}(D(i)).$$

In general, $(\beta\alpha)^*$ is not equal to the composite $\alpha^* \circ \beta^*$; the two functors are only naturally isomorphic. It is a strong type of natural isomorphism, however, which is encoded in the following definition.

Let **Ex** denote the category of exact categories and exact functors. An *exact pseudo-functor* $M:C - \to \mathbf{Ex}$ associates to each object c of C an exact category $M(c)$, and to each morphism $\alpha:c \to d$ of C an exact functor

$$M(\alpha):M(c) \to M(d).$$

The data for M also includes natural isomorphisms

$$\theta(\beta, \alpha):M(\beta\alpha) \overset{\cong}{\Rightarrow} M(\beta)M(\alpha)$$

for each composeable pair

$$c \xrightarrow{\alpha} d \xrightarrow{\beta} e$$

in C, and

$$\eta_c:M(1_c) \overset{\cong}{\Rightarrow} 1_{M(c)}$$

for each object c of C, such that the following diagrams of natural isomorphisms commute:

$$(5.1)\qquad \begin{tikzcd} M(\gamma\beta\alpha) \arrow[r, "\theta(\gamma\beta,\, \alpha)"] \arrow[d, "\theta(\gamma,\, \beta\alpha)"] & M(\gamma\beta,\, \alpha) \arrow[d, "\theta(\gamma,\, \beta)M(\alpha)"] \\ M(\gamma)M(\beta\alpha) \arrow[r, "M(\alpha)\theta(\beta,\, \alpha)"'] & M(\gamma)M(\beta)M(\alpha) \end{tikzcd}$$

$$(5.2)\qquad \begin{tikzcd} M(1_d\alpha) \arrow[r, "\theta(1_d,\, \alpha)"] \arrow[d, equal] & M(1_d)M(\alpha) \arrow[d, "\eta_d M(\alpha)"] \\ M(\alpha) \arrow[r, phantom, "="] & 1_{M(d)}M(\alpha) \end{tikzcd}$$

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(5.3)
$$\begin{tikzcd} M(\alpha 1_c) \arrow[r, "\theta(\alpha,\, 1_c)"] \arrow[d, equal] & M(\alpha)M(1_c) \arrow[d, "M(\alpha)\eta_c"] \\ M(\alpha) \arrow[r, phantom, "="] & M(\alpha)1_{M(c)} \end{tikzcd}$$

In other words, M is a sort of lax functor in the sense of [**9**], the distinction being the exactness and isomorphism requirements. The prefix "pseudo" follows the convention of [**13**].

Let $D:I \to \mathbf{Sch}$ be the diagram of schemes referred to above. Then D induces a diagram $D_*:I \to \mathbf{Add}$ of additive categories, where $D_*(i)$ is the category of $\mathbf{O}_{D(i)}$-modules, and $D(\alpha)_*$ is the direct image functor for $\alpha:i \to j$ in I. $D(\alpha)_*$ has a left adjoint $D(\alpha)^*$, which restricts to the inverse image functor

$$D(\alpha)^*:\mathbf{P}(j) \to \mathbf{P}(i)$$

on vector bundles. There are unique natural isomorphisms

$$\theta(\alpha,\beta):D(\beta\alpha)^* \overset{\cong}{\Rightarrow} D(\alpha)^*D(\beta)^*, \quad i \xrightarrow{\alpha} j \xrightarrow{\beta} k,$$
$$\eta_i:D(1_i)^* \overset{\cong}{\Rightarrow} 1_{\mathbf{P}D(i)}, \quad i \in I,$$

which reflect the fact that any two left adjoints of a given functor are naturally isomorphic via a uniquely determined isomorphism. One checks that the diagrams corresponding to (5.1), (5.2) and (5.3) commute in the same way. Thus, the data is specified for an exact contravariant pseudo-functor

$$\mathbf{P}D:I^{\mathrm{op}} \dashrightarrow \mathbf{Ex}$$

on the category I.

Of particular interest is the exact pseudo-functor

$$\mathbf{P}\pi_S:(\mathrm{et}|_S)^{\mathrm{op}} \dashrightarrow \mathbf{Ex}$$

that corresponds to the forgetful map

$$\pi_S:\mathrm{et}|_S \to \mathbf{Sch}$$

on the étale site $\mathrm{et}|_S$ of a scheme S. π_S is defined by

$$\pi_S(U \to S) = U$$

for étale maps $U \to S$. $\mathbf{P}$ restricts to an exact pseudo-functor on arbitrary scheme-theoretic Grothendieck sites in the same way.

A *pseudo-natural transformation*

$$(\omega, d):M' \to M$$

between exact pseudo-functors M' and M defined on the category C

consists of exact functors

$$\omega_c : M'(c) \to M(c),$$

one for each object c of C, and natural isomorphisms

$$d(\alpha): M(\alpha)\omega_c \overset{\cong}{\Rightarrow} \omega_d M'(\alpha)$$

for each morphism $\alpha: c \to d$ of C, such that the following diagrams of natural isomorphisms commute:

(5.4)
$$\begin{tikzcd} M(1_c)\omega_c \arrow[rr, "d(1_c)"] \arrow[dr, "\eta_c\omega_c"'] & & \omega_c M'(1_c) \arrow[dl, "\omega_c\eta_c"] \\ & \omega_c & \end{tikzcd}$$

(5.5)
$$\begin{tikzcd} M(\beta\alpha)\omega_c \arrow[rr, "d(\beta\alpha)"] \arrow[d, "\theta(\beta, \alpha)\omega_c"] & & \omega_e M'(\beta\alpha) \arrow[d, "\omega_e\theta(\beta\alpha)"'] \\ M(\beta)M(\alpha)\omega_c \arrow[r, "M(\beta)d(\alpha)"'] & M(\beta)\omega_d M'(\alpha) \arrow[r, "d(\beta)M'(\alpha)"'] & \omega_e M'(\beta)M(\alpha) \end{tikzcd}$$

Here,

$$c \xrightarrow{\alpha} d \xrightarrow{\beta} e$$

is a composeable pair of morphisms in C. Pseudo-natural transformations may be composed in the usual way [**9**]. If

$$(\xi, d'): M \to M''$$

is a second such object, then the composite

$$(\xi\omega, d' \circ d): M' \to M''$$

is defined by

$$(\xi\omega)_c = \xi_c\omega_c, \quad \text{and}$$
$$(d' \circ d)(\alpha) = (\xi_e d(\alpha))(d'(\alpha)\omega_c) \quad \text{for } \alpha: c \to e.$$

A category of exact pseudo-functors and pseudo-natural transformations is therefore defined, relative to the base category C.

Each of the functors Q_n^k, $Q^k L_n$, $Q^k F_n$ and Q^k Is_n is defined at an exact category $\mathscr{A}$ by specifying a type of functor from some finite category to $\mathscr{A}$. It follows that an exact pseudo-functor $M: C - \to \mathbf{Ex}$ has associated to it exact pseudo-functors $Q^k M_n$, $Q^k LM_n$, $Q^k FM_n$ and Q^k Is M_n. In effect, if $F: C - \to \mathbf{cat}$ is a pseudo-functor in the sense of [**13**], then so is each object F^I, where $F^I(c)$ is the category of functors from the index category I to $F(c)$. Here, **cat** is the category of small categories. Furthermore, any functor $J \to I$ induces a pseudo-natural transformation $F^I \to F^J$ in the obvious way. This proves

PROPOSITION 5.6. *Let $M:C - \to \mathbf{Ex}$ be an exact pseudo-functor. Then there are canonical simplicial exact pseudo-functors Q^kM, Q^kLM, Q^kFM, and Q^k Is M, and morphisms*

$$Q^kFM \to Q^kLM \to Q^{k+1}M$$

$$Q^kM \to Q^k \text{ Is } M \leftarrow Q^kFM$$

of such objects. The indicated morphisms are natural in M.

Any exact pseudo-functor $M:C - \to \mathbf{Ex}$ has associated to it a lax functor $\mathrm{Iso}(M):C - \to \mathbf{cat}$. Iso $M(c)$ is the groupoid of isomorphisms in the exact category $M(c)$. $M \mapsto \mathrm{Iso}(M)$ is functorial; any pseudo-natural transformation

$$(\omega, d):M \to N$$

restricts to a lax natural transformation

$$(\omega, d):\mathrm{Iso}(M) \to \mathrm{Iso}(N).$$

Suppose that $K:C - \to \mathbf{cat}$ is a lax functor on C. Then K can be rectified in the standard way. Explicitly, K may be used to construct a category $[K]$ and a functor $\pi_K:[K] \to C$. The objects of $[K]$ consist of pairs (c, A), where $c \in C$ and A is an object of $K(c)$. A morphism

$$(\alpha, f):(c, A) \to (d, B)$$

of $[K]$ consists of a morphism

$$\alpha:c \to d$$

of C and a map

$$f:K(\alpha)(A) \to B$$

of $K(d)$. The composite of the morphisms

$$(c, A) \xrightarrow{(\alpha, f)} (d, B) \xrightarrow{(\beta, g)} (e, C)$$

is the map

$$(\beta\alpha, g^*f):(c, A) \to (e, C),$$

where g^*f is defined to be the composite

$$K(\beta\alpha)A \xrightarrow{\theta(\beta, \alpha)} K(\beta)K(\alpha)A \xrightarrow{K(\beta)f} K(\beta)B \xrightarrow{g} C$$

in $K(e)$. The functor

$$\pi_K:[K] \to C$$

is projection onto the first factor in both objects and arrows.

Write $[K](c)$ for the comma category $\pi_K \downarrow c$. There are standard functors

$$F_c:[K](c) \to K(c),$$

$$G_c:K(c) \to [K](c),$$

where

$$F_c(\alpha:\pi_K(d, B) \to c) = K(\alpha)B,$$

and $G_c(A)$ is the map

$$1_c:\pi_K(c, A) \to c.$$

F_c is left adjoint to G_c, and so G_c induces a homotopy equivalence

$$BG_c:BK(c) \to B[K](c)$$

of classifying spaces. If $\alpha:c \to d$ is a map of C, then the diagram of functors

$$\begin{tikzcd} {[K](c)} \arrow[r, "\alpha_*"] & {[K](d)} \\ K(c) \arrow[u, "G_c"] \arrow[r, "K(\alpha)"'] & K(d) \arrow[u, "G_d"'] \end{tikzcd}$$

commutes up to a natural transformation

$$\omega_\alpha:\alpha_* G_c \to G_d K(\alpha).$$

ω_α has component at $B \in K(c)$ given by the diagram

$$\begin{tikzcd} \pi_K(c, A) \arrow[rr, "{\pi_K(\alpha, 1_{K(\alpha)A})}"] \arrow[dr, "\pi_K"'] & & \pi_K(d, K(\alpha)A) \arrow[dl, "\pi_L"] \\ & d & \end{tikzcd}$$

All of these constructions are functorial in lax natural transformations. Any lax natural transformation

$$(\omega, d):K \to L$$

induces a functor

$$\begin{tikzcd}[K] \arrow[rr, "\omega_*"] \arrow[dr, "\pi_K"'] & & [L] \arrow[dl, "\pi_L"] \\ & C & \end{tikzcd}$$

of categories fibred over C. ω_* sends

$$(\alpha, f):(c, A) \to (d, B)$$

to the morphism

$$(\alpha, f_*):(c, \omega_c A) \to (d, \omega_d B),$$

where f_* is the composite

$$L(\alpha)\omega_c A \xrightarrow{d(\alpha)} \omega_d K(\alpha) A \xrightarrow{\omega_d f} \omega_d B.$$

Note in particular that the induced diagram of functors

$$\begin{tikzcd}[K](c) \arrow[r, "\omega_*"] & [L](c) \\ K(c) \arrow[u, "G_c"] \arrow[r, "\omega_c"'] & L(c) \arrow[u, "G_c"'] \end{tikzcd}$$

commutes on the nose.

Feeding the data of Proposition 5.6 through this machine now gives (multi) simplicial set-valued functors $[Q^k M]$, $[Q^k LM]$, $[Q^k FM]$ and $[Q^k \text{ Is } M]$ for each exact pseudo-category M, where, for example,

$$[Q^k M](c) = B[\text{Iso } Q^k M](c) \quad \text{for } c \in C.$$

We also get natural transformations of the form

(5.7) $\quad [Q^k FM] \to [Q^k LM] \to [Q^{k+1} M]$, and

(5.8) $\quad [Q^k M] \to [Q^k \text{ Is } M] \leftarrow [Q^k FM]$.

Observe that the maps of (5.8) are pointwise weak equivalences.

Let $\mathbf{O}_c$ be the full subcategory of 0-objects of $M(c)$ for each object c of C. Then the exact pseudo-functor structure of M restricts to the $\mathbf{O}_c$, giving an exact pseudo-functor $\mathbf{O}$, and a pseudo-natural transformation $\mathbf{O} \rightarrowtail M$. This transformation induces pointwise monomorphisms of the form

(5.9) $\quad [Q^k ? \mathbf{O}] \rightarrowtail [Q^k ? M]$

for each of the functors appearing in (5.7) and (5.8). The natural transformations (5.9) also commute with the maps appearing in (5.7) and (5.8). Define, for example, the functor $[Q^k M]_*$ on C with values in the category of pointed simplicial sets by

$$[Q^kM]_* = [Q^kM]/[Q^k\mathbf{O}].$$

Then putting this all together gives

PROPOSITION 5.10. *Let $M:C \dashrightarrow \mathrm{Ex}$ be an exact pseudo-functor. Then there are functors and natural transformations*

(5.11) $[Q^kFM]_* \rightarrow [Q^kLM]_* \rightarrow [Q^{k+1}M]_*$

(5.12) $[Q^kM]_* \rightarrow [Q^k \text{ Is } M]_* \leftarrow [Q^kFM]_*$,

defined on C and taking values in the category of pointed simplicial sets, such that

(a) (5.11) *is a pointwise fibre homotopy sequence, and* $[Q^kLM]_*$ *is pointwise contractible, and*

(b) *the maps in* (5.12) *are pointwise weak equivalences. Furthermore if $\alpha:c \rightarrow d$ is a morphism of C, then there is a pointed homotopy commutative diagram*

$$\begin{tikzcd}
{[Q^kM]_*(c)} \arrow[r, "\alpha_*"] & {[Q^kM]_*(d)} \\
Q^k_*M(c) \arrow[r, "\alpha_*"] \arrow[u] & Q^k_*M(d) \arrow[u]
\end{tikzcd},$$

in which the vertical maps are weak equivalences. There are similar diagrams for the other functors appearing in (5.11) *and* (5.12).

Specializing the above to vector bundles on schemes étale over S gives a presheaf of infinite loop spaces $[Q^1\mathbf{P}]_*$ on $\mathrm{et}|_S$. One can show that the tensor product induces smash product pairings

$$[Q^k\mathbf{P}]_* \wedge [Q^l\mathbf{P}]_* \rightarrow [Q^{k+l}\mathbf{P}]_*$$

in the homotopy category of pointed simplicial presheaves over $\mathrm{et}|_S$. These pairings fit together naively, in the style of the diagrams (4.3) and (4.4).

REFERENCES

1. A. K. Bousfield and E. M. Friedlander, *Homotopy theory of Γ-spaces, spectra and bisimplicial sets*, Springer Lecture Notes in Math. *658* (1978), 80-150.
2. P. Gabriel and M. Zisman, *Calculus of fractions and homotopy theory* (Springer-Verlag, New York, 1967).
3. H. Gillet, *Riemann-Roch theorems for higher algebraic K-theory*, Advances in Math. *40* (1981), 203-289.
4. D. Grayson, *Higher algebraic K-theory II*, Springer Lecture Notes in Math. *551* (1976), 217-240.
5. ——— *Products in K-theory and intersecting algebraic cycles*, Inv. Math. *47* (1978), 71-83.

6. J. F. Jardine, *Simplicial presheaves*, to appear in J. Pure Applied Algebra.

7. ——— *Stable homotopy theory of simplicial presheaves*, to appear in Can. J. Math.

8. J.-L. Loday, *K-théorie algébrique et représentations de groupes*, Ann. scient. Éc. Norm. Sup., 4^e série, t. *9* (1976), 309-377.

9. J. P. May, *Pairings of categories and spectra*, J. Pure Applied Algebra *19* (1980), 299-346.

10. D. Quillen, *Homotopical algebra*, Springer Lecture Notes in Math. *43* (1967).

11. ——— *Higher algebraic K-theory I*, Springer Lecture Notes in Math. *341* (1973), 85-147.

12. G. Segal, *Categories and cohomology theories*, Topology *13* (1974), 293-312.

13. A. Grothendieck et al., *Revêtements étales et groupe fondamental* (*SGA* 1), Springer Lecture Notes in Math. *224* (1971).

14. K. Shimakawa, *Multiple categories and algebraic K-theory*, to appear in J. Pure Applied Algebra.

15. R. Thomason, *Algebraic K-theory and étale cohomology*, Ann. Scient. Éc. Norm. Sup., 4^e série *18* (1985), 437-552.

16. F. Waldhausen, *Algebraic K-theory of generalized free products*, Part I, Ann. Math. *108* (1978), 135-204.

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