# PARTITION OF LARGE SUBSETS OF SEMIGROUPS

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**Abstract.** It is known that in an infinite very weakly cancellative semigroup with size $\kappa$, any central set can be partitioned into $\kappa$ central sets. Furthermore, if $\kappa$ contains $\lambda$ almost disjoint sets, then any central set contains $\lambda$ almost disjoint central sets. Similar results hold for thick sets, very thick sets and piecewise syndetic sets. In this article, we investigate three other notions of largeness: quasi-central sets, C-sets, and J-sets. We obtain that the statement applies for quasi-central sets. If the semigroup is commutative, then the statement holds for C-sets. Moreover, if $\kappa^\omega = \kappa$, then the statement holds for J-sets.

**§1. Introduction.** As we know that there are many combinatorial notions of largeness in a semigroup $S$ that are studied extensively. In [1], Carlson et al. investigated partition problems of some notions of largeness. Based on the fact that any central set in $(\mathbb{N}, +)$ can be partitioned into infinitely many pairwise disjoint central sets, they extended this result to a large class of semigroups. To be precise, they showed that if $S$ is a very weakly cancellative semigroup with size $\kappa$, $\kappa \geq \omega$, then every central set in $S$ contains $\kappa$ disjoint central sets. Furthermore, if $\kappa$ contains $\lambda$ almost disjoint sets, then every central set in $S$ contains $\lambda$ almost disjoint central sets. They also observed that the same statement holds for thick sets. And if the semigroup is left cancellative, the conclusion applies for piecewise syndetic sets; if the size $\kappa$ of the semigroup is regular, then the conclusion holds for very thick sets. They also considered syndetic sets, but this situation is more complicated (see [1] for more details).

Besides these notions of largeness, there are many other notions whose properties of partition are not known yet. So in this article, we will focus on this question and investigate three notions of largeness: quasi-central sets, C-sets, and J-sets. And we obtain that the same conclusion holds for quasi-central sets (Corollary 2.4) as for central sets. If the semigroup is also commutative, then the conclusion holds for C-sets (Corollary 3.3). Moreover, if the size $\kappa$ of the semigroup satisfies $\kappa^\omega = \kappa$, then the conclusion holds for J-sets (Theorem 3.4).

Now let us introduce some notions, notations and basic facts that we will refer to. Most of this information can be found in [3]. Given a discrete semigroup $(S, \cdot)$, $\beta S$ is the Stone–Čech compactification of $S$ and there is a natural extension of $\cdot$ to $\beta S$ making $\beta S$ a compact right topological semigroup. For each $p \in \beta S$, the function $\rho_p : \beta \to \beta S$, defined by $\rho_p(q) = q \cdot p$, is continuous, and for each $x \in S$, $\lambda_x : \beta \to \beta S$, defined by $\lambda_x(p) = x \cdot p$, is also continuous. The topological basis

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of $\beta S$ is $\{U_A : \emptyset \neq A \subseteq S\}$, where $U_A = \{p \in \beta S : A \in p\}$. Given a compact right topological semigroup $S$, it has a smallest ideal $K(S)$, which is the union of all minimal left ideals of $S$ and also the union of all minimal right ideals of $S$. An idempotent in $K(S)$ is called minimal.

Let $(S, +)$ be a semigroup, $\langle x_n \rangle_{n=1}^{\infty}$ be a sequence in $S$, write $\mathrm{FS}(\langle x_n \rangle_{n=1}^{\infty}) = \{\sum_{n \in H} x_n : H \in \mathcal{P}_f(\mathbb{N})\}$, where $\mathcal{P}_f(\mathbb{N})$ is the set of nonempty finite subsets of $\mathbb{N}$ and $\sum_{n \in H} x_n$ is the sum in increasing order of indices. Given a subset $A$ of $S$, we call $A$ a quasi-central set [2] if there is an idempotent $p \in \beta S$ such that $A \in p \in clK(\beta S)$, where $clX$ is the topological closure of $X$. To introduce J-sets, we denote $S_L(a, H) = \{a + \sum_{t \in H} f(t) : f \in L\}$, where $a \in S$, $H \in \mathcal{P}_f(\mathbb{N})$ and $L \in \mathcal{P}_f({}^{\mathbb{N}}S)$. Then given a commutative semigroup $(S, +)$, $A \subseteq S$ is a J-set in $S$ if for any $L \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $S_L(a, H) \subseteq A$. $A \subseteq S$ is a C-set if there exist functions $\alpha : \mathcal{P}_f({}^{\mathbb{N}}S) \to S$ and $H : \mathcal{P}_f({}^{\mathbb{N}}S) \to \mathcal{P}_f(\mathbb{N})$ such that $\max H(F) < \min H(G)$ for every $F, G \in \mathcal{P}_f({}^{\mathbb{N}}S)$ satisfying $F \subsetneq G$, and $\sum_{i=1}^{m} (\alpha(G_i) + \sum_{t \in H(G_i)} f_i(t)) \in A$ whenever $m \in \mathbb{N}$, $G_1, \ldots, G_m \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $G_1 \subsetneq \cdots \subsetneq G_m$ and $f_i \in G_i$ for each $i \in \{1, \ldots, m\}$. $J(S) = \{p \in \beta S : \forall A \in p(A \text{ is a J-set in } S)\}$. $J(S)$ is a compact ideal of $\beta S$, and $A$ is a C-set if and only if there is an idempotent $p \in J(S)$ such that $A \in p$.

Let $(S, \cdot)$ be a semigroup. A subset $A$ of $S$ is called a left solution set of $S$ (respectively, a right solution set of $S$) if there are $a, b \in S$ such that $A = \{x \in S : ax = b\}$ (respectively, $A = \{x \in S : xa = b\}$). Let $S$ be an infinite semigroup with size $\kappa$. We say $S$ is very weakly left cancellative (respectively, very weakly right cancellative) if the union of less than $\kappa$ left solution sets of $S$ (respectively, right solution sets of $S$) has size less than $\kappa$. We say $S$ is very weakly cancellative if it is both very weakly left cancellative and very weakly right cancellative. Let $S$ be a nonempty set, an ultrafilter $p$ on $S$ is called uniform if for any $X \in p$, $|X| = |S|$. Given a semigroup $S$, we denote the set of uniform ultrafilters on $S$ by $U$. If $S$ is infinite very weakly cancellative, then $U$ is an ideal of $\beta S$ [1].

**§2. Disjoint quasi-central sets.** In this section, we establish the existence of almost disjoint families of quasi-central subsets of a given quasi-central set in an infinite very weakly cancellative semigroup. Let $X$ be an infinite set. We call $\mathcal{A}$ a set of almost disjoint subsets of $X$ if for any $A \in \mathcal{A}$, $A \subseteq X$ and $|A| = |X|$, and for distinct $A, B \in \mathcal{A}$, $|A \cap B| < |X|$. Then there is the following fact:

LEMMA 2.1 [1, Lemma 2.1]. *Let $\kappa$ be an infinite cardinal.*

(i) *If there is a family $\{A_\alpha : \alpha < \delta\}$ of almost disjoint subsets of $\kappa$, then there is a family $\{\mathcal{A}_\alpha : \alpha < \delta\}$ of almost disjoint subsets of $\mathcal{P}_f(\kappa)$ such that for any $F \in \mathcal{P}_f(\kappa)$ and any $\alpha < \delta$ there is some $G \in \mathcal{A}_\alpha$ such that $F \subseteq G$.*

(ii) *There is a family $\{\mathcal{B}_\alpha : \alpha < \kappa\}$ of pairwise disjoint subsets of $\mathcal{P}_f(\kappa)$, each with size $\kappa$, such that for any $F \in \mathcal{P}_f(\kappa)$ and any $\alpha < \delta$ there is some $G \in \mathcal{B}_\alpha$ such that $F \subseteq G$.*

LEMMA 2.2 [1, Lemma 3.1]. *If $S$ is very weakly left cancellative, $U$ is a left ideal of $\beta S$. If $S$ is very weakly cancellative, $U$ is an ideal of $\beta S$.*

If $(S, \cdot)$ is a semigroup, $I$ is an ideal of $\beta S$, and $A \subseteq S$, recall that $U_A = \{p \in \beta S : A \in p\}$. For convenience, we call $A$ an $I$-large subset of $S$ if there is an idempotent

$p \in I \cap U_A$; and we call $A$ a uniform $I$-large subset of $S$ if there is a uniform idempotent $p \in I \cap U_A$. Then we have the following facts, the argument of which is similar to that of [1, Theorem 3.3], but more general.

THEOREM 2.3. *Suppose $\kappa$ is an infinite cardinal, $S$ is a very weakly left cancellative semigroup with size $\kappa$, $I$ is an ideal of $\beta S$, and $C$ is uniform $I$-large in $S$.*

(i) *If there is a family of $\delta$ almost disjoint subsets of $\kappa$, then $C$ contains $\delta$ almost disjoint uniform $I$-large sets.*
(ii) *$C$ contains $\kappa$ disjoint uniform $I$-large sets.*

PROOF. The proof of (ii) is the same as (i). So here we only prove (i). Since $C$ is uniform $I$-large, we pick a uniform idempotent $p \in I \cap U_C$. Define $C^\star = \{s \in C : s^{-1}C \in p\}$, so by [3, Lemma 4.14], for any $s \in C^\star$, $s^{-1}C^\star \in p$. For each $F \in \mathcal{P}_f(C^\star)$, define $S_F = C^\star \cap \bigcap_{s\in F} s^{-1}C^\star$. So $S_F \in p$.

Let $V = \bigcap_{F\in\mathcal{P}_f(C^\star)} U_{S_F}$. So $p \in V$. Now let us show that $V$ is a semigroup of $\beta S$. Observe that for each $F \in \mathcal{P}_f(C^\star)$ and each $s \in S_F$, if $H = \{s\} \cup Fs$, then $sS_H \subseteq S_F$. Since for each $t \in S_H$, we have $st \in C^\star$, and since for each $r \in F$, $rs \in H$, so $rst \in C^\star$. Hence $st \in S_F$, which means $sS_H \subseteq S_F$. Then by [3, Theorem 4.20], $V$ is a semigroup of $\beta S$.

Now well order $\mathcal{P}_f(C^\star)$ by $<$ as a $\kappa$-sequence. Now we define $x_F$ for each $F \in \mathcal{P}_f(C^\star)$ such that $Fx_F \cap Hx_H = \emptyset$ and $x_F \neq x_H$ whenever $F$ and $H$ are distinct elements of $\mathcal{P}_f(C^\star)$. Assume we have obtained $\{x_F : F < H\}$. Since $S$ is very weakly left cancellative, $|\{y \in S : Hy \cap \bigcup_{F<H} Fx_F \neq \emptyset\}| < \kappa$, while $S_H \in p$ so $S_H$ has size $\kappa$. Then we pick $x_H \in S_H \setminus (\{y \in S : Hy \cap \bigcup_{F<H} Fx_F \neq \emptyset\} \cup \{x_F : F < H\})$.

By Lemma 2.1(i), there is a family $\{\mathcal{A}_\alpha : \alpha < \delta\}$ of almost disjoint subsets of $\mathcal{P}_f(C^\star)$ such that for any $F \in \mathcal{P}_f(C^\star)$ and any $\alpha < \delta$ there is some $G \in \mathcal{A}_\alpha$ such that $F \subseteq G$. Let $A_\alpha = \bigcup_{F\in\mathcal{A}_\alpha} Fx_F$ for each $\alpha \in \delta$. Notice that $\{A_\alpha : \alpha < \delta\}$ is an almost disjoint family of subsets of $C$. Now let us show each $A_\alpha$ is uniform $I$-large.

Fix some $\alpha < \delta$. Observe that $\{H : H \in \mathcal{A}_\alpha \wedge F \subseteq H\}$ has size $\kappa$ for each $F \in \mathcal{P}_f(C^\star)$, so $X_F = \{x_H : H \in \mathcal{A}_\alpha \wedge F \subseteq H\}$ has size $\kappa$ and $\{X_F : F \in \mathcal{P}_f(C^\star)\}$ has the $\kappa$-uniform finite intersection property [3, Definition 3.60]. Then by [3, Theorem 3.62], we take a uniform ultrafilter $q \in \beta S$ such that $\{X_F : F \in \mathcal{P}_f(C^\star)\} \subseteq q$. For any $F \in \mathcal{P}_f(C^\star)$, if $x_H \in X_F$, then $H \in \mathcal{A}_\alpha$ and $F \subseteq H$, so $S_H \subseteq S_F$, while $x_H \in S_H$, hence $X_F \subseteq S_F$, which implies that $q \in V$.

For each $s \in C^\star$, and each $H \in \mathcal{A}_\alpha$ satisfying $s \in H$, we have $sx_H \in Hx_H \subseteq A_\alpha$, so $X_{\{s\}} \subseteq s^{-1}A_\alpha$. Hence $s^{-1}A_\alpha \in q$, which means $sq \in U_{A_\alpha}$. Since $s$ is arbitrary in $C^\star$, we have $C^\star q \subseteq U_{A_\alpha}$, so $clC^\star q \subseteq U_{A_\alpha}$. Note that $V \subseteq U_{C^\star} = clC^\star$, so $Vq \subseteq U_{A_\alpha}$. Also note that $Vq$ is a left ideal of $V$, so we can pick an idempotent $r \in Vq$ which is minimal in $V$, hence $r \in U_{A_\alpha} \cap K(V)$. Observe that $V \cap I \neq \emptyset$ since $p \in V \cap I$, so $V \cap I$ is an ideal of $V$, and thus $K(V) \subseteq V \cap I \subseteq I$, so $r \in U_{A_\alpha} \cap I$. By Lemma 2.2, $U$ is a left ideal of $\beta S$, then $r \in Vq \subseteq \beta Sq \subseteq \beta SU \subseteq U$, so $r$ is uniform, therefore $A_\alpha$ is uniform $I$-large. ⊣

If $S$ is a semigroup, recall that $clK(\beta S)$ is an ideal of $\beta S$, and quasi-central sets in $S$ are exactly $clK(\beta S)$-large sets. Then we have the following corollary of Theorem 2.3.

COROLLARY 2.4. *Suppose $\kappa$ is an infinite cardinal and $S$ is a very weakly cancellative semigroup with size $\kappa$.*

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1. *If $\kappa$ contains $\delta$ almost disjoint subsets, then every quasi-central set in $S$ contains $\delta$ almost disjoint quasi-central subsets.*
2. *Every quasi-central set in $S$ contains $\kappa$ disjoint quasi-central subsets.*

PROOF. Take a quasi-central set $A$ in $S$. Then there is some idempotent $p \in clK(\beta S) \cap U_A$. Note that $U$ is a closed ideal of $\beta S$, so $clK(\beta S) \subseteq U$, which implies that $p$ is uniform, so $A$ is uniform $clK(\beta S)$-large. Then we can apply Theorem 2.3 to obtain the result. ⊣

**§3. Disjoint C-sets and J-sets.** In this section, we investigate two other kinds of large sets: J-sets and C-sets. Recall that given a commutative semigroup $(S, +)$, $L \in \mathcal{P}_f({}^{\mathbb{N}}S)$, $a \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$, we have defined $S_L(a, H) = \{a + \sum_{t \in H} f(t) : f \in L\}$.

LEMMA 3.1. *Let $(S, +)$ be a commutative semigroup and let $A \subseteq S$. Then $A$ is a J-set in $S$ if and only if for each $L \in \mathcal{P}_f({}^{\mathbb{N}}S)$, there exist $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $S_L(a, H) \subseteq A$.*

PROOF. The sufficiency is immediate. For the necessity, let $L \in \mathcal{P}_f({}^{\mathbb{N}}S)$ and pick $g \in L$. Let $L' = \{g\} \cup \{g + f : f \in L\}$, pick $b \in S$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $S_{L'}(b, H) \subseteq A$, and let $a = b + \sum_{t \in H} g(t)$. Then $a \in A$ and $S_L(a, H) \subseteq A$. ⊣

THEOREM 3.2. *Suppose $(S, +)$ is an infinite commutative very weakly cancellative semigroup with size $\kappa$. Then every J-set in $S$ has size $\kappa$.*

PROOF. Assume that there is a J-set $A$ such that $|A| < \kappa$. Let us construct an injective sequence in $A$ of length $\kappa$, so that a contradiction appears.

Assume we have already obtained $\langle a_\xi \rangle_{\xi < \delta}$ for some $\delta < \kappa$, which is an injective sequence in $A$, let us define $a_\delta$. For each $(a, \xi) \in A \times \delta$, let $B_{a,\xi} = \{x \in S : a + x = a_\xi\}$, which is a left solution set. Then let $B = \bigcup_{(a,\xi) \in A \times \delta} B_{a,\xi}$. Since $S$ is very weakly cancellative, $|B| < \kappa$. Now we need to build a sequence $\langle b_n \rangle_{n=1}^\infty$ which satisfies $\mathrm{FS}(\langle b_n \rangle_{n=1}^\infty) \subseteq S \setminus B$, and which will be used to define $a_\delta$. First take $b_1 \in S \setminus B$. Assume we have obtained $\langle b_i \rangle_{i=1}^n$ for some $n \in \mathbb{N}$ such that $\mathrm{FS}(\langle b_i \rangle_{i=1}^n) \subseteq S \setminus B$. For each $y \in \mathrm{FS}(\langle b_i \rangle_{i=1}^n)$, let $B_y = \bigcup_{(a,\xi) \in A \times \delta} B_{y,a,\xi}$, where $B_{y,a,\xi} = \{x \in S : a + y + x = a_\xi\}$. Hence each $B_y$ has size less than $\kappa$, so $|\bigcup_{y \in \mathrm{FS}(\langle b_i \rangle_{i=1}^n)} B_y| < \kappa$. Then take $b_{n+1} \in S \setminus (B \cup (\bigcup_{y \in \mathrm{FS}(\langle b_i \rangle_{i=1}^n)} B_y))$.

Then for any $z = b_{i_1} + \cdots + b_{i_m} \in \mathrm{FS}(\langle b_i \rangle_{i=1}^{n+1})$, if $i_m \neq n+1$, then $z \in \mathrm{FS}(\langle b_i \rangle_{i=1}^n)$ so by hypothesis $z \in S \setminus B$. Otherwise, $i_m = n+1$. If $m = 1$, then $z = b_{n+1} \in S \setminus B$. Otherwise, $z = y + b_{n+1}$ for some $y \in \mathrm{FS}(\langle b_i \rangle_{i=1}^n)$. Note that $b_{n+1} \notin B_y$, so there is no $(a, \xi) \in A \times \delta$ such that $a + y + b_{n+1} = a_\xi$. That is, $z \notin B$. Therefore, $\mathrm{FS}(\langle b_i \rangle_{i=1}^{n+1}) \subseteq S \setminus B$.

Finally, we obtain $\langle b_n \rangle_{n=1}^\infty$ such that $\mathrm{FS}(\langle b_n \rangle_{n=1}^\infty) \subseteq S \setminus B$. Since $A$ is a J-set, for $\langle b_n \rangle_{n=1}^\infty$, by Lemma 3.1, there is some $a \in A$ and $H \in \mathcal{P}_f(\mathbb{N})$ such that $a + \sum_{i \in H} b_i \in A$. Now let $z = \sum_{i \in H} b_i$ and $a_\delta = a + z$, so $a_\delta \in A$ and $a_\delta$ is not equal to any $a_\xi$, $\xi < \delta$. (If there is some $\xi < \delta$ such that $a_\delta = a_\xi$, then $a + z = a_\xi$, so $z \in B$ while $z \in \mathrm{FS}(\langle b_n \rangle_{n=1}^\infty) \subseteq S \setminus B$, which is a contradiction.) Hence $\langle a_\xi \rangle_{\xi < \delta + 1}$ is an injective sequence in $A$.

By induction, we obtain an injective $\kappa$-sequence in $A$; this is a contradiction. ⊣

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Therefore, by Theorem 3.2, we know that in any infinite commutative very weakly cancellative semigroup $S$, $J(S) \subseteq U$. Then we have the following result.

COROLLARY 3.3. *Suppose $\kappa$ is an infinite cardinal and $S$ is a commutative very weakly cancellative semigroup with size $\kappa$.*

1. *If $\kappa$ contains $\delta$ almost disjoint subsets, then every C-set in $S$ contains $\delta$ almost disjoint C-sets.*
2. *Every C-set in $S$ contains $\kappa$ disjoint C-sets.*

PROOF. Since $S$ is commutative very weakly cancellative, every C-set is uniform $J(S)$-large. Then by Theorem 2.3 we deduce the result. ⊣

As for J-sets, we also have a similar result.

THEOREM 3.4. *Suppose $\kappa$ is an infinite cardinal satisfying $\kappa^\omega = \kappa$ and $S$ is a commutative very weakly cancellative semigroup with size $\kappa$.*

1. *If $\kappa$ contains $\delta$ almost disjoint subsets, then every J-set in $S$ contains $\delta$ almost disjoint J-sets.*
2. *Every J-set in $S$ contains $\kappa$ disjoint J-sets.*

PROOF. Since the proofs of the two items are essentially the same, here we only provide the proof of the first item. Let $A$ be a J-set. Since $\kappa^\omega = \kappa$, $|\mathcal{P}_f({}^{\mathbb{N}}S)| = \kappa$. Enumerate $\mathcal{P}_f({}^{\mathbb{N}}S)$ as $\langle L_\sigma \rangle_{\sigma<\kappa}$. We will inductively build two $\kappa$-sequences $\langle a_\sigma \rangle_{\sigma<\kappa}$ and $\langle H_\sigma \rangle_{\sigma<\kappa}$ such that for each $\sigma < \kappa$, $a_\sigma \in S$, $H_\sigma \in \mathcal{P}_f(\mathbb{N})$, $S_{L_\sigma}(a_\sigma, H_\sigma) \subseteq A$, and for $\alpha < \sigma < \kappa$, $S_{L_\alpha}(a_\alpha, H_\alpha) \cap S_{L_\sigma}(a_\sigma, H_\sigma) = \emptyset$.

Since $A$ is a J-set, pick $a_0 \in S$ and $H_0 \in \mathcal{P}_f(\mathbb{N})$ such that $S_{L_0}(a_0, H_0) \subseteq A$. Let $0 < \sigma < \kappa$ and assume that $\langle a_\alpha \rangle_{\alpha<\sigma}$ and $\langle H_\alpha \rangle_{\alpha<\sigma}$ have been chosen. Let $S_\sigma = \bigcup_{\alpha<\sigma} S_{L_\alpha}(a_\alpha, H_\alpha)$ and note that $|S_\sigma| < \kappa$.

CLAIM 1. *Let $C = \{(a, H) : a \in S, H \in \mathcal{P}_f(\mathbb{N})$ and $S_{L_\sigma}(a, H) \cap S_\sigma \neq \emptyset\}$. Then $|C| < \kappa$.*

PROOF. Note that, since $\kappa = \kappa^\omega$ and $\kappa < \kappa^{\mathrm{cf}(\kappa)}$, we have that $\mathrm{cf}(\kappa) > \omega$. For $H \in \mathcal{P}_f(\mathbb{N})$ let $D_H = \{a \in S : S_{L_\sigma}(a, H) \cap S_\sigma \neq \emptyset\}$. Then $C = \bigcup\{D_H \times \{H\} : H \in \mathcal{P}_f(\mathbb{N})\}$ so $|C| \leq \sum_{H\in\mathcal{P}_f(\mathbb{N})} |D_H|$. Given $H \in \mathcal{P}_f(\mathbb{N})$, $D_H \subseteq \bigcup_{x\in S_\sigma} \bigcup_{f\in L_\sigma}$ $\{a \in S : x = a + \sum_{t\in H} f(t)\}$ and each $\{a \in S : x = a + \sum_{t\in H} f(t)\}$ is a right solution set, so we have $|D_H| < \kappa$. Therefore, since $|\mathcal{P}_f(\mathbb{N})| = \omega$ and $\mathrm{cf}(\kappa) > \omega$, we have $|C| < \kappa$ as claimed. ⊣

CLAIM 2. *Let $D = \{(a, H) : a \in S, H \in \mathcal{P}_f(\mathbb{N})$ and $S_{L_\sigma}(a, H) \subseteq A\}$. Then $|D| = \kappa$.*

PROOF. Since $|D| \geq |\{a \in S : (\exists H \in \mathcal{P}_f(\mathbb{N}))(S_{L_\sigma}(a, H) \subseteq A)\}|$, it suffices by Theorem 3.2 to show that

$$\{a \in S : (\exists H \in \mathcal{P}_f(\mathbb{N}))(S_{L_\sigma}(a, H) \subseteq A)\}$$

is a J-set.

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So let $L \in \mathcal{P}_f(^{\mathbb{N}}S)$ and let $L' = \{f + g : f \in L \text{ and } g \in L_\sigma\}$. Since $A$ is a J-set, pick $b \in S$ and $K \in \mathcal{P}_f(\mathbb{N})$ such that

$$\{b + \sum_{t \in K} f(t) + \sum_{t \in K} g(t) : f \in L \text{ and } g \in L_\sigma\} \subseteq A.$$

Then $S_L(b, K) \subseteq \{a \in S : (\exists H \in \mathcal{P}_f(\mathbb{N}))(S_{L_\sigma}(a, H) \subseteq A)\}$. ⊣

Then we take $(a_\sigma, H_\sigma) \in D \setminus C$, which is as desired.

By Lemma 2.1(i), we obtain a family $\{\mathcal{A}_\alpha : \alpha < \delta\}$ of almost disjoint subsets of $\mathcal{P}_f(\kappa)$ such that for any $F \in \mathcal{P}_f(\kappa)$ and any $\alpha < \delta$ there is some $G \in \mathcal{A}_\alpha$ such that $F \subseteq G$. For any $\xi < \delta$, let $D_\xi = \bigcup\{S_{L_\sigma}(a_\sigma, H_\sigma) : \sigma < \kappa \text{ and } L_\sigma \in \mathcal{A}_\xi\}$; since $|\mathcal{A}_\xi| = \kappa$, $|D_\xi| = \kappa$. For any $\alpha < \beta < \delta$, $D_\alpha \cap D_\beta = \bigcup\{S_{L_\sigma}(a_\sigma, H_\sigma) : \sigma < \kappa$ and $L_\sigma \in \mathcal{A}_\alpha \cap \mathcal{A}_\beta\}$ so $|D_\alpha \cap D_\beta| < \kappa$. Finally we let $\xi < \delta$ and show that $D_\xi$ is a J-set. Let $L \in \mathcal{P}_f(^{\mathbb{N}}S)$ and pick $M \in \mathcal{P}_f(^{\mathbb{N}}S)$ such that $M \in \mathcal{A}_\xi$ and $L \subseteq M$. Then $M = L_\beta$ for some $\beta < \kappa$ so $S_L(a_\beta, H_\beta) \subseteq S_{L_\beta}(a_\beta, H_\beta) \subseteq D_\xi$. ⊣

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