ASYMPTOTICS OF FIRST-PASSAGE PERCOLATION ON ONE-DIMENSIONAL GRAPHS

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Abstract

In this paper we consider first-passage percolation on certain one-dimensional periodic graphs, such as the $\mathbb{Z} \times \{0, 1, \ldots, K - 1\}^{d-1}$ nearest neighbour graph for $d, K \geq 1$. We expose a regenerative structure within the first-passage process, and use this structure to show that both length and weight of minimal-weight paths present a typical one-dimensional asymptotic behaviour. Apart from a strong law of large numbers, we derive a central limit theorem, a law of the iterated logarithm, and a Donsker theorem for these quantities. In addition, we prove that the mean and variance of the length and weight of minimizing paths are monotone in the distance between their end-points, and further show how the regenerative idea can be used to couple two first-passage processes to eventually coincide. Using this coupling we derive a 0–1 law.

Keywords: First-passage percolation; renewal theory; classical limit theorem

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1. Introduction

First-passage percolation can be thought of as a discrete model for the spread of an infectious entity. The model has been studied extensively since its introduction by Hammersley and Welsh [15], most notably in the case when the underlying discrete structure is given by the $\mathbb{Z}^d$ nearest neighbour lattice for some $d \geq 2$ (see, e.g. [17] for the state of the art in the mid 1980s, and [13] for a hint of the development since). Although these studies have generated a range of tools and techniques, there are many conjectured properties of the model that so far have not been rigorously verified. Among these properties we find monotonicity of travel times, fluctuations around the asymptotic shape, and the existence of infinite geodesics (see [16] for a more comprehensive list of open problems). The incomplete picture in the challenging two and higher-dimensional case is the main motivation behind the present study, in which first-passage percolation on essentially one-dimensional periodic graphs is considered. Of particular interest will be subgraphs of the usual $\mathbb{Z}^d$ lattice, such as the $\mathbb{Z} \times \{0, 1, \ldots, K-1\}^{d-1}$ nearest neighbour graph, for some $K, d \geq 1$, below referred to as the (K, d)-*tube*. This is illustrated in Figure 1.

The asymptotic behaviour of first-passage percolation on the class of graphs considered in this study is found to be typically one-dimensional, and differs in several aspects from what is expected in higher dimensions. Apart from a usual strong law of large numbers, we further prove that the first-passage process obeys a central limit theorem, a law of the iterated logarithm, and a Donsker theorem. In addition to these classical limit theorems, we are able to obtain a more precise description of the process, showing that both the mean and variance of travel times grow monotonically with respect to the graph distance between vertices. This behaviour
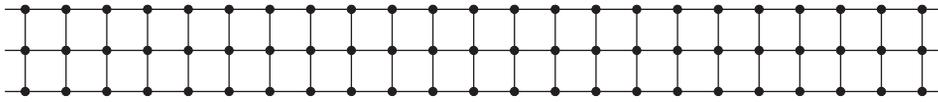
FIGURE 1: A graph of particular interest is the $\mathbb{Z} \times \{0, 1, \ldots, K-1\}^{d-1}$ nearest neighbour graph, referred to as the (K, d)-tube. We think of this graph as constructed from a sequence of vertical stretches consisting of three vertices and two edges.

is plausible to hold also in higher dimensions, but remains so far unknown except in a certain special case [12]. We further note that an analogous behaviour to the one described above also holds for the ℓ^1-length of the path with minimal weight between two vertices. Finally, the regenerative idea is used to construct a coupling between two first-passage processes with different initial configurations. As an application of the coupling, we deduce a 0–1 law.

The periodicity of the graphs we consider induces a form of translation invariance of the first-passage process. Together with the one-dimensional nature of the graphs, this allows for the identification of a suitable regenerative structure. The regenerative structure may then be exploited to study the first-passage process with the help of classical renewal theory. This paper aims at giving a clear demonstration of how the identification of a regenerative behaviour can provide detailed information about a random process. The same regenerative idea has been found useful in the study of first-passage percolation on the integer lattice in two and more dimensions [3].

The asymptotic results derived in the present study greatly generalizes earlier results by Schlemm [28], who proved a central limit theorem for first-passage percolation on the (2, 2)-tube with exponential weights. A central limit theorem and Donsker theorem have also been obtained independently, and by different methods, by Chatterjee and Dey [7], which we further comment upon below. Before we present the contributions of this paper, we introduce some basic definitions and provide the reader with some background.

1.1. Description of model

Let $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ be a connected graph, and associate to the edges of the graph nonnegative independent, identically distributed (i.i.d.) random weights $\{\tau_e\}_{e \in \mathbb{E}}$, referred to as *passage times*. Passage times are interpreted as the time it takes an infection to traverse the edges. To avoid trivialities we assume throughout that the passage time distribution $P_\tau(\cdot) := P(\tau_e \in \cdot)$ does not concentrate all mass at a single point. Let us refer to a *path* as an alternating sequence of vertices and edges; $v_0, e_1, v_1, \ldots, e_m, v_m$, such that the vertex v_k is the endpoint of the edges e_k and e_{k+1}. A path with one endpoint in U and the other in V, where $U, V \subset \mathbb{V}$, will be referred to as a path from U to V. We will repeatedly abuse notation and identify a path with its set of edges. For a path Γ, we define its passage time as $T(\Gamma) := \sum_{e \in \Gamma} \tau_e$, and define the *travel time* between two sets $U, V \subset \mathbb{V}$ as

$$T(U, V) := \inf\{T(\Gamma) \colon \Gamma \text{ is a path from } U \text{ to } V\}.$$

In the case $U = \{u\}$ or $V = \{v\}$, we simply suppress brackets.

First-passage percolation refers to the process started with a finite set $I \subset \mathbb{V}$ of infected vertices, from which the infection spreads to adjacent vertices with delays indicated by the passage times. The corresponding set of infected vertices at time t, is given by

$$B_t := \{v \in \mathbb{V} \colon T(I, v) \leq t\}.$$

1.2. Higher-dimensional background

The foremost characteristic feature of first-passage percolation is its subadditive behaviour, meaning that for all vertices u, v, and w of the graph we have

$$T(u, v) \leq T(u, w) + T(w, v).$$

Its importance was realised by Hammersley and Welsh [15], and further inspired Kingman [20] to derive the subadditive ergodic theorem. Consider in the following first-passage percolation on the $\mathbb{Z}^d$ lattice (for $d \geq 2$), and let Y denote the minimum of $2d$ independent random variables distributed according to P_τ. When $\mathrm{E}[Y] < \infty$, it follows from the subadditive ergodic theorem that there is a constant $\mu(\boldsymbol{e}_1)$, referred to as the *time constant*, such that

$$\lim_{n \to \infty} \frac{T(I, \boldsymbol{n})}{n} = \mu(\boldsymbol{e}_1) \quad \text{almost surely and in } L^1,$$

where $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$, and $\boldsymbol{n} = n\boldsymbol{e}_1$. A necessary and sufficient condition for simultaneous convergence in all directions was provided by Cox and Durrett [8], in inspiration of a result due to Richardson [26]. For the sake of convenience, let us identify B_t by the set in $\mathbb{R}^d$ obtained by centering a unit cube around each point $z \in B_t$. Under the assumption that $\mathrm{E}[Y^d] < \infty$, the result of Cox and Durrett, known as the shape theorem, states that: if $\mu(\boldsymbol{e}_1) > 0$, then there exists a deterministic compact and convex set $B^* \subset \mathbb{R}^d$ with nonempty interior such that for all $\varepsilon > 0$, almost surely,

$$(1 - \varepsilon)B^* \subset \frac{1}{t}B_t \subset (1 + \varepsilon)B^* \quad \text{for large enough } t. \tag{1.1}$$

If $\mu(\boldsymbol{e}_1) = 0$ and $K \subset \mathbb{R}^d$ is compact, then, almost surely, $K \subset (1/t)B_t$ for all t large enough. It is further known [17, Theorem 6.1] that $\mu(\boldsymbol{e}_1) = 0$ if and only if $P_\tau(0) \geq p_c(d)$, where $p_c(d)$ is the critical value for independent bond percolation on the $\mathbb{Z}^d$ lattice.

The nature of fluctuations around the asymptotic shape have turned out to be harder to understand. Apart from a result of Kesten and Zhang [19] in the 'critical' case when $d = 2$ and $P_\tau(0) = p_c(2) = 1/2$, precise rigorous results remain essentially nonexistent. In an earlier study, Kesten [18] showed that for $d \geq 2$, if $P_\tau(0) < p_c$ and $\mathrm{E}[\tau_e^2] < \infty$, then there are constants $C_1 > 0$ and $C_2 < \infty$ such that

$$C_1 \leq \mathrm{var}(T(I, \boldsymbol{n})) \leq C_2 n \quad \text{for all } n \geq 1.$$

It is not believed that first-passage percolation should obey Gaussian scaling for $d \geq 2$. The first result in this direction was obtained by Benjamini *et al.* [6], who gave an upper bound on $\mathrm{var}(T(I, \boldsymbol{n}))$ of order $n/\log n$ for $d \geq 2$. Their argument was restricted to the case of $\{a, b\}$-valued passage times, where $0 < a < b < \infty$, but was later extended to more general distributions in [5], [9]. However, the predicted truth suggests that for $d = 2$ the correct order of growth is $n^{2/3}$, and it is not clear which behaviour to expect in higher dimensions (see, e.g. [22] and references therein). In two dimensions, it is known that the variance of $T(I, \boldsymbol{n})$ grows at least logarithmically in n, except for certain degenerate cases; see [22], and also [23].

1.3. First-passage percolation on essentially one-dimensional periodic graphs

This study considers first-passage percolation on essentially one-dimensional periodic graphs, which we define as follows.

Definition. The class of *essentially one-dimensional periodic graphs* consists of all connected graphs $\mathcal{G}$ that can be constructed in the following manner. Let $\{\mathcal{G}_n\}_{n \in \mathbb{Z}}$ be a sequence of identical copies of some finite connected graph, each with vertex set $\mathbb{V}_{\mathcal{G}_n} = \{v_{n,1}, \ldots, v_{n,K}\}$ and edge set $\mathbb{E}_{\mathcal{G}_n} = \{e_{n,1}, \ldots, e_{n,L}\}$. Fix a nonempty set $J \subset \{(i,j) : 1 \leq i, j \leq K\}$, and connect $\mathcal{G}_n$ to $\mathcal{G}_{n+1}$ for each n by adding an edge $e(v_{n,i}, v_{n+1,j})$ between $v_{n,i}$ and $v_{n+1,j}$, for each $(i,j) \in J$. We will write $\mathbb{E}_{\mathcal{G}_n}^*$ for $\mathbb{E}_{\mathcal{G}_n} \cup \{e(v_{n,i}, v_{n+1,j}) : (i,j) \in J\}$, and say that a vertex v in the resulting graph $\mathcal{G} = (\mathbb{V}, \mathbb{E})$ is at *level n* if $v \in \mathbb{V}_{\mathcal{G}_n}$.

Due to periodicity, each essentially one-dimensional periodic graph can be constructed in more than one way. We will, in the sequel, assume that each graph comes with a construction given, which we typically want to be minimal in some sense, although this is not essential.

We will show that first-passage percolation on any essentially one-dimensional periodic graph $\mathcal{G}$ has a typical one-dimensional asymptotic behaviour. To give the reader a flavour of what is meant by that, we state some results here, and describe further results below. Recall that throughout we assume that the passage-time distribution does not concentrate all mass at a single point. In what follows $v_{n,i}$ will denote an arbitrary vertex at level n. We will prove that there are nonnegative finite constants $\mu = \mu(\mathcal{G}, P_\tau)$ and $\sigma = \sigma(\mathcal{G}, P_\tau)$, such that the following holds.

Theorem 1.1. (Law of large numbers.) *If* $\mathrm{E}[\tau_e^r] < \infty$ *for some* $r \geq 1$*, then*

$$\lim_{n \to \infty} \frac{T(I, v_{n,i})}{n} = \mu \quad \text{almost surely and in } L^r.$$

Theorem 1.2. (Central limit theorem.) *If* $\mathrm{E}[\tau_e^2] < \infty$*, then*

$$\frac{T(I, v_{n,i}) - \mu n}{\sigma \sqrt{n}} \xrightarrow{\mathrm{D}} \chi \quad \text{in distribution,}$$

as $n \to \infty$*, where* χ *has a standard normal distribution.*

Let $\mathcal{L}(\{x_n\}_{n \geq 1})$ denote the set of limit points of a real-valued sequence $\{x_n\}_{n \geq 1}$.

Theorem 1.3. (Law of the iterated logarithm.) *If* $\mathrm{E}[\tau_e^2] < \infty$*, then*

$$\mathcal{L}\left(\left\{\frac{T(I, v_{n,i}) - \mu n}{\sigma \sqrt{2n \log \log n}}\right\}_{n \geq 3}\right) = [-1, 1] \quad \text{almost surely.}$$

In particular, almost surely,

$$\limsup_{n \to \infty} \frac{T(I, v_{n,i}) - \mu n}{\sigma \sqrt{2n \log \log n}} = 1 \quad \text{and} \quad \liminf_{n \to \infty} \frac{T(I, v_{n,i}) - \mu n}{\sigma \sqrt{2n \log \log n}} = -1.$$

Remarks. (a) We have at this stage preferred to state simple moment conditions, but we point out that they can be relaxed somewhat in certain cases (see Remark 3.1).

(b) As a consequence of the regenerative structure explored in Section 2, μ and σ are in (2.5) given by explicit equations. It will be clear that their values may depend on $\mathcal{G}$ and P_τ, but not on I or i.

(c) Since $T(I, v_{n,i})$ differs from $\min_{v \in \mathbb{V}_{\mathcal{G}_n}} T(I, v)$ and $\max_{v \in \mathbb{V}_{\mathcal{G}_n}} T(I, v)$ by at most a finite number of passage times, the conclusions in Theorems 1.1, 1.2, and 1.3 hold also for these quantities.

The almost sure and L^1-convergence in Theorem 1.1 could easily be derived from the subadditive ergodic theorem. However, it is instructive for the understanding of our approach to give an alternative proof. As the classical central limit theorem for i.i.d. sequences extends to Donsker's theorem, Theorem 1.2 also extends to a functional version (see Theorem 3.1).

At a comparison with higher dimensions, Theorem 1.1 is the one-dimensional analogue to the shape theorem. Theorems 1.2 and 1.3, on the other hand, point out a one-dimensional behaviour that is not generally expected in higher dimensions. Indeed, Theorem 1.3 gives the precise order of fluctuations around the time constant. Restrict for a moment attention to (K, d)-tubes, and let μ_K and σ_K denote the constants associated therewith. In comparison with (1.1), Theorem 1.3 implies that if $B^* = B^*(t) := [-\mu_K^{-1}, \mu_K^{-1}] \times [0, K/t]^{d-1}$, then for every $\lambda > \sigma_K \sqrt{2/\mu_K}$,

$$(1 - \lambda \sqrt{t^{-1} \log \log t}) B^* \subset \tfrac{1}{t} B_t \subset (1 + \lambda \sqrt{t^{-1} \log \log t}) B^*$$

for large enough t, almost surely. Moreover, both inclusions fail for $\lambda < \sigma_K \sqrt{2/\mu_K}$. The claim follows from a straightforward inversion argument, which is found in [2].

Theorem 1.2 was also obtained by Schlemm [28] in the particular case of the $(2, 2)$-tube with exponential passage times. A variant of Theorem 1.2 was proved by other means in an independent work by Chatterjee and Dey [7]. Their main result concerns first-passage percolation on the (K, d)-tube, where $K = K(n)$ is allowed to depend on n. They showed that for every $r > 2$ there exists $\alpha = \alpha(d, r)$ such that if $E[\tau_e^r] < \infty$ and $K = K(n) = o(n^\alpha)$, then $\{T(\mathbf{0}, \mathbf{n})\}_{n \geq 1}$ will obey a Gaussian central limit theorem. Their result extends our Theorem 1.2, but assumes a slightly stronger moment condition. We emphasise that the present paper was prepared simultaneously and independently of theirs, and appeared, in part, as [1].

1.4. Identification of a regenerative structure

The idea of how to identify a suitable regenerative sequence arises naturally for first-passage percolation with exponentially distributed passage times, which we illustrate below for the $(2, 2)$-tube.

Equip the $(2, 2)$-tube with i.i.d. exponential passage times, and let both vertices at level zero be initially infected. At any fixed time t, given the infected component B_t, each edge with exactly one endpoint in the infected component is equally likely to be next passed by the infection. Thus, at each level, with probability at least $1/2$, both vertices will become infected before any vertex at the next level. It follows that with probability one, at some level r, both vertices will become infected before any vertex at level $r + 1$. Denote by ρ the first level for which this happens, and τ_ρ the time at which this happens. By the lack-of-memory property, the time it takes for the infection, from this moment, to reach m further levels has the same distribution as the time it would take to reach level m, i.e.

$$T(I, v_{\rho+m,i}) - \tau_\rho \overset{\mathrm{D}}{=} T(I, v_{m,i}). \tag{1.2}$$

In fact, at infinitely many levels, both vertices at that level will be infected before any vertex at higher levels. If we repeat the argument, we generate a sequence of (regenerative) levels $\{\rho_k\}_{k \geq 1}$ (see Figure 2), with corresponding sequence of instants $\{\tau_{\rho_k}\}_{k \geq 1}$, such that (1.2) holds.

Since passage times are i.i.d., the consecutive differences $\rho_{k+1} - \rho_k$ will be i.i.d., as well as the differences $\tau_{\rho_{k+1}} - \tau_{\rho_k}$. The usefulness of the regenerative structure is the following. Note
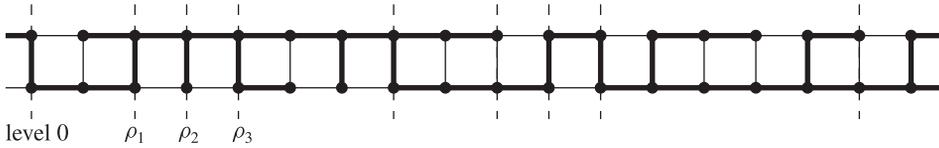
FIGURE 2: A realisation of the spread of an infection on the $(2, 2)$-tube, where the thicker edges indicate those used to reach uninfected vertices. The broken lines indicate levels at which both vertices became infected before any vertex ahead.

that the level and the time of the nth regeneration may be written as i.i.d. sums, i.e.,

$$\rho_n = \sum_{k=0}^{n-1} \rho_{k+1} - \rho_k \quad \text{and} \quad \tau_{\rho_n} = \sum_{k=0}^{n-1} \tau_{\rho_{k+1}} - \tau_{\rho_k},$$

where $\rho_0 = 0$ and $\tau_{\rho_0} = 0$. This link will enable the first-passage process to be studied via the classical theory for i.i.d. sequences.

1.5. Consequences of the regenerative structure

The main results of this paper are derived from the 'regenerative' nature of first-passage percolation on essentially one-dimensional periodic graphs. What is meant by a regenerative behaviour will be properly defined in Section 2. Once a regenerative structure has been identified, its asymptotics may be studied with renewal theory for random walks. A general account for this theory is given in [14]. Among the consequences thereof, we have already mentioned Theorems 1.1, 1.2, and 1.3, which we will easily derive in Section 3, based on this structure.

In Section 4 we will show that $\mathrm{E}[T(I, v_{n+1,i}) - T(I, v_{n,i})] \to \mu$ as $n \to \infty$. Since $\mu > 0$, this proves monotonicity of $\mathrm{E}[T(I, v_{n,i})]$ for large n. More remarkable, since it is not a general consequence of renewal theory, is the fact that fluctuations of $T(I, v_{n,i})$ are also monotone for large n, which we also prove. For small n the local geometry of the graph plays a larger role. We remind the reader of a counter-example due to van den Berg [29], showing that quantities of this kind are not always monotone for all values of $n \in \mathbb{N}$.

There is little hope of finding a closed form expression for the time constant on the $\mathbb{Z}^d$ lattice, for $d \geq 2$. Determining similar quantities for one-dimensional graphs, such as the (K, d)-tube, remains very difficult in practice. However, some results in this direction have been obtained for certain 'width-two stretches', such as the $(2, 2)$-tube (see [11], [24], [25], and [27]). We are, in addition, interested in how the constants μ_K and σ_K associated with the (K, d)-tube compare to similar properties of the lattice. For fixed d, a trivial coupling argument shows that $\mu_K \geq \mu_{K+1}$ for all $K \geq 1$. In Section 5 we will show that the inequality is in fact strict, and that

$$\lim_{K \to \infty} \mu_K = \mu(\boldsymbol{e}_1).$$

In the light of the subdiffusive behaviour in dimensions two and above, we would expect that $\lim_{K \to \infty} \sigma_K = 0$. However, we have not been able to verify this (see also Remark 5.1).

Apart from studying the temporal progression of the first-passage process, describing its spatial history has received considerable attention (see [16]). A path between u and v attaining $T(u, v)$ is commonly called a *geodesic*. Let $N(u, v)$ denote the length of the geodesic between u and v (or, say, the shortest if several). For essentially one-dimensional periodic graphs the

sequence $\{N(I, v_{n,i})\}_{n\geq 1}$ will obey an asymptotic behaviour analogous as that described above for $\{T(I, v_{n,i})\}_{n\geq 1}$. More precisely, for any passage-time distribution not concentrating all mass at a single point, analogous conclusions of Theorems 1.1, 1.2, and 1.3 above, as well as of Theorems 3.1, 4.1, and 4.2 to come, will hold also for the sequence $\{N(I, v_{n,i})\}_{n\geq 1}$. That is, the conclusions will hold without the need for a moment assumption.

Finally, in Section 6 we show how the regenerative structure can be used to construct a coupling between two first-passage processes, started with different initial conditions, so that they eventually coincide. As an application of the coupling, we prove a 0–1 law. It is not known for which other graphs a 0–1 law analogous to the one obtained here holds. But, we give an example showing that the analogous 0–1 law cannot hold on the binary tree.

Remark. Some details are left out of this version of the paper in order to keep the presentation concise. A longer version, with further details and additional results, is found in [2].

2. Regenerative behaviour

Consider first-passage percolation with general passage time distribution on any essentially one-dimensional periodic graph. With a regenerative sequence we will refer to the following.

Definition. We say that a sequence $\{X_k\}_{k\geq 1}$ of random variables is a *regenerative sequence* if there exists an increasing sequence of random variables $\{\lambda_k\}_{k\geq 0}$ such that $\{\lambda_k - \lambda_{k-1}\}_{k\geq 1}$ and $\{X_{\lambda_k} - X_{\lambda_{k-1}}\}_{k\geq 1}$ form i.i.d. sequences. We call $\{\lambda_k\}_{k\geq 0}$ a sequence of *regenerative levels*.

In what follows, an edge *at* level n refers to an edge in $\mathbb{E}_{\mathcal{G}_n}$. An edge *between* levels n and $n+m$ refers to an edge in $\mathbb{E}^*_{\mathcal{G}_n} \cup \cdots \cup \mathbb{E}^*_{\mathcal{G}_{n+m-1}} \cup \mathbb{E}_{\mathcal{G}_{n+m}}$. We first define our regenerative event. Let M be a positive integer and denote the set of edges between level n and $n+2M$ by E_n. Fix a path γ_n of shortest length between $\mathbb{V}_{\mathcal{G}_n}$ and $\mathbb{V}_{\mathcal{G}_{n+2M}}$, i.e. between two vertices at level n and $n+2M$, respectively. Define the subset $\hat{E}_n$ of E_n as

$$\hat{E}_n := \gamma_n \cup \mathbb{E}_{\mathcal{G}_n} \cup \mathbb{E}_{\mathcal{G}_{n+2M}}. \tag{2.1}$$

Furthermore, let $m_\tau := \inf\{x \geq 0 : P_\tau([0, x]) > 0\}$ and $M_\tau := \sup\{x \geq 0 : P_\tau([x, \infty)) > 0\}$, and note that $0 \leq m_\tau < M_\tau \leq \infty$ since we consider passage-time distributions not concentrated at a single point. Fix t' and t'' such that $m_\tau < t' < t'' < M_\tau$, and define the 'regenerative' event

$$A_n := \{\tau_e \leq t', \text{ for all } e \in \hat{E}_n\} \cap \{\tau_e \geq t'', \text{ for all } e \in E_n \setminus \hat{E}_n\}. \tag{2.2}$$

The event A_n is depicted in Figure 3. Trivially $P(A_n) > 0$. The vertex at level $n + M$ first reached via γ_n will be of particular interest, so we introduce the following notation.

Definition. Let $\hat{v}_n$ denote the vertex at level n first reached via γ_{n-M}.
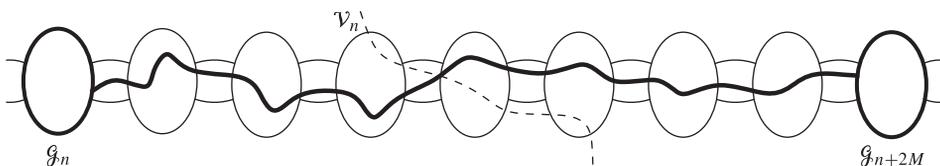

FIGURE 3: If A_n occurs, the thick edges symbolising $\hat{E}_n$ are 'quick'.

We will consider random variables conditioned on the occurrence of events like A_n. Two random variables X and Y will be said to be *conditionally independent* given A, if the random variables X conditioned on A, and Y conditioned on A, are independent.

Lemma 2.1. *For every t' and t'' such that $m_\tau < t' < t'' < M_\tau$, there exists $M \in \mathbb{N}$, such that for all $u \in \bigcup_{k \leq n} \mathbb{V}_{\mathcal{G}_k}$ and $v \in \bigcup_{k \geq n+2M} \mathbb{V}_{\mathcal{G}_k}$:*

(a) *If A_n occurs, then $T(\Gamma) > T(u, v)$ for any path Γ between u and v not visiting $\hat{v}_{n+M}$, and*

$$T(u, v) = T(u, \hat{v}_{n+M}) + T(\hat{v}_{n+M}, v). \tag{2.3}$$

(b) *Given A_n, $T(u, \hat{v}_{n+M})$ and $T(\hat{v}_{n+M}, v)$ are conditionally independent, and, in addition, $T(u, \hat{v}_{n+M})$ is conditionally independent of the passage times of edges beyond level $n + 2M$, and $T(\hat{v}_{n+M}, v)$ is conditionally independent of the passage times of edges before level n.*

Proof. It suffices to prove the lemma for $u \in \mathbb{V}_{\mathcal{G}_n}$ and $v \in \mathbb{V}_{\mathcal{G}_{n+2M}}$. For given t' and t'', choose an integer $M > t'L/(t'' - t')$, where L denotes the cardinality of $\mathbb{E}_{\mathcal{G}_n}$. Set $\beta :=$ $\mathrm{dist}(\hat{v}_{n+M}, \mathbb{V}_{\mathcal{G}_{n+2M}})$, where $\mathrm{dist}(v, V)$ denotes the smallest number of edges one has to pass in order to reach a vertex of V from v, and define (see Figure 3)

$$\mathcal{V}_n := \left\{ v \in \bigcup_{j=n}^{n+2M} \mathbb{V}_{\mathcal{G}_j} : \mathrm{dist}(v, \mathbb{V}_{\mathcal{G}_{n+2M}}) = \beta \right\}.$$

We will prove that, given A_n,

$$T(u, \hat{v}_{n+M}) < T(u, w) \quad \text{and} \quad T(\hat{v}_{n+M}, v) < T(w, v) \tag{2.4}$$

for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. This proves that $T(\Gamma) > T(u, v)$ for all paths Γ between u and v that does not visit $\hat{v}_{n+M}$, since each path from u to v has to pass some vertex in $\mathcal{V}_n$. Thus, (2.3) holds. That $T(u, \hat{v}_{n+M})$ and $T(\hat{v}_{n+M}, v)$ are conditionally independent given A_n is easily seen from the following observation. When A_n occurs, it follows from (2.4) that $T(u, \hat{v}_{n+M})$ is the infimum of $T(\Gamma)$ over all paths Γ from u to $\hat{v}_{n+M}$ that intersects $\mathcal{V}_n$ only at $\hat{v}_{n+M}$, whereas $T(\hat{v}_{n+M}, v)$ is the infimum of $T(\Gamma)$ over all paths Γ from $\hat{v}_{n+M}$ to v that intersects $\mathcal{V}_n$ only at $\hat{v}_{n+M}$. Hence, the infima of passage times are taken over paths in disjoint parts of the graph. The remaining statement in (b) follows similarly.

To deduce (2.4), condition on A_n. First note that, by definition of γ_n and $\mathcal{V}_n$, $T(w', \hat{v}_{n+M}) < T(w', w)$ for any vertex w' visited by γ_n, and $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. Let γ_n^- denote the part of the path γ_n between $\mathbb{V}_{\mathcal{G}_n}$ and $\hat{v}_{n+M}$. Let Γ be any path from u to $\mathcal{V}_n$ disjoint from γ_n^-. Note that

$$T(u, \hat{v}_{n+M}) \leq (L + |\gamma_n^-|)t' \quad \text{and} \quad T(\Gamma) \geq |\gamma_n^-|t''.$$

(Here γ_n^- is identified with its set of edges.) Thus, by the choice of M,

$$T(\Gamma) - T(u, \hat{v}_{n+M}) \geq (t'' - t')|\gamma_n^-| - t'L \geq (t'' - t')M - t'L > 0.$$

This proves that $T(u, \hat{v}_{n+M}) < T(u, w)$ for all $w \in \mathcal{V}_n \setminus \{\hat{v}_{n+M}\}$. The proof of the remaining inequality in (2.4) is similar. QED.
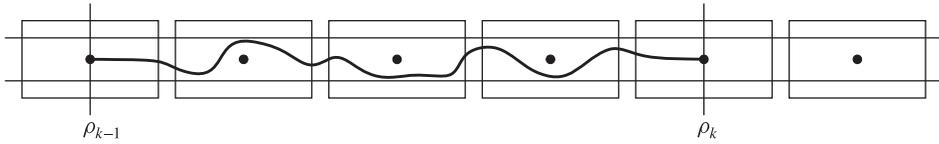
FIGURE 4: A schematic in which boxes indicate locations of the sequence $\{A_{n_k}\}_{k\geq 0}$, vertical lines indicate the sequence $\{\rho_k\}_{k\geq 0}$, and dots indicate $\{\hat{v}_{n_k}\}_{k\geq 0}$. The distance between the two vertical lines is S_k, and the thick curve indicates τ_{S_k}.

We assume from now on that t', t'' and M are chosen in accordance with Lemma 2.1. Let $\rho_I := \max\{n \in \mathbb{Z} : \mathbb{V}_{\hat{g}_n} \cap I \neq \varnothing\}$ denote the furthest initially infected level. Define

$$n_k := \rho_I + k(2M + 1) \quad \text{for } k \in \mathbb{Z},$$

and note that the sequence of events $\{A_{n_k}\}_{k\in\mathbb{Z}}$ is readily seen to be i.i.d. Let $\kappa = \min\{k \geq 0 : A_{n_k} \text{ occurs}\}$ and set $\rho_0 := n_\kappa + M$. Define further

$$\rho_k := M + \min\{n_m : n_m > \rho_{k-1} \text{ and } A_{n_m} \text{ occurs}\} \quad \text{for } k \geq 1.$$

(And analogously for $k < 0$.) By Lemma 2.1 we find that each path along which any vertex at level $\rho_k + M$ and beyond is infected has to pass the vertex $\hat{v}_{\rho_k}$. We will thus refer to the points $\hat{v}_n$, for which $n = \rho_k$ for some $k \geq 0$, as *regeneration points*.

For $k \geq 1$, let S_k denote the distance (measured in levels) between two regeneration points, and τ_{S_k} denotes the passage time between two regeneration points. That is,

$$S_k := \rho_k - \rho_{k-1} \quad \text{and} \quad \tau_{S_k} := T(\hat{v}_{\rho_{k-1}}, \hat{v}_{\rho_k}).$$

A schematic of the passage time between two regeneration points is shown in Figure 4.

By Lemma 2.1 we see that $\tau_{S_k} = T(I, \hat{v}_{\rho_k}) - T(I, \hat{v}_{\rho_{k-1}})$, which immediately gives that

$$\rho_n = \rho_0 + \sum_{k=1}^n S_k \quad \text{and} \quad T(I, \hat{v}_{\rho_n}) = T(I, \hat{v}_{\rho_0}) + \sum_{k=1}^n \tau_{S_k}.$$

Lemma 2.2. *Assume that t', t'' and M are chosen in accordance with Lemma 2.1. Then, $\{(\tau_{S_k}, S_k)\}_{k\in\mathbb{Z}}$ forms a sequence of i.i.d. $[0, \infty) \times \mathbb{Z}_+$-valued random variables.*

Proof. It is easily seen that $\{S_k\}_{k\in\mathbb{Z}}$ is an i.i.d. sequence of geometrically distributed random variables, multiplied by a factor $2M + 1$, since the events A_{n_k} are pairwise independent with equal success probabilities. For the same reason does the distribution of τ_{S_k} not depend on k? Independence of τ_{S_k} and τ_{S_l} for $k < l$ follows from Lemma 2.1 part (b), since τ_{S_l} is independent of ρ_k. QED.

Proposition 2.1. *It holds that $\{T(I, \hat{v}_n)\}_{n\geq 1}$ is a regenerative sequence. Moreover, if t', t'', and M are chosen in accordance with Lemma 2.1, then $\{\rho_n\}_{n\geq 0}$ is a sequence of regenerative levels, and*

$$T(I, v_{\rho_n+m,i}) - T(I, \hat{v}_{\rho_n}) \stackrel{\mathrm{D}}{=} T(I, v_{\rho_1+m,i}) - T(I, \hat{v}_{\rho_1}), \quad \text{for all } m \geq M, n \geq 1,$$

where superscript D indicates that the equality holds in distribution.

Proof. That $\{T(I, \hat{v}_n)\}_{n \geq 1}$ is regenerative with regenerative levels $\{\rho_n\}_{n \geq 0}$ follows from Lemma 2.2. By Lemma 2.1, $T(I, v_{\rho_n+m,i}) - T(I, \hat{v}_{\rho_n})$ equals $T(\hat{v}_{\rho_n}, v_{\rho_n+m,i})$ for $m \geq M$, whose distribution is independent of n, by the i.i.d. nature of the A_{n_k}. QED.

Let $\mu_\tau := \mathrm{E}[\tau_{S_k}]$ and $\mu_S := \mathrm{E}[S_k]$ denote the expected passage time and distance between two regeneration points, respectively, and define

$$\mu := \frac{\mu_\tau}{\mu_S} \quad \text{and} \quad \sigma^2 := \frac{\mathrm{var}(\tau_{S_k} - \mu S_k)}{\mu_S}. \tag{2.5}$$

The constants μ and σ^2 are those featured in the introduction. The next result shows that $\mathrm{E}[\tau_e^\alpha] < \infty$, for $\alpha = 1, 2$ respectively, is sufficient for $0 < \mu < \infty$ and $0 < \sigma^2 < \infty$ to hold.

Proposition 2.2. *Assume that P_τ is not concentrated at a single point. Then,*

(a) *there exists $\alpha > 0$ such that $\mathrm{E}[e^{\alpha S_k}] < \infty$.*

Assume further that there are $p \geq 1$ (edge) disjoint paths from $\hat{v}_0$ to $\hat{v}_1$. Let Y_p denote the minimum of p independent random variables distributed according to P_τ. Then,

(b) *if $\mathrm{E}[Y_p^\alpha] < \infty$, for some $\alpha > 0$, we have $\mathrm{E}[\tau_{S_k}^\alpha] < \infty$.*

Proof. (a) Recall that $S_k/(2M+1)$ is geometrically distributed with parameter $p_A = P(A_n)$. So

$$\mathrm{E}[e^{\alpha S_k}] = \sum_{n=1}^\infty e^{\alpha(2M+1)n}(1 - p_A)^{n-1} p_A = e^{\alpha(2M+1)} p_A \sum_{n=1}^\infty (e^{\alpha(2M+1)}(1 - p_A))^{n-1},$$

which is finite given that $e^{\alpha(2M+1)}(1 - p_A) < 1$.

(b) Let $\eta = \{\eta_e\}_{e \in \mathbb{E}}$ denote the family of indicators where η_e takes on the values $-1, 0, 1$, depending on whether $\{\tau_e \leq t'\}$, $\{\tau_e \in (t', t'')\}$ or $\{\tau_e \geq t''\}$. Independently of $\{\tau_e\}_{e \in \mathbb{E}}$, let $\{\tilde{\tau}_e\}_{e \in \mathbb{E}}$ be an i.i.d. collection of random variables with distribution given by $P(\tilde{\tau}_e \in \cdot) = P(\tau_e \in \cdot \mid \tau_e \geq t'')$, and define $\{\sigma_e\}_{e \in \mathbb{E}}$ as

$$\sigma_e := \begin{cases} \tau_e & \text{if } \eta_e = 1, \\ \tilde{\tau}_e & \text{otherwise.} \end{cases}$$

Note that $\{\sigma_e\}_{e \in \mathbb{E}}$ is an i.i.d. family independent of η, but that η determines $\{A_{n_k}\}_{k \in \mathbb{Z}}$, and hence $\{\rho_j\}_{j \geq 0}$. In particular, $\{\sigma_e\}_{e \in \mathbb{E}}$ and $\{\rho_j\}_{j \geq 0}$ are independent. Let $T'(u, v)$ denote the passage time between u and v in $\mathbb{V}$ with respect to $\{\sigma_e\}_{e \in \mathbb{E}}$. By construction $\tau_e \leq \sigma_e$, so that $T(u, v) \leq T'(u, v)$. We next show that for every $j \in \mathbb{Z}$ and $\alpha > 0$, and for some $C_1 < \infty$,

$$\mathrm{E}[T'(\hat{v}_{j-1}, \hat{v}_j)^\alpha] \leq C_1 \mathrm{E}[Y_p^\alpha]. \tag{2.6}$$

Note that $P(\sigma_e > t) \leq C_2 P(\tau_e > t)$ for some finite C_2. Let $\Gamma_j^{(1)}, \ldots, \Gamma_j^{(p)}$ denote the p disjoint paths from $\hat{v}_{j-1}$ to $\hat{v}_j$. Let λ denote the length of the longest of these paths. Then (2.6)

follows immediately from

$$P(T'(\hat{v}_{j-1}, \hat{v}_j)^\alpha > t) \leq \prod_{i=1,\ldots,p} P(T'(\Gamma_j^{(i)}) > t^{1/\alpha})$$

$$\leq \prod_{i=1,\ldots,p} \lambda P\left(\sigma_e > \frac{t^{1/\alpha}}{\lambda}\right)$$

$$\leq (\lambda C_2)^p P\left(\tau_e > \frac{t^{1/\alpha}}{\lambda}\right)^p$$

$$= (\lambda C_2)^p P\left(Y_p^\alpha > \frac{t}{\lambda^\alpha}\right),$$

where the second inequality follows since $T'(\Gamma_j^{(i)}) \geq s$ implies that some edge in $\Gamma_j^{(i)}$ has $\sigma_e > s/\lambda$.

Set $\Lambda_n := \{S_k = (2M + 1)n\}$. By (2.6), subadditivity and above domination, we deduce that

$$E[\tau_{S_k}^\alpha] \leq \sum_{n=1}^\infty E\left[\left(\sum_{j=\rho_{k-1}+1}^{\rho_k} T'(\hat{v}_{j-1}, \hat{v}_j)\right)^\alpha \Big| \Lambda_n\right] P(\Lambda_n)$$

$$\leq \sum_{n=1}^\infty n^\alpha \sum_{j=1}^n E[T'(\hat{v}_{j-1}, \hat{v}_j)^\alpha \mid \Lambda_n] P(\Lambda_n)$$

$$\leq C_1 \sum_{n=1}^\infty n^{\alpha+1} E[Y_p^\alpha] P(\Lambda_n)$$

$$\leq C_1 E[Y_p^\alpha] E[S_k^{\alpha+1}], \tag{2.7}$$

where the second inequality follows since for any nonnegative numbers a_j we have

$$\left(\sum_{j=1}^n a_j\right)^\alpha \leq \left(n \max_j a_j\right)^\alpha \leq n^\alpha \sum_{j=1}^n a_j^\alpha.$$

Thus, $E[\tau_{S_k}^\alpha] < \infty$ by (2.7) and part (a). QED.

3. Asymptotics for first-passage percolation

The regenerative structure examined in Section 2 will prove helpful when deriving the asymptotic behaviour for the sequence $\{T(I, \hat{v}_n)\}_{n\geq 1}$. It will essentially suffice to study the sequence $\{T(I, \hat{v}_{\rho_k})\}_{k\geq 0}$, which we stop in a suitable way. The resulting object is sometimes referred to as a stopped random walk, for which a general theory has been developed (see, e.g. [14]). Although we occasionally rely on this theory in order to keep the presentation concise, it is often both easy and instructive to derive our results directly from the regenerative behaviour.

Assume throughout that t', t'', and M are chosen in accordance with Lemma 2.1. We will without further comment use the fact that if $X_n \to X$ and $\eta_n \to \infty$ almost surely as $n \to \infty$, then $X_{\eta_n} \to X$ almost surely as $n \to \infty$. We also remind the reader that for any i.i.d. sequence $\{X_n\}_{n\geq 1}$, a simple application of the Borel–Cantelli lemmas shows that

$$\lim_{n\to\infty} \frac{X_n^\alpha}{n} = 0 \quad \text{almost surely} \quad \Longleftrightarrow \quad E[|X_1|^\alpha] < \infty. \tag{3.1}$$

In order to approximate $T(I, \hat{v}_n)$, we will stop the regenerating sequence when A_{n_k} occurs for the least k such that $n_k \geq n$. In terms of the sequence of regenerative levels, we define

$$\nu(n) := \min\{m \geq 0 \colon \rho_m \geq n + M\}.$$

Lemma 3.1. *It hold that* $\{\nu(n)\}_{n \geq 0}$ *is a nondecreasing sequence such that almost surely*

$$\lim_{n \to \infty} \frac{n}{\nu(n)} = \mu_S \quad and \quad \lim_{n \to \infty} \frac{\rho_{\nu(n)}}{n} = 1.$$

Proof. It is clear that $\nu(n) \uparrow \infty$ as $n \to \infty$, almost surely. Lemma 2.2 and Proposition 2.2 ensure that $\{S_k\}_{k \geq 1}$ forms an i.i.d. sequence with finite mean. Since $\rho_{\nu(n)-1} < n + M \leq \rho_{\nu(n)}$, we have

$$\frac{\rho_{\nu(n)}}{\nu(n)} - \frac{S_{\nu(n)}}{\nu(n)} < \frac{n + M}{\nu(n)} \leq \frac{\rho_{\nu(n)}}{\nu(n)}.$$

This, together with the classical law of large numbers and (3.1) proves the first statement. Since $\rho_{\nu(n)}/n = (\rho_{\nu(n)}/\nu(n))(\nu(n)/n)$, the second statement follows from the law of large numbers and the first statement. QED.

To prove Theorems 1.1, 1.2, and 1.3 it suffices to consider the sequence $\{T(I, \hat{v}_n)\}_{n \geq 1}$, since $T(I, v_{n,i})$ and $T(I, \hat{v}_n)$ differ by at most a finite sum of random variables. In fact, it will be sufficient to study $\{T(I, \hat{v}_{\rho_{\nu(n)}})\}_{n \geq 1}$, and then apply the following lemma. Recall that Y_p denotes the minimum of p independent random variables distributed according to P_τ.

Lemma 3.2. *Assume that there are* $p \geq 1$ *(edge) disjoint paths from* $\hat{v}_0$ *to* $\hat{v}_1$. *For every* $\alpha > 0$,

(a) $\lim_{n \to \infty} |\rho_{\nu(n)} - n|^\alpha / n = 0$ *almost surely.*

(b) *If* $\mathrm{E}[Y_p^\alpha] < \infty$, *then* $\lim_{n \to \infty} |T(I, \hat{v}_n) - T(I, \hat{v}_{\rho_{\nu(n)}})|^\alpha / n = 0$ *almost surely.*

Proof. Since $\rho_{\nu(n)} - n \leq S_{\nu(n)} + M$, part (a) can be derived via (3.1) and part (a) of Proposition 2.2. By subadditivity

$$|T(I, \hat{v}_{\rho_{\nu(n)}}) - T(I, \hat{v}_n)| \leq \sum_{j=n+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j) \leq \sum_{j=\rho_{\nu(n)-1}-M+1}^{\rho_{\nu(n)}} T(\hat{v}_{j-1}, \hat{v}_j),$$

which in the proof of Proposition 2.2 was seen to have finite moment of the same order as Y_p. Thus, (b) also follows from (3.1). QED.

3.1. Proof of point-wise limit theorems

Proof of Theorem 1.1. We first show almost sure convergence. Lemma 2.2 and Proposition 2.2 show that $\{\tau_{S_k}\}_{k \geq 1}$ are i.i.d. with finite mean. Thus, as $n \to \infty$,

$$\frac{T(I, \hat{v}_{\rho_{\nu(n)}})}{n} = \frac{T(I, \hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k}}{\nu(n)} \frac{\nu(n)}{n} \to \frac{\mu_\tau}{\mu_S} \quad \text{almost surely,}$$

by the classical law of large numbers and Lemma 3.1. Thus by Lemma 3.2, as $n \to \infty$,

$$\frac{T(I, \hat{v}_n)}{n} = \frac{T(I, \hat{v}_{\rho_{\nu(n)}})}{n} + \frac{T(I, \hat{v}_n) - T(I, \hat{v}_{\rho_{\nu(n)}})}{n} \to \frac{\mu_\tau}{\mu_S} \quad \text{almost surely.}$$

We next prove uniform integrability, as L^1-convergence then follows as a consequence of the almost sure convergence. Assume that $E[\tau_e^r] < \infty$. Since the distributions of $T(I, \hat{v}_{\rho_0})$ and $(T(I, v_{n,i}) - T(I, \hat{v}_{\rho_{v(n)}}))^r$ are independent of n and have finite mean, it suffices to show that $\{((1/n)T(\hat{v}_{\rho_0}, \hat{v}_{\rho_{v(n)}}))^r\}_{n \geq 1}$ is uniformly integrable. However, by subadditivity

$$\frac{1}{n}T(\hat{v}_{\rho_0}, \hat{v}_{\rho_{v(n)}}) \leq \frac{1}{n}T(\hat{v}_{\rho_0}, \hat{v}_{\rho_n}) \leq \frac{1}{n}\sum_{j=1}^{n} T(\hat{v}_{\rho_{j-1}}, \hat{v}_{\rho_j}),$$

so uniform integrability follows from uniform integrability for i.i.d. sequences. L^r-convergence is now immediate from the almost sure convergence and uniform integrability. QED.

Theorem 1.2 will be deduced from the following result sometimes referred to as *Anscombe's theorem*. For a proof, we refer the reader to [14, Theorem 1.3.1].

Lemma 3.3. (Anscombe's theorem.) *Let $\{\xi_k\}_{k \geq 1}$ be an i.i.d. sequence with mean zero and variance σ_ξ^2. Assume further that $\eta(n)/n \xrightarrow{\text{P}} \theta$ in probability as $n \to \infty$. Then, as $n \to \infty$,*

$$\frac{1}{\sigma_\xi \sqrt{\theta n}} \sum_{k=1}^{\eta(n)} \xi_k \xrightarrow{\text{D}} \chi \quad \text{in distribution,}$$

where χ has a standard normal distribution.

Proof of Theorem 1.2. By Lemma 2.2 and Proposition 2.2, $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$ is an i.i.d. sequence with zero mean and finite variance. Anscombe's theorem and Lemma 3.1 give convergence in distribution of the former term in the right-hand side of

$$\frac{T(I, \hat{v}_n) - \mu n}{\sigma\sqrt{n}} = \frac{T(I, \hat{v}_{\rho_{v(n)}}) - \mu\rho_{v(n)}}{\sigma\sqrt{n}} + \frac{T(I, \hat{v}_n) - T(I, \hat{v}_{\rho_{v(n)}}) - \mu(n - \rho_{v(n)})}{\sigma\sqrt{n}},$$

to a standard normal distribution, as $n \to \infty$. The latter vanishes according to Lemma 3.2.

Proof of Theorem 1.3. Recall that $\tau_{S_k} - \mu S_k$ are i.i.d. for $k \geq 1$, with zero mean and finite variance, due to Lemma 2.2 and Proposition 2.2. Trivially

$$\frac{T(I, \hat{v}_n) - \mu n}{\sigma\sqrt{2n \log\log n}} = \frac{T(I, \hat{v}_{\rho_{v(n)}}) - \mu\rho_{v(n)}}{\sigma\sqrt{\mu_S 2v(n) \log\log v(n)}} \sqrt{\mu_S \frac{v(n)}{n}} \sqrt{\frac{\log\log v(n)}{\log\log n}}$$

$$+ \frac{T(I, \hat{v}_n) - T(I, \hat{v}_{\rho_{v(n)}}) - \mu(n - \rho_{v(n)})}{\sigma\sqrt{2n \log\log n}}.$$

Since $v(n)$ is nondecreasing, and for each $m \in \mathbb{Z}_+$, there is an $n \in \mathbb{Z}_+$ such that $v(n) = m$, it follows from (an extended version of) the law of the iterated logarithm for i.i.d. sequences that

$$\mathcal{L}\left(\left\{\frac{T(I, \hat{v}_{\rho_{v(n)}}) - \mu\rho_{v(n)}}{\sigma\sqrt{\mu_S 2v(n) \log\log v(n)}}\right\}_{n:v(n)\geq 3}\right) = [-1, 1] \quad \text{almost surely,}$$

and, in particular, that almost surely

$$\limsup_{n\to\infty} \frac{T(I, \hat{v}_{\rho_{v(n)}}) - \mu\rho_{v(n)}}{\sigma\sqrt{\mu_S 2v \log\log v}} = 1 \quad \text{and} \quad \liminf_{n\to\infty} \frac{T(I, \hat{v}_{\rho_{v(n)}}) - \mu\rho_{v(n)}}{\sigma\sqrt{\mu_S 2v \log\log v}} = -1.$$

Lemma 3.1 gives that $\mu_S \nu(n)/n \to 1$, almost surely, as $n \to \infty$, and we further conclude that

$$\lim_{n \to \infty} \frac{\log \log \nu(n)}{\log \log n} = \lim_{n \to \infty} \frac{\log(\log n + \log(\nu(n)/n))}{\log \log n} = 1 \quad \text{almost surely.}$$

An application of Lemma 3.2 now completes the proof.

3.2. Functional Donsker theorem

Let $D = D[0, \infty)$ denote the set of right-continuous functions with left-hand limits on $[0, \infty)$, and let $\mathcal{D}$ denote the σ-algebra generated by the open sets in D with Skorokhod's J_1-topology. A sequence $\{P_n\}_{n \geq 1}$ of probability measures on $(D, \mathcal{D})$ is said to *converge weakly* to P if

$$\int_D f \, dP_n \to \int_D f \, dP,$$

for all bounded continuous f from D to $\mathbb{R}$; Denote this $P_n \overset{J_1}{\Rightarrow} P$. In spirit with Donsker's theorem, we prove weak convergence of travel times to Wiener measure W.

Theorem 3.1. (Functional Donsker theorem.) *If* $E[\tau_e^2] < \infty$, *then*

$$\frac{T(I, v_{\lfloor nt \rfloor, i}) - \mu \lfloor nt \rfloor}{\sigma \sqrt{n}} \overset{J_1}{\Rightarrow} W \quad \text{as } n \to \infty.$$

As for the point-wise central limit theorem, there is an Anscombe version of Donsker's theorem (cf. [14, Theorem 5.2.1]), from which we will deduce Theorem 3.1.

Lemma 3.4. *Let* $\{\xi_k\}_{k \geq 1}$ *be an i.i.d. sequence of random variables with zero mean and variance* σ_ξ^2. *Assume further that* $\{\eta(n)\}_{n \geq 0}$ *is a nondecreasing sequence of positive, integer valued random variables such that* $\eta(n)/n \to \theta$ *almost surely as* $n \to \infty$. *Then,*

$$\frac{1}{\sigma_\xi \sqrt{\theta n}} \sum_{k=1}^{\eta(\lfloor nt \rfloor)} \xi_k \overset{J_1}{\Rightarrow} W \quad \text{as } n \to \infty.$$

Proof of Theorem 3.1. Lemma 2.2 and Proposition 2.2 ensure that $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$ are i.i.d. with zero mean and finite variance. From Lemma 3.4 it follows that

$$\frac{T(I, \hat{v}_{\rho_{\nu(\lfloor nt \rfloor)}}) - \mu \rho_{\nu(\lfloor nt \rfloor)}}{\sigma \sqrt{n}} \overset{J_1}{\Rightarrow} W \quad \text{as } n \to \infty.$$

It suffices to prove that, as $n \to \infty$,

$$\sup_{0 \leq t \leq b} \left| \frac{T(I, v_{\lfloor nt \rfloor, i}) - T(I, \hat{v}_{\rho_{\nu(\lfloor nt \rfloor)}}) - \mu(\lfloor nt \rfloor - \rho_{\nu(\lfloor nt \rfloor)})}{\sigma \sqrt{n}} \right| \to 0 \quad \text{almost surely.} \quad (3.2)$$

It follows Lemma 3.2 that, as $n \to \infty$,

$$\left| \frac{T(I, v_{n,i}) - T(I, \hat{v}_{\rho_{\nu(n)}}) - \mu(n - \rho_{\nu(n)})}{\sigma \sqrt{n}} \right| \to 0 \quad \text{almost surely.} \quad (3.3)$$

For any sequence of real numbers $\{x_n\}_{n \geq 1}$ and $b \in \mathbb{R}_+$ it holds that

$$\lim_{n \to \infty} \frac{x_n}{\sqrt{n}} = 0 \quad \Longleftrightarrow \quad \lim_{n \to \infty} \frac{\max_{k \leq bn} |x_k|}{\sqrt{n}} = 0.$$

This fact improves (3.3), and we get

$$\frac{\max_{k \le bn} |T(I, v_{k,i}) - T(I, \hat{v}_{\rho_{\nu(k)}}) - \mu(k - \rho_{\nu(k)})|}{\sigma \sqrt{n}} \to 0 \quad \text{almost surely,}$$

as $n \to \infty$. But this is equivalent to (3.2), so the proof is complete.

Remark 3.1. A closer look at the proofs of Theorems 1.1, 1.2, 1.3, and 3.1 reveals that finite moment of τ_{S_k}, for $k \ge 1$, of correct order is sufficient for the various modes of convergence for the sequence $\{T(\hat{v}_0, \hat{v}_n)\}_{n \ge 1}$. According to Proposition 2.2, $E[\tau_e^\alpha] < \infty$ may be replaced by $E[Y_p^\alpha] < \infty$ (for given α), where p is the number of (edge) disjoint paths from $\hat{v}_0$ to $\hat{v}_1$, and Y_p denotes the minimum of p independent random variables distributed according to P_τ.

4. Monotonicity of mean and variance

We consider next the question of monotonicity of expected travel times and their fluctuations. Indeed, the theory for stopped random walks tells us that $E[T(I, \hat{v}_{\rho_{\nu(n)}})] = \mu n + C$ for some constant C, and that $\text{var}(T(I, \hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + o(n)$ for large n (see [14, Theorem 4.2.4]). The former is not far from showing that $E[T(I, \hat{v}_n)]$ is monotone in n. However, an essential additional effort is needed in order to improve the latter statement to the form $\text{var}(T(I, \hat{v}_{\rho_{\nu(n)}})) = \sigma^2 n + C'$, and then prove monotonicity also for $\text{var}(T(I, \hat{v}_n))$. More precisely, we will prove the following.

Theorem 4.1. Let $E[\tau_e] < \infty$. There exists $C = C(i, I, \mathcal{G}, P_\tau) \in \mathbb{R}$ such that

$$E[T(I, v_{n,i})] = \mu n + C + o(1) \quad \text{as } n \to \infty.$$

Theorem 4.2. Let $E[\tau_e^2] < \infty$. There exists $C' = C'(i, I, \mathcal{G}, P_\tau) \in \mathbb{R}$ such that

$$\text{var}(T(I, v_{n,i})) = \sigma^2 n + C' + o(1) \quad \text{as } n \to \infty.$$

In order to keep the presentation neat, we will for the rest of this section require a slight variation in the definitions made in Section 2. We will introduce an auxiliary shift variable Δ, in order to make 'regeneration possible' equally likely for every level. Let Δ be uniformly distributed on $\{0, 1, \ldots, 2M\}$, and redefine n_k (for this section only) to equal $\rho_I + \Delta + k(2M+1)$ for $k \in \mathbb{Z}$, with the sequential changes to ρ_k, S_k, τ_{S_k} and $\nu(n)$. In addition, as it is just a matter of vertex labelling, we will for simplicity assume that $\rho_I = 0$.

Lemma 4.1. Assume that $\rho_I = 0$. Then for $n \ge 0$, $\quad E[\nu(n)] = n/\mu_S$.

Proof. We may interpret $\nu(n)$ as the number of regeneration points before (but not including) level $n + M$, that is, the number of $k \ge 0$ such that A_{n_k} occurs for $n_k < n$. Since $n_0 = \Delta$, this number is at most $n_A = \lfloor (n + 2M - \Delta)/(2M+1) \rfloor$. The shift Δ is independent of $\{\tau_e\}_{e \in \mathbb{E}}$, and conditioned on Δ, we can think of $\nu(n)$ as the number of successes in n_A independent Bernoulli trials, each with success probability $p_A = P(A_{n_k})$. Conditioning on Δ, we see that

$$E[\nu(n)] = p_A E\left[\left\lfloor \frac{n + 2M - \Delta}{2M + 1} \right\rfloor\right]. \tag{4.1}$$

If $n - \rho_I = (2M + 1)k$, for some $k \ge 0$, we realise from (4.1) that

$$E[\nu(n)] = \frac{p_A}{2M + 1}(2M + 1)k = \frac{n}{\mu_S},$$

where the latter equality follows since S_k is geometrically distributed with parameter p_A, multiplied by a factor $2M + 1$, that is, $\mu_S = (2M + 1)/p_A$. Again from (4.1), we realise that as n increases from $(2M + 1)k$ to $(2M + 1)k + 2M$, then $E[\nu(n)]$ will have to increase with $p_A/(2M + 1)$ for each step. QED.

The proofs of Theorems 4.1 and 4.2 will both make use of Wald's lemma.

Lemma 4.2. (Wald's lemma.) *Let $\xi_1, \xi_2, \ldots$ be i.i.d. random variables with mean μ_ξ, and set $S_n = \sum_{k=1}^n \xi_k$. Let N be a stopping time with $E[N] < \infty$.*

(a) $E[S_N] = \mu_\xi E[N]$.

(b) *If $\sigma_\xi^2 = \text{var}(\xi_1) < \infty$, then $E[(S_N - \mu_\xi N)^2] = \sigma_\xi^2 E[N]$.*

(c) *If X is independent of $\xi_1, \xi_2, \ldots$, then $E[X S_N] = \mu_\xi E[X N]$. In particular, $\text{cov}(X, S_N) = \mu_\xi \text{cov}(X, N)$.*

The third part of the lemma is a slight extension of the first part, and proved in an analogous way. If $\mathcal{F}_n = \sigma(\{(\rho_0, T(I, \hat{v}_{\rho_0})), (S_1, \tau_{S_1}), \ldots, (S_n, \tau_{S_n})\})$, then it is immediate from the definition that $\nu(n)$ is a stopping time with respect to the sequence of σ-algebras $\{\mathcal{F}_n\}_{n \geq 1}$.

Proof of Theorem 4.1. Without loss of generality, assume that $\rho_I = 0$. Wald's lemma and Lemma 4.1 gives

$$E[T(I, \hat{v}_{\rho_{\nu(n)}})] = E\left[T(I, \hat{v}_{\rho_0}) + \sum_{k=1}^{\nu(n)} \tau_{S_k}\right] = E[T(I, \hat{v}_{\rho_0})] + \mu n.$$

It remains to prove that there is a finite constant $C = C(i, I, \mathcal{G}, P_\tau)$ such that

$$E[T(I, v_{n,i}) - T(I, \hat{v}_{\rho_{\nu(n)}})] \to C \quad \text{as } n \to \infty. \tag{4.2}$$

Arguments of the type that we use to prove (4.2) will be used repeatedly in the proof of Theorem 4.2. For this reason, we present the argument in detail here. To make the argument clear, we will define a random variable to which $T(I, v_{n,i}) - T(I, \hat{v}_{\rho_{\nu(n)}})$ will converge in distribution. The limit C will then equal the expectation of this random variable.

Recall that $n_k = \Delta + k(2M + 1)$ for $k \geq 0$, set $m_{n,k} := n - (2M + 1)k$ for $k \geq 1$, and let

$$r_+ := M + \min\{n_k \geq 0 : A_{n_k} \text{ occurs}\},$$
$$r_0 := M + \max\{m_{0,k} < 0 : A_{m_{0,k}} \text{ occurs}\}.$$

Observe that r_+ denotes the first element of the sequence $\{\rho_k\}_{k \geq 0}$ greater than zero, whereas r_0 is not defined along the same subsequence of the integers as $\{\rho_k\}_{k \geq 0}$. We introduce

$$Y_{k,i} := T(\hat{v}_{r_0}, v_{k,i}) \quad \text{and} \quad Y_+ := T(\hat{v}_{r_0}, \hat{v}_{r_+}),$$

and the events

$$D_{T,n} := \{A_{m_{n,k}} \text{ occurs for some } k \text{ such that } 0 \leq m_{n,k} < n\},$$
$$D_{Y,n} := \{A_{m_{0,k}} \text{ occurs for some } k \text{ such that } 0 \leq m_{0,k} + n < n\}.$$

Clearly $P(D_{T,n}) = P(D_{Y,n}) \to 1$ as $n \to \infty$. Moreover,

$$(T(I, v_{n,i}) - T(I, \hat{v}_{\rho_{\nu(n)}})) \cdot \mathbf{1}_{D_{T,n}} \overset{\text{D}}{=} (Y_{0,i} - Y_+) \cdot \mathbf{1}_{D_{Y,n}}.$$

So, if we let $T^* = T(I, v_{n,i}) - T(I, \hat{v}_{\rho v(n)})$ and $Y^* = Y_{0,i} - Y_+$, then as $n \to \infty$,

$$T^* = T^*(\mathbf{1}_{D_{T,n}} + \mathbf{1}_{D_{T,n}^c}) \overset{\mathrm{D}}{=} Y^* + T^* \cdot \mathbf{1}_{D_{T,n}^c} - Y^* \cdot \mathbf{1}_{D_{Y,n}^c} \overset{\mathrm{D}}{\to} Y^*. \qquad (4.3)$$

If, in addition, $\{T(I, v_{n,i}) - T(I, \hat{v}_{\rho v(n)})\}_{n \geq 1}$ is uniformly integrable, then the convergence carries over in mean, and we would have proved (4.2) with $C = \mathrm{E}[Y_{0,i} - Y_+]$.

To deduce uniform integrability, note that subadditivity gives

$$T(I, v_{n,i}) - T(I, \hat{v}_{\rho v(n)}) \leq T(v_{n,i}, \hat{v}_n) + \sum_{j=n+1}^{\rho v(n)} T(\hat{v}_{j-1}, \hat{v}_j). \qquad (4.4)$$

But, the distribution of the right-hand side of (4.4) does not depend on n. Thus, it suffices to see that it has finite expectation. This is easily achieved in an analogous way as in (2.7) in the proof of Proposition 2.2, conditioning on $\Lambda_k = \{\rho v(n) - n = k\}$. We omit the details. QED.

The proof of Theorem 4.2 needs considerable extra work, due to arising covariance terms. Moment convergence arguments similar to the one used to prove (4.2) will be used repeatedly.

Proof of Theorem 4.2. Assume that $\rho_I = 0$. To begin with,

$$\begin{aligned}
\mathrm{var}(T(I, v_{n+2M,i})) = {} & \mathrm{var}(T(I, \hat{v}_{\rho v(n)}) - \mu \rho v(n)) \\
& + \mathrm{var}(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho v(n)}) + \mu \rho v(n)) \\
& + 2\,\mathrm{cov}(T(I, \hat{v}_{\rho v(n)}) - \mu \rho v(n), T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho v(n)})) \\
& + 2\mu\,\mathrm{cov}(T(I, \hat{v}_{\rho v(n)}), \rho v(n)) - 2\mu^2\,\mathrm{var}(\rho v(n)). \qquad (4.5)
\end{aligned}$$

We will have to treat each of the terms on the right-hand side one by one. Consider the first term in the right-hand side in (4.5), and note that

$$\begin{aligned}
\mathrm{var}(T(I, \hat{v}_{\rho v(n)}) - \mu \rho v(n)) = {} & \mathrm{var}\left(T(I, \hat{v}_{\rho_0}) - \mu \rho_0 + \sum_{k=1}^{v(n)} (\tau_{S_k} - \mu S_k) \right) \\
= {} & \mathrm{E}\left[\left(\sum_{k=1}^{v(n)} (\tau_{S_k} - \mu S_k) \right)^2 \right] + \mathrm{var}(T(I, \hat{v}_{\rho_0}) - \mu \rho_0) \\
& + 2\,\mathrm{cov}\left(\sum_{k=1}^{v(n)} (\tau_{S_k} - \mu S_k), T(I, \hat{v}_{\rho_0}) - \mu \rho_0 \right).
\end{aligned}$$

So, an application of parts (b) and (c) of Wald's lemma, together with Lemma 4.1, yield

$$\begin{aligned}
\mathrm{var}(T(I, \hat{v}_{\rho v(n)}) - \mu \rho v(n)) = {} & \mathrm{var}(\tau_{S_k} - \mu S_k)\,\mathrm{E}[v(n)] + \mathrm{var}(T(I, \hat{v}_{\rho_0}) - \mu \rho_0) \\
& + \mathrm{E}[\tau_{S_k} - \mu S_k]\,\mathrm{cov}(v(n), T(I, \hat{v}_{\rho_0}) - \mu \rho_0) \\
= {} & \sigma^2 n + \mathrm{var}(T(I, \hat{v}_{\rho_0}) - \mu \rho_0).
\end{aligned}$$

Next, to conclude that $\mathrm{var}(\rho v(n))$ is constant, interpret $\rho v(n)$ as the level of the first regeneration after level n. Since a regeneration is equally likely to occur at any level, due to the shift variable Δ, it follows that $\mathrm{var}(\rho v(n)) = \mathrm{var}(\rho v(n) - n)$ is independent of n, and therefore constant.

The remaining three terms in the right-hand side of (4.5) will in some way or other need an argument similar to that used to prove (4.2). Recall the notation used for that purpose.

Claim 1. *We claim that* $\mathrm{var}(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}) + \mu\rho_{v(n)}) = \mathrm{var}(Y_{2M,i} - Y_+ + \mu r_+) + o(1)$.

To prove the claim, note that we, in an analogous way as in (4.3), may divide into cases whether $D_{T,n}$ and $D_{Y,n}$ occur or not, to show that as $n \to \infty$

$$T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}) + \mu(\rho_{v(n)} - n) \xrightarrow{\mathrm{D}} Y_{2M,i} - Y_+ + \mu r_+.$$

We require uniform integrability of $\{(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}) + \mu(\rho_{v(n)} - n))^2\}_{n \geq 1}$. This can be proved similar to the uniform integrability needed for (4.2), via (4.4). It follows that for $r = 1, 2$, as $n \to \infty$

$$\mathrm{E}[(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}) + \mu(\rho_{v(n)} - n))^r] \to \mathrm{E}[(Y_{2M,i} - Y_+ + \mu r_+)^r],$$

from which the claim follows.

Claim 2. *We claim that*

$$\mathrm{cov}(T(I, \hat{v}_{\rho_{v(n)}}), \rho_{v(n)}) = \mathrm{cov}(Y_+, r_+) + o(1).$$

To prove the second claim, set $r_n := M + \max\{m_{n,k} < n : A_{m_{n,k}} \text{ occurs}\}$, and rewrite

$$\mathrm{cov}(T(I, \hat{v}_{\rho_{v(n)}}), \rho_{v(n)}) = \mathrm{cov}(T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}), \rho_{v(n)} - n)$$
$$+ \mathrm{cov}(T(I, \hat{v}_{r_n}), \rho_{v(n)} - n).$$

It is easy to see that $\rho_{v(n)} - n \overset{\mathrm{D}}{=} r_+$ for $n \geq 0$. Partitioning on whether $D_{T,n}$ and $D_{Y,n}$ occur or not, we find that, as $n \to \infty$,

$$T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}) \xrightarrow{\mathrm{D}} Y_+ \quad \text{and} \quad (T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}))(\rho_{v(n)} - n) \xrightarrow{\mathrm{D}} Y_+ r_+.$$

The uniform integrability of $\{(\rho_{v(n)} - n)^2\}_{n \geq 1}$ and $\{(T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}))^2\}_{n \geq 1}$ is possible to deduce, in a similar way as before, by conditioning on $\Lambda_k = \{\rho_{v(n)} - r_n = k\}$. This implies that also $\{(T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}))(\rho_{v(n)} - n)\}_{n \geq 1}$ is uniformly integrable. We conclude that, as $n \to \infty$

$$\mathrm{cov}(T(I, \hat{v}_{\rho_{v(n)}}) - T(I, \hat{v}_{r_n}), \rho_{v(n)} - n) \to \mathrm{E}[Y_+ r_+] - \mathrm{E}[Y_+]\mathrm{E}[r_+] = \mathrm{cov}(Y_+, r_+).$$

On the event $D_{T,n}$, $T(I, \hat{v}_{r_n})$ depends on passage times below level n, but not on Δ, whereas $\rho_{v(n)} - n$ is independent of passage times below level n, and hence on $D_{T,n}$. It follows that

$$\mathrm{E}[T(I, \hat{v}_{r_n})(\rho_{v(n)} - n)\mathbf{1}_{D_{T,n}}] = \mathrm{E}[T(I, \hat{v}_{r_n})\mathbf{1}_{D_{T,n}}]\mathrm{E}[r_+].$$

In particular,

$$\begin{aligned}
\mathrm{cov}(T(I, \hat{v}_{r_n}), \rho_{v(n)} - n) &= \mathrm{E}[T(I, \hat{v}_{r_n})(\rho_{v(n)} - n)\mathbf{1}_{D_{T,n}}] \\
&\quad - \mathrm{E}[T(I, \hat{v}_{r_n})\mathbf{1}_{D_{T,n}}]\mathrm{E}[r_+] \\
&\quad + \mathrm{E}[T(I, \hat{v}_{r_n})(\rho_{v(n)} - n)\mathbf{1}_{D_{T,n}^c}] \\
&\quad - \mathrm{E}[T(I, \hat{v}_{r_n})\mathbf{1}_{D_{T,n}^c}]\mathrm{E}[r_+] \\
&= \mathrm{E}[T(I, \hat{v}_{r_n})(\rho_{v(n)} - n - \mathrm{E}[r_+])\mathbf{1}_{D_{T,n}^c}].
\end{aligned}$$

As $n \to \infty$, this expression vanishes, since on $D_{T,n}^c$ then $r_n \leq M$, and we may obtain an upper bound on $T(I, \hat{v}_{r_n})(\rho_{v(n)} - n - \mathrm{E}[r_+])$ as in the proof of Proposition 2.2. The claim follows.

Claim 3. *We claim that*

$$\lim_{n \to \infty} \mathrm{cov}(T(I, \hat{v}_{\rho_{\nu(n)}}) - \mu \rho_{\nu(n)}, T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{\nu(n)}}))$$

exists finitely.

Once this final claim is proved, we have proved existence of a finite constant C' such that

$$\mathrm{var}(T(I, v_{n+2M,i})) = \sigma^2 n + C' + o(1) \quad \text{as } n \to \infty,$$

and hence the theorem. To see that the claim is true, we will 'reverse' our point of view in the following sense. The sequence $\{\tau_{S_k} - \mu S_k\}_{k=1}^{\nu(n)}$ has until now been considered as a sequence started at $k = 1$ and stopped at $k = \nu(n)$. But, we can equally well see $\{\tau_{S_k} - \mu S_k\}_{k=0}^{\nu(n)-1}$ as a sequence in the opposite direction (where $S_0 = \rho_0 - \rho_{-1}$ and $\tau_{S_0} = T(\hat{v}_{\rho_{-1}}, \hat{v}_{\rho_0})$). That is, as a sequence started at the first point of regeneration 'left' of level $n + M$, and stopped at the first point of regeneration 'left' of level 0.

Let $T^* = T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{\nu(n)}})$. On the event $\{\nu(n) \geq 1\}$, T^* may be expressed as $T(\hat{v}_{\rho_{\nu(n)-1}}, v_{n+2M,i}) - T(\hat{v}_{\rho_{\nu(n)-1}}, \hat{v}_{\rho_{\nu(n)}})$ and is independent of $\tau_{S_k} - \mu S_k$ for $k < \nu(n)$. The event $\{\nu(n) \geq 1\}$ is itself independent of $\{\tau_{S_k} - \mu S_k\}_{k \geq 1}$. This allows us to apply the first, and later, the third part of Wald's lemma to obtain

$$\mathrm{cov}\left(\sum_{k=1}^{\nu(n)} \tau_{S_k} - \mu S_k, T^*\right) = \mathrm{E}\left[\sum_{k=1}^{\nu(n)} (\tau_{S_k} - \mu S_k) T^*\right]$$

$$= \mathrm{E}\left[\sum_{k=1}^{\nu(n)} (\tau_{S_k} - \mu S_k) T^* \mathbf{1}_{\{\nu(n) \geq 1\}}\right]$$

$$= \mathrm{E}[(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}) T^* \mathbf{1}_{\{\nu(n) \geq 1\}}]$$
$$- \mathrm{E}[(\tau_{S_0} - \mu S_0) T^* \mathbf{1}_{\{\nu(n) \geq 1\}}].$$

Let $r_- := M + \max\{n_k < 0 : A_{n_k} \text{ occurs}\}$, $Y_- := T(\hat{v}_{r_-}, \hat{v}_{r_+})$ and $Z_{k,i} := T(\hat{v}_{r_-}, v_{k,i})$. Note that

$$(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}) T^* \mathbf{1}_{\{\nu(n) \geq 1\}} \overset{\mathrm{D}}{=} (Y_- - \mu(r_+ - r_-))(Z_{2M,i} - Y_-) \mathbf{1}_H,$$

where $H = \{A_{n_k} \text{ occurs for some } \Delta - n \leq n_k < 0\}$. That the random variables Y_- and $Z_{2M,i}$ have finite variance is concluded as in the proof of Proposition 2.2 (conditioning on $\Lambda_n = \{r_+ - r_- = n\}$). Thus, an application of the monotone convergence theorem shows that

$$\mathrm{E}[(\tau_{S_{\nu(n)}} - \mu S_{\nu(n)}) T^* \mathbf{1}_{\{\nu(n) \geq 1\}}] = \mathrm{E}[(Y_- - \mu(r_+ - r_-))(Z_{2M,i} - Y_-)] + o(1),$$

which is then finite. Moreover, due to the conditional independence between $\tau_{S_0} - \mu S_0$ and T^*, given $\{\nu(n) \geq 1\}$, it follows that

$$\mathrm{E}[(\tau_{S_0} - \mu S_0) T^* \mathbf{1}_{\{\nu(n) \geq 1\}}]$$

$$= \mathrm{E}[(\tau_{S_0} - \mu S_0) \mathbf{1}_{\{\nu(n) \geq 1\}}] \mathrm{E}[T^* \mathbf{1}_{\{\nu(n) \geq 1\}}] \frac{1}{P(\nu(n) \geq 1)}$$

which vanishes as $n \to \infty$, again via an application of the monotone convergence theorem.

It remains for us to measure the correlation between $T(I, \hat{v}_{\rho_0}) - \mu \rho_0$ and T^*. Let $Z'_{2M,i}$ and Y'_- be defined in the same way as $Z_{2M,i}$ and Y_- above, but now for a set of passage times $\{\tau'_e\}_{e \in \mathbb{E}}$

independent of $\{\tau_e\}_{e \in \mathbb{E}}$ (that defines $T(I, \hat{v}_{\rho_0}) - \mu\rho_0$), but with the same Δ. By conditioning on the events $\{v(n) \geq 1\}$ (with respect to $\{\tau_e\}_{e \in \mathbb{E}}$) and H (with respect to $\{\tau'_e\}_{e \in \mathbb{E}}$), we see that

$$T^* \xrightarrow{\text{D}} (Z'_{2M,i} - Y'_-) \quad \text{and} \quad T^*(T(I, \hat{v}_{\rho_0}) - \mu\rho_0) \xrightarrow{\text{D}} (Z'_{2M,i} - Y'_-)(T(I, \hat{v}_{\rho_0}) - \mu\rho_0),$$

as $n \to \infty$. That $\{(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}))^2\}_{n \geq 1}$ is uniformly integrable was argued for during the proof of the first claim, and $T(I, \hat{v}_{\rho_0}) - \mu\rho_0$ has finite variance. Consequently $\{(T(I, v_{n+2M,i}) - T(I, \hat{v}_{\rho_{v(n)}}))(T(I, \hat{v}_{\rho_0}) - \mu\rho_0)\}_{n \geq 1}$ is also uniform integrable, and we have that

$$\text{cov}(T(I, \hat{v}_{\rho_0}) - \mu\rho_0, T^*) = \text{cov}(T(I, \hat{v}_{\rho_0}) - \mu\rho_0, Z'_{2M,i} - Y'_-) + o(1).$$

This ends the proof of the claim, and hence the theorem.

5. Comparison between time constants

We begin with a coupling that allows us to compare the first-passage process on a graph with its subgraphs. Let $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$ be i.i.d. passage times associated to the edges of the $\mathbb{Z}^d$ lattice, and denote by $T_K(u, v)$ the passage time with respect to $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$ between u and v when only paths visiting vertices in $\mathbb{Z} \times \{0, \ldots, K-1\}^{d-1}$ are allowed. This produces a simultaneous coupling of the passage time on (K, d)-tubes for all $K \geq 1$. Trivially, $T_{K+1}(u, v) \leq T_K(u, v)$ for any u and v in $\mathbb{Z} \times \{0, \ldots, K-1\}^{d-1}$.

Recall that μ_K denotes the time constant associated with the (K, d)-tube, and assume that P_τ has finite mean, or that $K \geq 3$ and Y_{2d} has finite mean, so that μ_K is well defined.

Proposition 5.1. *For all $K \geq 1$, $\mu_{K+1} < \mu_K$.*

Proof. Let A_n^K be the event defined in (2.2) with respect to the (K, d)-tube, for γ_n chosen to be the straight line segment between the two points $(n, K, 0, \ldots, 0)$ and $(n+2M, K, 0, \ldots, 0)$. It follows from Lemma 2.1 that if A_n^{K+1} occurs, then

$$\delta := T_K(n e_1, (n+2M)e_1) - T_{K+1}(n e_1, (n+2M)e_1) > 0.$$

Thus, if $m_k = (2M+1)k$, then

$$T_{K+1}(\mathbf{0}, m_k e_1) + \delta \sum_{j=0}^{k-1} \mathbf{1}_{A_{m_j}^{K+1}} \leq T_K(\mathbf{0}, m_k e_1)$$

for all $k \geq 0$. The claimed statement now follows by dividing by m_k, and sending k to infinity. QED.

We modify the above coupling slightly, and let $\tilde{T}_K(u, v)$ denote the passage time with respect to $\{\tau_e\}_{e \in \mathbb{E}_{\mathbb{Z}^d}}$, between u and v, over paths restricted to the $\mathbb{Z} \times \{-K, \ldots, K\}^{d-1}$ nearest neighbour graph. This produces a simultaneous coupling of the passage time on $(2K+1, d)$-tubes.

Proposition 5.2. *Let $\lim_{K \to \infty} \mu_K = \mu(e_1)$.*

Proof. Clearly $\tilde{T}_K(\mathbf{0}, \mathbf{n}) \geq \tilde{T}_{K+1}(\mathbf{0}, \mathbf{n})$, and $T(\mathbf{0}, \mathbf{n}) = \lim_{K \to \infty} \tilde{T}_K(\mathbf{0}, \mathbf{n})$. An application of the monotone convergence theorem shows that

$$\text{E}[T(\mathbf{0}, \mathbf{n})] = \lim_{K \to \infty} \text{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})] = \inf_{K \geq 0} \text{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})].$$

Since the limit $\lim_{n\to\infty}(1/n)a_n$ exists and equals $\inf_{n\geq 1}(1/n)a_n$ for any subadditive real-valued sequence $\{a_n\}_{n\geq 1}$, we have for any $K \geq 1$ (including $K = \infty$, corresponds to the $\mathbb{Z}^d$ lattice) that

$$\mu_{2K+1} = \lim_{n\to\infty} \frac{1}{n} \mathrm{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})] = \inf_{n\geq 1} \frac{1}{n} \mathrm{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})].$$

Thus, since μ_K is nonincreasing in K

$$\lim_{K\to\infty} \mu_{2K+1} = \inf_{K\geq 0} \inf_{n\geq 1} \frac{1}{n} \mathrm{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})] = \inf_{n\geq 1} \inf_{K\geq 0} \frac{1}{n} \mathrm{E}[\tilde{T}_K(\mathbf{0}, \mathbf{n})] = \mu(\mathbf{e}_1).$$

QED.

Remark 5.1. It is not clear that the sequence $\{\sigma_K\}_{K\geq 1}$ should also be decreasing in K. However, the subdiffusive behaviour discovered in [6] suggests that we should have $\lim_{K\to\infty} \sigma_K = 0$. Although we do not know how to prove this, we mention that the argument of [6] can be adapted to deduce this behaviour for the corresponding quantities for the (K, d)-*cylinder*, obtained from the (K, d)-tube by connecting opposing vertices on the boundary. The argument breaks down in the case of (K, d)-tubes due to the lack of symmetry.

6. Exact coupling and a 0–1 law

A *coupling* of two random variables $X \sim P$ and $Y \sim P'$ on a measurable space $(E, \mathcal{E})$, is a joint distribution $\hat{P}$ of (X, Y), i.e. a measure on $(E^2, \mathcal{E}^2)$, such that its marginal distributions coincide with P and P'. We call a coupling of two time-dependent random elements $\{X_t\}_{t\geq 0}$ and $\{Y_t\}_{t\geq 0}$ *exact* if with probability 1 there exists a $T_c < \infty$ such that $X_t = Y_t$ for all $t \geq T_c$.

We will present an exact coupling of the evolutions of two first-passage processes started from different initial configurations. Let $\mathcal{R}_+$ denote the Borel σ-algebra on $[0, \infty)$. Then, $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ are random elements on $([0, \infty)^{\mathbb{E}}, \mathcal{R}_+^{\mathbb{E}})$, each with distribution given by the product measure $P_\tau^{\mathbb{E}}$. Let $\mathbb{E}_n$ denote the set of edges between level $-n$ and n, but not including edges between two vertices at level $-n$ and n. In the same manner $\mathbb{E}_n^c$ denotes the set of edges at and before level $-n$, as well as at level n and beyond.

Proposition 6.1. (Coupling, continuous times.) *Let I and I' be finite subsets of the set of vertices $\mathbb{V}$ of an essentially one-dimensional periodic graph $\mathcal{G}$. Assume that the passage time distribution P_τ has an absolutely continuous component (with respect to Lebesgue measure). For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e\in\mathbb{E}_m^c}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e\in\mathbb{E}_m}$ and $\{\tau'_e\}_{e\in\mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e\in\mathbb{E}}$ and $\{\tau'_e\}_{e\in\mathbb{E}}$ are given by the product measure $P_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e\in\mathbb{E}})$ and $(I', \{\tau'_e\}_{e\in\mathbb{E}})$, respectively, then with probability 1 there exists an $N_c < \infty$ such that*

$$T(I, v_{n,i}) = T'(I', v_{n,i}) \quad and \quad B_t = B'_t,$$

for all i, all $n \geq N_c$ and all $t \geq N_c$.

When the passage time distribution P_τ is discrete, i.e. $P_\tau(\Lambda) = 1$ for the set of point masses

$$\Lambda := \{t_j \in [0, \infty) : P_\tau(t_j) > 0\},$$

the statement of Proposition 6.1 is not true in general (see the remark at the end of Section 6.2). In the discrete case, we will therefore restrict our attention to the case of (K, d)-tubes.

Proposition 6.2. (Coupling, discrete times.) *Let I and I' be finite subsets of the set of vertices $\mathbb{V}$ of the (K, d)-tube, for some K, and $d \geq 2$. Assume that the passage time distribution P_τ is such that $P_\tau(\Lambda) = 1$ for the set of point masses Λ, and that either of the following hold:*

(a) *there are $t_j \in \Lambda$ and integers n_j for j in some finite set of indices J^*, such that*

$$\sum_{j \in J^*} n_j \text{ is odd} \quad and \quad \sum_{j \in J^*} n_j t_j = 0;$$

(b) *$\mathrm{dist}(\boldsymbol{x}, \boldsymbol{y})$ is even, for all $\boldsymbol{x} \in I$, $\boldsymbol{y} \in I'$.*

For any $m \geq 0$, there exists a coupling of $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ such that if $\{\tau_e\}_{e \in \mathbb{E}_m}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m}$ each have distribution $P_\tau^{\mathbb{E}_m}$, then the marginal distributions of $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$ are given by the product measure $P_\tau^{\mathbb{E}}$, and such that if first-passage percolation is performed with $(I, \{\tau_e\}_{e \in \mathbb{E}})$ and $(I', \{\tau'_e\}_{e \in \mathbb{E}})$, respectively, then with probability one there exists an $N_c < \infty$ such that

$$T(I, v_{n,i}) = T'(I', v_{n,i}) \quad and \quad B_t = B'_t,$$

for all i, all $n \geq N_c$ and all $t \geq N_c$.

Before we construct the couplings, we focus on the promised 0–1 law that follows from Propositions 6.1 and 6.2. Let $\mathcal{F}_t := \sigma(\{B_s\}_{0 \leq s \leq t})$, $\mathcal{T}_t = \sigma(\{B_s\}_{s \geq t})$ and let $\mathcal{T} = \bigcap_{t \geq 0} \mathcal{T}_t$. As before, B_s denotes the set of infected vertices at time s, and, loosely speaking, we may think of $\mathcal{T}_t$ as the σ-algebra of events $A \in \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ that do not depend on the times at which vertices were infected before time t.

Theorem 6.1. (0–1 law.) *Consider first-passage percolation performed under the assumptions of either Propositions 6.1 or 6.2. Then $P(A) \in \{0, 1\}$ for any event $A \in \mathcal{T}$.*

For the proof we will use *Lévy's 0–1 law*. It states that for σ-algebras $\{\mathcal{F}_t\}_{t \geq 0}$ such that $\mathcal{F}_t \uparrow \mathcal{F}_\infty$ as $t \to \infty$, if $A \in \mathcal{F}_\infty$, then $P(A|\mathcal{F}_t) \to \mathbf{1}_A$, as $n \to \infty$, almost surely. A proof for the discrete case can be found in [10]. The continuous case follows via the martingale convergence theorem.

Proof of Theorem 6.1 from Propositions 6.1 and 6.2. Consider two infections with the respective sets of passage times $\{\tau_e\}_{e \in \mathbb{E}}$ and $\{\tau'_e\}_{e \in \mathbb{E}}$. For $t \geq 0$, let $\mathcal{F}_t$ and $\mathcal{F}'_t$ be σ-algebras generated by their respective realisations up to time t. Let

$$v_t = \max\{n \geq 0 : (B_t \cup B'_t) \cap (\mathbb{V}_{\mathring{g}_n} \cup \mathbb{V}_{\mathring{g}_{-n}}) \neq \varnothing\}$$

denote the furthest level (in positive or negative direction) infected at time t. Clearly $v_t < \infty$ almost surely, for every $t < \infty$.

For any fixed $t \geq 0$, by Propositions 6.1 and 6.2, there is a coupling of $\{\tau_e\}_{e \in \mathbb{E}_{v_t+1}^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_{v_t+1}^c}$, such that there exists an almost surely finite time N_c, such that $B_s = B'_s$ for all $s \geq N_c$. Since $A \in \mathcal{T}_{N_c}$, the outcome of A only depends on B_s for $s \geq N_c$. In particular

$$P(A \mid \mathcal{F}_t) = P(A \mid \mathcal{F}'_t).$$

Thus, $P(A \mid \mathcal{F}_t)$ is nonrandom and equals $P(A)$, for all $t \geq 0$. But, according to Lévy's 0–1 law, $P(A \mid \mathcal{F}_t) \to \mathbf{1}_A$ as $t \to \infty$, almost surely. So $P(A) = \mathbf{1}_A \in \{0, 1\}$ almost surely. QED.

6.1. Exact coupling of time-delayed infections on $\mathbb{Z}$

Before proving Propositions 6.1 and 6.2, we shall first provide a coupling of two infections on $\mathbb{Z}$, where one is delayed for some time T_{delay}. This lemma will figure as a key step in the proof of Propositions 6.1 and 6.2.

Lemma 6.1. *Let T_{delay} be a nonnegative constant, and assume that either of the following hold:*

(a) *P_τ has an absolutely continuous component (with respect to Lebesgue measure).*

(b) *P_τ is such that for some finite index set J, there are nonnegative integers n_j and n'_j, such that $\sum_{j\in J} n_j = \sum_{j\in J} n'_j$, and for atoms $t_j \in \Lambda$ of P_τ*

$$\sum_{j\in J} n_j t_j = \sum_{j\in J} n'_j t_j + T_{\text{delay}}. \tag{6.1}$$

Then, there exists a coupling of $\{\tau_k\}_{k\geq 1}$ and $\{\tau'_k\}_{k\geq 1}$ such that their marginal distributions are that of i.i.d. random variables with distribution P_τ, and such that almost surely

$$\sum_{k=1}^n \tau_k = T_{\text{delay}} + \sum_{k=1}^n \tau'_k \quad \text{for large } n. \tag{6.2}$$

The key to proving this lemma is to identify suitable random walks. The identification of the random walk in case (a) heavily exploits ideas similar to those found in [21, Chapter III.5]. In case (b), a multi dimensional random walk will be based on condition (6.1). This walk is then coupled with techniques described in [21, Chapter II.12–17].

Proof of case (a). Let $[a, b]$ be an interval on which P_τ has density $\geq c$, for some $c > 0$. Define

$$\delta := \max\left\{ d \geq 0 : d \leq \frac{b-a}{2},\ d = \frac{T_{\text{delay}}}{m} \text{ for some } m \in \mathbb{N} \right\},$$

and couple $\{\tau_k\}_{k\geq 1}$ and $\{\tau'_k\}_{k\geq 1}$ in the following way. With probability $1 - c2\delta$ we choose $\tau_k = \tau'_k$, drawn from the distribution

$$\tilde{P}_\tau(\cdot) := \frac{P_\tau(\cdot) - c\lambda(\cdot \cap [a, a+2\delta])}{1 - c2\delta},$$

where λ denotes the Lebesgue measure. With the remaining probability $c2\delta$, draw τ_k uniformly on the interval $[a, a + 2\delta]$, and choose τ'_k as

$$\tau'_k = \begin{cases} \tau_k + \delta & \text{if } \tau_k \leq a + \delta, \\ \tau_k - \delta & \text{if } \tau_k > a + \delta. \end{cases}$$

It is immediate that τ'_k is also uniformly distributed on $[a, a + 2\delta]$. Thus, it is easy to see that the marginal distribution of both τ_k and τ'_k is P_τ, and this is indeed a coupling of the two infections. In Figure 5 the times at which the infections spread is illustrated.

The coupling is constructed so that each time τ_k and τ'_k are chosen differently, the difference $D_n := T_{\text{delay}} + \sum_{k=1}^n (\tau'_k - \tau_k)$ will jump $\pm\delta$. Since $T_{\text{delay}} = m\delta$, for some integer m, then $\{D_n\}_{n\geq 1}$ constitutes a simple random walk on $\delta\mathbb{Z}$. Let N_c denote the first n for which D_n hits zero. From this moment on, τ_k and τ'_k are chosen identically, and (6.2) holds for $n \geq N_c$. That the coupling is successful follows from the recurrence of one-dimensional (lazy) simple random walks. QED.
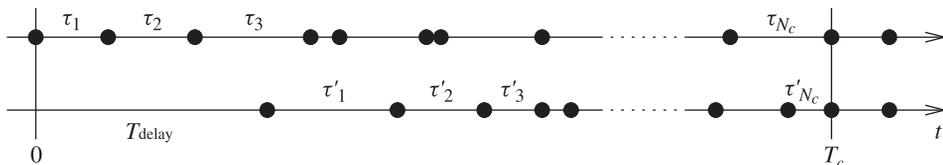
FIGURE 5: The dots represent the times at which the respective infection spreads. In this realisation $\tau_1 = \tau'_1 - \delta$, $\tau_2 = \tau'_2$ and $\tau_3 = \tau'_3 + \delta$.

Proof of case (b). By assumption, for some set $\{t_j\}_{j \in J} \subseteq \Lambda$ of atoms for the distribution P_τ, there are nonnegative integers n_j and n'_j such that $\sum_{j \in J} n_j = \sum_{j \in J} n'_j$ and (6.1) hold. It is easily seen that we may assume that J, n_j, and n'_j are chosen such that for each $j \in J$, exactly one of the integers n_j and n'_j is positive. We introduce integer valued random variables

$$X^n_j = \#\{k \le n : \tau_k = t_j\} - n_j,$$
$$Y^n_j = \#\{k \le n : \tau'_k = t_j\} - n'_j.$$

Define $Z^n_j = X^n_j - Y^n_j$. It is clear that from (6.1) we can conclude that (6.2) holds, if $Z^n_j = 0$ for all $j \in J$ and $\tau_k = \tau'_k$ for all $k \le n$ such that $\tau_k \notin \{t_j\}_{j \in J}$ or $\tau'_k \notin \{t_j\}_{j \in J}$.

Let $J_n = \{j \in J : Z^n_j \ne 0\}$, let $p_j = P_\tau(t_j)$, and $q_n = \sum_{j \in J_n} p_j$. In particular, $J_0 = J$. Couple $\{\tau_k\}_{k \ge 1}$ and $\{\tau'_k\}_{k \ge 1}$ by choosing τ_k and τ'_k identically from the distribution

$$\tilde{P}_\tau(\cdot) := \frac{1}{1 - q_{k-1}} \left(P_\tau(\cdot) - \sum_{j \in J_{k-1}} p_j \mathbf{1}_{\{t_j\}}(\cdot) \right)$$

with probability $1 - q_{k-1}$. With remaining probability q_{k-1} we choose τ_k and τ'_k independently with distribution $P(\tau = t_j) = p_j / q_{k-1}$, for $j \in J_{k-1}$. The marginal distribution of τ_k and τ'_k is readily seen to be P_τ, whence this is a coupling of $\{\tau_k\}_{k \ge 1}$ and $\{\tau'_k\}_{k \ge 1}$.

Note that $\tau_k = \tau'_k$ for all k such that $\tau_k \notin \{t_j\}_{j \in J}$ and $\tau'_k \notin \{t_j\}_{j \in J}$. For each fixed $j \in J$, $\{Z^n_j\}_{n \ge 0}$ will, as n increases, jump ± 1 with equal probability. Hence, for fixed j, $\{Z^n_j\}_{n \ge 0}$ constitutes a (lazy) simple random walk on $\mathbb{Z}$. Note that if n^* denotes the first n such that $Z^n_j = 0$, then, by definition, $j \in J_n$ for $n < n^*$, but $j \notin J_n$ for $n \ge n^*$.

By assumption we have that

$$0 = \sum_{j \in J} (n_j - n'_j) = \sum_{j \in J} Z^0_j = \sum_{j \in J} Z^n_j \quad \text{for all } n \ge 0.$$

It follows that $|J_n| \ne 1$ for all n. There will therefore always be a positive probability to choose $\tau_{n+1} \ne \tau'_{n+1}$ as long as $Z^n_j \ne 0$ for some j. By the recurrence of one-dimensional simple random walks, we conclude that $N_c = \min\{n \ge 0 : Z^n_j = 0 \text{ for all } j \in J\}$ is almost surely finite. QED.

6.2. Exact coupling of two infections

In order to prove Propositions 6.1 and 6.2, we will arrange matters so that Lemma 6.1 can be applied. We first outline the general idea. It follows from the regenerative behaviour that if $\tau_e = \tau'_e$ for all $e \in \mathbb{E}$, then there is a real number T_d such that

$$T(I, v_{n,i}) - T'(I', v_{n,i}) = T_d$$

for all i and all n large enough. The idea will be to assign identical passage times for both infections, that is $\tau_e = \tau'_e$, except for certain edges which we make sure both infections have to pass. This generates a sequence of edges for which we invoke Lemma 6.1. This will complete the coupling for positive levels n. The opposite direction is treated analogously.

To make this precise, recall the notation in (2.1) and (2.2). Introduce the notation $\hat{e}_{n+M}$ for the edge in γ_n with endpoints $\hat{v}_{n+M}$ and u, where $\hat{v}_{n+M}$ is the vertex in $\mathbb{V}_{\mathcal{G}_{n+M}}$ first reached by γ_n, and u the vertex first reached after $\hat{v}_{n+M}$ by γ_n. Define the event

$$A_n^* := \{\tau_e \leq t' \text{ for all } e \in \hat{E}_n \setminus \{\hat{e}_{n+M}\}\} \cap \{\tau_e \geq t'' \text{ for all } e \in E_n \setminus \hat{E}_n\}.$$

Note that $A_n = A_n^* \cap \{\tau_{\hat{e}_{n+M}} \leq t'\}$ for A_n as defined in (2.2).

Proof of Proposition 6.1. By assumption, P_τ has an absolutely continuous component, so suppose that $[a, b]$ is an interval on which P_τ has density $\geq c > 0$. Let $a < t' < t'' < b$ and choose M in accordance with Lemma 2.1. We may further assume that $I \cup I'$ contains no vertex beyond level m. Let $l_k := m + k(2M + 1)$ for $k \geq 0$. Couple $\{\tau_e\}_{e \in \mathbb{E}_m^c}$ and $\{\tau'_e\}_{e \in \mathbb{E}_m^c}$ by choosing $\tau_e = \tau'_e$ with distribution P_τ, independently for all e at level m or beyond such that $e \neq \hat{e}_{l_k+M}$ for some $k \geq 0$. Independently, for $k \geq 0$, let

$$(\xi_k, \xi'_k) = \begin{cases} (\theta_k, \theta'_k) & \text{with probability } P_\tau([0, t']), \\ (\eta_k, \eta_k) & \text{with probability } 1 - P_\tau([0, t']), \end{cases}$$

where θ_k and θ'_k are to be coupled below, so that they both have marginal distribution $P_\tau(\cdot | \tau \leq t')$, and η_k has distribution $P_\tau(\cdot | \tau > t')$. For the set of edges $\{\hat{e}_{l_k+M}, \text{ for } k \geq 0\}$, we choose the pair

$$(\tau_{\hat{e}_{l_k+M}}, \tau'_{\hat{e}_{l_k+M}}) = \begin{cases} (\xi_k, \xi'_k) & \text{if } A_{l_k}^* \text{ occurs} \\ (\tau_k, \tau_k) & \text{otherwise,} \end{cases}$$

where τ_k is distributed according to P_τ, independently for all k. It follows from the coupling that the marginal distributions of both τ_e and τ'_e is P_τ, for every edge e.

Note that the only edges for which τ_e and τ'_e may differ are the edges $\hat{e}_{l_k+M}$ for $k \geq 0$ such that A_{l_k} occurs. Let κ_j denote the index k for which A_{l_k} occurs for the jth time. That

$$(\tau_{\hat{e}_{l_k+M}}, \tau'_{\hat{e}_{l_k+M}}) = (\theta_k, \theta'_k) \tag{6.3}$$

is equivalent to occurrence of A_{l_k}. Since $P(A_{l_k}) > 0$, there is an infinite sequence $\{\kappa_j\}_{j \geq 1}$ so that (6.3) holds. By Lemma 2.1 we conclude that $\hat{e}_{l_{\kappa_j}}$ has to be passed by both infections, and that

$$T(I, \hat{v}_{l_{\kappa_j}+M}) - T'(I', \hat{v}_{l_{\kappa_j}+M}) = T(I, \hat{v}_{l_{\kappa_1}+M}) - T'(I', \hat{v}_{l_{\kappa_1}+M}) + \sum_{i=1}^{j-1} \theta_{\kappa_i} - \theta'_{\kappa_i}.$$

Apply Lemma 6.1 to $\{\theta_{\kappa_j}\}_{j \geq 1}$ and $\{\theta'_{\kappa_j}\}_{j \geq 1}$, with distribution $P_\tau(\cdot | \tau \leq t')$, and $T_{delay} = |T(I, \hat{v}_{l_{\kappa_1}+M}) - T'(I', \hat{v}_{l_{\kappa_1}+M})|$. Since $P_\tau(\cdot | \tau \leq t')$ is absolutely continuous on $[a, t']$, it follows that $T(I, v_{n,i}) = T'(I', v_{n,i})$ for all i, and all n large enough. Both infections can be coupled analogously in the negative direction, which completes the construction. QED.

The proof of Proposition 6.2 is a bit more involved, since before applying Lemma 6.1 we need to make sure that the geodesics attain 'correct' length. We outline the additional steps here, and refer the reader to [2] for the remaining details.
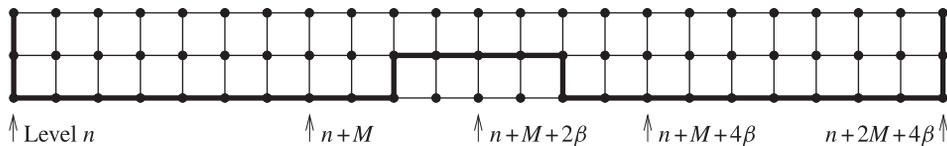
FIGURE 6: If D_n occurs, then the infection is forced to follow the thick line, whereas if C_n had occurred, then the infection would have followed the straight line segment.

In the coupling constructed above, the difference between $N(I, v_{n,i})$ and $N'(I', v_{n,i})$ is constant for all i and all but finitely many $n \geq 0$. In order to succeed with the coupling in the discrete case, we first need to couple the infections so that this eventual difference is either zero or the odd number $n^* = \sum_{j \in J^*} n_j$ figuring in the assumption of Proposition 6.2. This is accomplished as follows. First, assign identical passage times for both infections. Second, define two events C_n and D_n such that if one occurs for either infection, then the difference in length $N(I, v_{n,i}) - N'(I', v_{n,i})$ between the two geodesics to $v_{n,i}$ changes by two for all i and all but finitely many $n \geq 0$ (cf. Figure 6).

Repeating this procedure, we cause the difference to perform a random walk on either $2\mathbb{Z}$ or $2\mathbb{Z} + 1$. If $\text{dist}(x, y)$ is even for all $x \in I$ and $y \in I'$, then the walk lives on $2\mathbb{Z}$, will eventually hit zero, and applying Lemma 6.1 will be easy. If this is not the case, then the random walk may live on $2\mathbb{Z} + 1$, and the additional condition (a) is necessary.

Once the difference between geodesics have attained the right value, the coupling continues along the lines of the continuous case. However, we need to consider a variant of the event A_n^*, since the infections may have passed an edge with value M_τ in order to reach this stage.

Remark. If $\text{dist}(x, y)$ is odd, for all $x \in I$, $y \in I'$, then condition (a) of Proposition 6.2 is necessary. To see this, assume that an exact coupling is possible. In particular, $T(I, v) = T'(I', v)$ for some vertex v. But, if one infection has an even number of edges to pass in order to reach v, the other has an odd number of edges to pass. Thus,

$$0 = T(I, v) - T'(I', v) = \sum_{j \in J} n_j t_j - \sum_{j \in J} n'_j t_j,$$

for integers n_j and n'_j such that $\sum_{j \in J}(n_j - n'_j)$ is odd. Hence, condition (a) holds.

Remark. Condition (a) of Proposition 6.2 is not sufficient for the existence of an exact coupling on arbitrary essentially one-dimensional periodic graphs. The distribution $P_\tau(1) = P_\tau(1 + \frac{3}{5}) = \frac{1}{2}$ satisfies the condition, but it is not always possible to exactly couple two infections on the graph with vertex set $\mathbb{Z} \times \{0, 1\}$ and two vertices are connected if their Euclidean distance is less than or equal to $\sqrt{2}$. However, both condition (a) and (b) of Proposition 6.2 could be dropped for e.g. the class of triangular graphs with vertex set $\mathbb{Z} \times \{0, 1, \ldots, K-1\}$ and where two vertices at Euclidean distance is 1 and every two vertices (n, m) and $(n+1, m+1)$ for any $n \in \mathbb{Z}$ and $m = 0, 1, \ldots, K - 2$, are connected by an edge.

6.3. No exact coupling possible on trees

We have seen that there is an exact coupling of two first-passage percolation infections on any essentially one-dimensional periodic graph when the passage time distribution has an absolutely continuous component. We also saw how this sort of coupling gave rise to a 0–1 law. We may ask whether a continuous component is sufficient for an analogous coupling, and

corresponding 0–1 law, on any graph? We will answer this question now by showing that the binary tree $\mathbb{T}^2$ constitutes a counterexample. $\mathbb{T}^2$ is the infinite graph that does not contain any circuit, and where each vertex has three neighbours. The graph is completely homogeneous and one vertex, called the *root*, is chosen for reference. Let $\{\tau_e\}_{e \in \mathbb{E}}$ be a set of independent and exponentially distributed passage times associated with the edge set $\mathbb{E}$ of $\mathbb{T}^2$, and analogous to before, let

$$B_t = \{v \in \mathbb{V} \colon T(\text{root}, v) \leq t\}.$$

The following argument is based on the theory of continuous branching processes. Define the front line of the infection at time t as

$$F_t := \#\{v \notin B_t \colon v \text{ shares an edge with some } u \in B_t\}.$$

Note that $F_0 = 3$ and that F_t increases by one, when B_t does. Hence, F_t can be seen as a continuous time branching process with F_t individuals at time t. Each individual gives birth, with probability one, to two children (and dies) after an exponentially distributed time, independent of one another. It is well-known (see, e.g. [4, Theorems III.7.1–2]) that, for some Malthusian parameter $\lambda > 0$,

$$\text{there exists } W := \lim_{t \to \infty} F_t \mathrm{e}^{-\lambda t}, \quad \text{almost surely,} \tag{6.4}$$

and that $\mathrm{E}[W] = 3$. Let τ_{e_1}, τ_{e_2} and τ_{e_3} denote the passage time of the edges connected to the root, and let $\tilde{F}_t$ denote F_t conditioned on $\{\tau_{e_1}, \tau_{e_2}, \tau_{e_3} \geq 1\}$. Then, by the lack-of-memory property of the exponential distribution, we have that $\tilde{F}_{t+1} \overset{\mathrm{D}}{=} F_t$ for any $t \geq 0$. Thus, by (6.4) we have almost surely

$$\lim_{t \to \infty} \tilde{F}_t \mathrm{e}^{-\lambda t} \overset{\mathrm{D}}{=} \mathrm{e}^{-\lambda} \lim_{t \to \infty} F_t \mathrm{e}^{-\lambda t} = \mathrm{e}^{-\lambda} W,$$

and we conclude that W is almost surely nonconstant. Note that the event

$$\left\{ W = \lim_{t \to \infty} F_t \mathrm{e}^{-\lambda t} \leq x \right\} \in \mathcal{T} \quad \text{for every } x.$$

Then, a 0–1 law analogous to Theorem 6.1 cannot hold for first-passage percolation on $\mathbb{T}^2$, since this would imply that $P(W \leq x) \in \{0, 1\}$, i.e. that W is almost surely constant.

Acknowledgements

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