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A SCHWARZ LEMMA FOR V-HARMONIC MAPS AND THEIR APPLICATIONS

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Abstract

We establish a Schwarz lemma for V-harmonic maps of generalised dilatation between Riemannian manifolds. We apply the result to obtain corresponding results for Weyl harmonic maps of generalised dilatation from conformal Weyl manifolds to Riemannian manifolds and holomorphic maps from almost Hermitian manifolds to quasi-Kähler and almost Kähler manifolds.

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1. Introduction

The classical Schwarz–Pick lemma says that any holomorphic map from the unit disk in the complex plane into itself decreases the Poincaré metric. Ahlfors [1], Chern [4] and Lu [10] generalised this result to more general domains and targets. In 1978, Yau [15] showed that a holomorphic map from a Kähler manifold with Ricci curvature bounded below to a Hermitian manifold with bisectional curvature bounded above by a negative constant is distance nonincreasing up to a constant depending only on these bounds. Later, in [14], Yang and Chen improved Yau's result. Recently, Tosatti [12] established a Schwarz lemma for holomorphic maps between almost Hermitian manifolds with curvature and torsion conditions on the canonical connection.

We can also consider the case of harmonic maps. In [5, 6], Goldberg *et al.* considered harmonic maps of bounded dilatation between Riemannian manifolds. Shen [11] considered generalised dilatation and improved the result in [5].

In this paper, using an Omori–Yau maximum principle from [3] (see Theorem 2.1), we establish a Schwarz lemma, that is, a distance decreasing theorem up to a constant depending on the dimensions of the manifolds, the bounds of the curvatures and the order of dilatation, for a V-harmonic map of generalised dilatation between

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Riemannian manifolds (Theorem 3.3). By combining this with the Maclaurin inequality, we obtain a corollary for intermediate volume elements (Corollary 3.5). Since Weyl harmonic maps from conformal Weyl manifolds to Riemannian manifolds are V-harmonic, we also have a Schwarz lemma for Weyl harmonic maps of generalised dilatation (Theorem 3.4). Finally, since any holomorphic map between almost Hermitian manifolds naturally has generalised dilatation, we can apply Theorem 3.3 to the holomorphic map between almost Hermitian manifolds when the target manifold is a quasi-Kähler manifold (Theorem 4.3). When the target manifold is an almost Kähler manifold, by replacing the sectional curvature with the holomorphic bisectional curvature, we can obtain a distance decreasing theorem up to a constant depending only on the bounds of the curvatures of the two manifolds (Theorem 4.4). Some results in this paper improve the corresponding results in [5, 8] (see Remarks 3.6 and 4.7).

2. Preliminaries for V-harmonic maps

For convenience, we denote by ∇ the Levi-Civita connections of all manifolds and the induced connections derived from Levi-Civita connections, and by $\langle \cdot , \cdot \rangle$ the inner products on all manifolds and bundles.

Let (M^m, g) and (N^n, h) be two Riemannian manifolds and V a C^1 vector field on M. Following [2], a smooth map $u : M \to N$ is called V-harmonic if it satisfies

$$\tau_V(u) = \tau(u) + du(V) = 0, \tag{2.1}$$

where $\tau(u)$ is the tension field of the map u. Examples of V-harmonic maps include Hermitian harmonic maps, Weyl harmonic maps and affine harmonic maps (see [2]).

Let $\Delta_V \cdot = \Delta \cdot + \langle V, \nabla \cdot \rangle$ and denote by $\mathrm{Ric}_V = \mathrm{Ric}^M - \frac{1}{2} L_V g$ the Bakry–Émery Ricci tensor of M, where L_V is the Lie derivative. For a smooth map $f : M \to N$, from [13], we have the Bochner formula:

$$\begin{aligned}\tfrac{1}{2}\Delta|du|^2 &= \langle \nabla_{e_i}\tau(u), du(e_i)\rangle + |\nabla\, du|^2 \\ &\quad + \mathrm{Ric}^M(e_i, e_j)\langle du(e_i), du(e_j)\rangle - \langle R^N(du(e_i), du(e_j))\, du(e_j), du(e_i)\rangle,\end{aligned} \tag{2.2}$$

where $\{e_i\}$ is a local orthonormal frame of TM. From (2.1),

$$\begin{aligned}\langle \nabla_{e_i}\tau(u), du(e_i)\rangle &= -\langle \nabla_{e_i}(du(V)), du(e_i)\rangle \\ &= -\langle (\nabla_{e_i} du)(V), du(e_i)\rangle - \langle du(\nabla_{e_i} V), du(e_i)\rangle \\ &= -\langle (\nabla_V\, du)(e_i), du(e_i)\rangle - \langle du(\nabla_{e_i} V), du(e_i)\rangle \\ &= -\langle \nabla_V(du(e_i)), du(e_i)\rangle + \langle du(\nabla_V e_i - \nabla_{e_i} V), du(e_i)\rangle \\ &= -\tfrac{1}{2} V|du|^2 - \tfrac{1}{2} L_V g(e_i, e_j)\langle du(e_i), du(e_j)\rangle.\end{aligned} \tag{2.3}$$

Substituting (2.3) into (2.2) and setting $e(u) = |du|^2$, we obtain the Bochner formula for V-harmonic maps:

$$\begin{aligned}\tfrac{1}{2}\Delta_V e(u) &= |\nabla\, du|^2 + \mathrm{Ric}_V(e_i, e_j)\langle du(e_i), du(e_j)\rangle \\ &\quad - \langle R^N(du(e_i), du(e_j))du(e_j), du(e_i)\rangle.\end{aligned} \tag{2.4}$$

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We also have the following Omori–Yau maximum principle for the operator Δ_V.

THEOREM 2.1 [3, Theorem 1]. *Let (M^m, g) be a complete Riemannian manifold and V a C^1 vector field on M. Suppose* $\mathrm{Ric}_V \geq -F(r)$*, where r is the distance function on M from a fixed point $x_0 \in M$, and $F : \mathbb{R} \to \mathbb{R}$ is a positive continuous function satisfying*

$$\varphi(t) := \int_{\rho_0+1}^{t} \frac{dr}{\int_{\rho_0}^{r} F(s)\, ds + 1} \to +\infty \quad (t \to +\infty)$$

for some positive constant ρ_0. Let $f \in C^2(M)$ with $\lim_{x\to\infty} f(x)/\varphi(r(x)) = 0$. Then there exist points $\{x_j\} \subset M$, such that

$$\lim_{j\to\infty} f(x_j) = \sup f, \quad \lim_{j\to\infty} |\nabla f|(x_j) = 0 \quad \textit{and} \quad \lim_{j\to\infty} \Delta_V f(x_j) \leq 0.$$

3. A Schwarz lemma for V-harmonic maps of bounded dilatation

Let $u : (M^m, g) \to (N^n, h)$ be a smooth map between Riemannian manifolds. For a point $x \in M$, the linear map $du : T_xM \to T_{u(x)}N$ and its transpose $du^\dagger : T_{u(x)}N \to T_xM$ are related by

$$\langle du(X), Y\rangle_h = \langle X, du^\dagger(Y)\rangle_g \quad \text{for } X \in \Gamma(TM), Y \in \Gamma(TN).$$

This gives a linear map $du^\dagger \circ du : T_xM \to T_xM$. Let $\{e_i\}_{i=1}^m$ and $\{\tilde{e}_\alpha\}_{\alpha=1}^n$ be local orthonormal frames of TM and TN, respectively and write $du^\dagger \circ du(e_i) = U_{ij}e_j$. Then

$$U_{ij} = \langle du^\dagger \circ du(e_i), e_j\rangle = \langle du(e_i), du(e_j)\rangle = u_i^\alpha u_j^\alpha.$$

In other words, we have a matrix equation, $(U_{ij}) = (u_i^\alpha)(u_i^\alpha)^\dagger$, where (u_i^α) is a semi-positive define symmetric matrix of order m. Therefore, the eigenvalues of $du^\dagger \circ du$ are nonnegative, say, $\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x) \geq 0$. Let $k = \min\{m, n\}$. The rank of $du^\dagger \circ du$ is less than or equal to k.

REMARK 3.1. Let Q^N denote the third term on the right-hand side of the Bochner formula (2.4). When the sectional curvature of the target manifold is bounded above by a negative constant $-B$, then

$$Q^N \geq B\Big(|du|^4 - \sum_{i,j}\langle du(e_i), du(e_j)\rangle^2\Big).$$

However, from the right-hand side of (2.4),

$$Q^N \geq C'|du|^4$$

is usually impossible, where C' is some positive constant.

In order to overcome the obstacle noted in Remark 3.1, we need to add some conditions on u. To this end, we can introduce the so-called bounded dilatation and generalised dilatation.

DEFINITION 3.2 (see [5, 11]). A smooth map $u : M \to N$ has bounded dilatation of order β, if there is a positive number β such that $\lambda_1(x) \leq \beta^2\lambda_2(x)$ for every $x \in M$. The map u has generalised dilatation of order β, if there is a positive number β such that $\lambda_1(x) \leq \beta^2(\lambda_2(x) + \cdots + \lambda_m(x))$ for every $x \in M$.

From this definition, bounded dilatation is generalised dilatation of the same order and generalised dilatation of order β is bounded dilatation of order $\sqrt{(m-1)}\beta$. Hence, the condition of generalised dilatation yields a smaller bound (see [11]). For V-harmonic maps of generalised dilatation between Riemannian manifolds, from the Bochner formula (2.4) and Theorem 2.1, we have the following Schwarz lemma.

THEOREM 3.3. *Let (M^m, g) be a complete Riemannian manifold with* $\mathrm{Ric}_V \geq -A$, *where A is a constant. Let (N^n, h) be a Riemannian manifold with sectional curvature bounded above by a negative constant $-B$. Let $u : M \to N$ be a nonconstant V-harmonic map of generalised dilatation of order β. Then, $A > 0$ and*

$$u^*h \leq \frac{Ak^2\beta^4}{2B(1+\beta^2)}g,$$

where $k = \min\{m, n\}$. In particular, if $A \leq 0$, then any V-harmonic map of generalised dilatation from M to N is constant.

PROOF. Choose a local orthonormal frame $\{e_1, \ldots, e_m\}$ formed by eigenvectors corresponding to the eigenvalues $\{\lambda_1, \ldots, \lambda_m\}$ of $du^\dagger \circ du$. By the curvature conditions and the Bochner formula (2.4), combined with the definition of sectional curvature,

$$\begin{aligned}
\Delta_V e(u) &= 2|\nabla\, du|^2 + 2\mathrm{Ric}_V(e_i, e_j)\langle du(e_i), du(e_j)\rangle \\
&\qquad - 2\mathrm{Sec}^N(du(e_i), du(e_j))(\langle du(e_i), du(e_i)\rangle\langle du(e_j), du(e_j)\rangle - \langle du(e_i), du(e_j)\rangle^2) \\
&\geq -2Ae(u) + 2B\sum_{i,j=1}^{m}(\langle du(e_i), du(e_i)\rangle\langle du(e_j), du(e_j)\rangle - \langle du(e_i), du(e_j)\rangle^2). \qquad (3.1)
\end{aligned}$$

Since u is of bounded dilatation,

$$\begin{aligned}
&\sum_{i,j=1}^{m}(\langle du(e_i), du(e_i)\rangle\langle du(e_j), du(e_j)\rangle - \langle du(e_i), du(e_j)\rangle^2) \\
&\quad = \sum_{i=1}^{m}\langle du^\dagger \circ du(e_i), e_i\rangle \sum_{j=1}^{m}\langle du^\dagger \circ du(e_j), e_j\rangle - \sum_{i,j=1}^{m}\langle du^\dagger \circ du(e_i), e_j\rangle^2 \\
&\quad = \sum_{i=1}^{m}\lambda_i \sum_{j=1}^{m}\lambda_j - \sum_{l=1}^{m}\lambda_l^2 = 2\sum_{1\leq i<j\leq m}\lambda_i\lambda_j \geq 2\lambda_1\sum_{j=2}^{m}\lambda_j \geq \frac{2}{\beta^2}\lambda_1^2 \\
&\quad \geq \frac{2}{\beta^2}\Big(\frac{\lambda_1 + \cdots + \lambda_k}{k}\Big)^2. \qquad (3.2)
\end{aligned}$$

On the other hand,

$$e(u) = \sum_{i=1}^{m}\langle du(e_i), du(e_i)\rangle = \sum_{i=1}^{m}\langle du^\dagger \circ du(e_i), e_i\rangle = \sum_{i=1}^{m}\lambda_i = \sum_{i=1}^{k}\lambda_i. \qquad (3.3)$$

Combining (3.1), (3.2) and (3.3),

$$\Delta_V e(u) \geq -2Ae(u) + \frac{4B}{k^2\beta^2}e(u)^2. \qquad (3.4)$$

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Now, choose $f = -1/\sqrt{e(u)+C}$, where $C > 0$ is a constant. Since $e(u) \geq 0$, we have $-1/\sqrt{C} \leq f < 0$, that is, f is bounded. By Theorem 2.1, there exist points $\{x_j\} \subset M$ such that

$$\lim_{j\to\infty} f(x_j) = \sup f, \quad \lim_{j\to\infty} |\nabla f|(x_j) = 0 \quad \text{and} \quad \lim_{j\to\infty} \Delta_V f(x_j) \leq 0. \tag{3.5}$$

However,

$$\nabla f = \frac{\nabla e(u)}{2(e(u)+C)^{3/2}} \quad \text{and} \quad \Delta_V f = -\frac{3|\nabla e(u)|^2}{4(e(u)+C)^{5/2}} + \frac{\Delta_V e(u)}{2(e(u)+C)^{3/2}}. \tag{3.6}$$

By (3.5), (3.6) and the definition of f, we have $\lim_{j\to\infty} e(u)(x_j) = \sup e(u)$ and

$$\lim_{j\to\infty} \frac{\Delta_V e(u)(x_j)}{(e(u)(x_j)+C)^2} \leq 0. \tag{3.7}$$

Combining (3.4) and (3.7),

$$\lim_{j\to\infty} \frac{2B}{k^2\beta^2} \frac{(e(u)(x_j))^2 - Ae(u)(x_j)}{(e(u)(x_j)+C)^2} \leq 0. \tag{3.8}$$

If $\lim_{j\to\infty} e(u)(x_j) = \sup u = +\infty$, then (3.8) gives a contradiction, so, $\sup u < +\infty$. Using (3.8) again,

$$\frac{2B}{k^2\beta^2}(\sup e(u))^2 - A \sup e(u) \leq 0,$$

and it follows that

$$e(u) \leq \frac{Ak^2\beta^2}{2B}. \tag{3.9}$$

Now, $e(u) = \mathrm{tr}_g u^*h = \mathrm{tr}(U_{ij}) = \sum_{i=1}^m \lambda_i$, so

$$\lambda_1 + \frac{1}{\beta^2}\lambda_1 \leq \sum_{i=1}^m \lambda_i \leq \frac{Ak^2\beta^2}{2B}.$$

Thus, $\lambda_1 \leq Ak^2\beta^4/(2B(1+\beta^2))$, which completes the proof. □

Now we consider Weyl harmonic maps from conformal Weyl manifolds to Riemannian manifolds. Firstly, we recall the related facts briefly (see [9, Section 2]). Let (M, c) be a conformal manifold and ∇^W its Weyl structure. There is a 1-form Θ, called the Higgs field, such that $\nabla^W g = \Theta \otimes g$ for $g \in c$. Let (N, h) be a Riemannian manifold. A smooth map $u : (M, c, \nabla^W) \to (N, h)$ is called Weyl harmonic if $\tau^W(u) = \tau(g, \nabla^W, \nabla) = 0$. We know that

$$\tau^W(u) = \tau(u) - \left(\frac{m-2}{2}\right) du(\Theta^\sharp), \tag{3.10}$$

where $^\sharp$ maps a 1-form to its dual vector field. Gauduchon showed that there is a unique (up to homothety) metric $g \in c$ whose Higgs field Θ is co-closed with respect to g. The Weyl Laplacian on functions is given by $\Delta^W = \mathrm{tr}_g \nabla^W d$. From (3.10), Weyl harmonic maps are V-harmonic, where $V = -\frac{1}{2}(m-2)\Theta^\sharp$.

THEOREM 3.4. *Let (M^m, c, ∇^W) be a conformal Weyl manifold, endowed with a Gauduchon metric $g \in c$, and denote by Θ its Higgs field. Suppose that*

$$\mathrm{Ric}^W + \frac{m-2}{4}(|\Theta|^2 g - \Theta \otimes \Theta) \geq -A,$$

where A is a constant and Ric^W denotes the Ricci tensor of ∇^W. Let (N^n, h) be a Riemannian manifold with sectional curvature bounded above by a negative constant $-B$. Let $u : M \to N$ be a Weyl harmonic map of generalised dilatation of order β. Then either u is constant, or $A > 0$ and u is distance decreasing up to a constant depending on A, B, m, n, β.

PROOF. From [9, Lemma 3.1],

$$\Delta^W e(u) = |\nabla\, du|^2 - \langle R^N(du(e_i), du(e_j))\, du(e_j), du(e_i)\rangle$$
$$+ \mathrm{Ric}^W(X_\alpha, X_\alpha) + \frac{m-2}{4}(|\Theta|^2|X_\alpha|^2 - \Theta(X_\alpha)^2),$$

where $X_\alpha = (u^*\tilde{\omega}^\alpha)^\sharp$ and the systems $\{e_i\}$ and $\{\tilde{\omega}^\alpha\}$ are local orthonormal frames of TM and T^*N, respectively. By the curvature condition and since u is of generalised dilatation,

$$\Delta^W e(u) \geq \frac{4B}{k^2\beta^2} e(u)^2 - A|X_\alpha|^2 = \frac{4B}{k^2\beta^2} e(u)^2 - A|u_i^\alpha e_i|^2 = \frac{4B}{k^2\beta^2} e(u)^2 - Ae(u). \tag{3.11}$$

By (3.10), for a function f on M,

$$\Delta^W f = \Delta f + df(V) = \Delta f + \langle V, \nabla f\rangle = \Delta_V f. \tag{3.12}$$

On the other hand, from the proof of [9, Lemma 3.1],

$$\mathrm{Ric}^W(X,X) + \frac{m-2}{4}(|\Theta|^2|X|^2 - \Theta(X)^2) = \mathrm{Ric}(X,X) + \frac{m-2}{2}(\nabla_X\Theta)(X),$$

where $X \in \Gamma(TM)$ and Ric denotes the Ricci tensor of a Gauduchon metric g. But

$$\frac{m-2}{2}(\nabla_X\Theta)(X) = \frac{m-2}{2}\langle \nabla_X\Theta^\sharp, X\rangle = -\langle \nabla_X V, X\rangle = -\frac{1}{2} L_V g(X,X).$$

It follows that

$$\mathrm{Ric}_V(X,X) = \mathrm{Ric}^W(X,X) + \frac{m-2}{4}(|\Theta|^2 g(X,X) - \Theta(X)^2).$$

Hence, $\mathrm{Ric}_V \geq -A$. Theorem 2.1, combined with (3.11) and (3.12), gives the desired result. □

As in [6], we can also consider the intermediate volume elements. If $r \leq k$, du can be extended to the linear map $\wedge^r du : \bigwedge^r TM \to \bigwedge^r TN$ given by

$$(\wedge^r du)(X_1 \wedge \cdots \wedge X_r) = du(X_1) \wedge \cdots \wedge du(X_r),$$

where $X_i \in \Gamma(TM)$. Then (see [6]),

$$|\wedge^r du|^2 = \sum_{1 \leq i_1 < \cdots < i_r \leq m} \lambda_{i_1} \cdots \lambda_{i_r}. \tag{3.13}$$

In particular, $|\wedge^1 du|^2 = |du|^2 = e(u)$. Observe that $|\wedge^r du|$ bounds the ratio of r-dimensional volume elements. Hence, from the Maclaurin inequality together with (3.9) and (3.13), we have the following corollary for intermediate volume elements.

COROLLARY 3.5. *Under the conditions of Theorem 3.3,*

$$|\wedge^r du|^2 \leq \binom{k}{r}\Big(\frac{Ak\beta^2}{2B}\Big)^r,$$

where $r = 1, \dots, k = \min\{m, n\}$.

REMARK 3.6. Theorem 3.3 and Corollary 3.5 generalise [5, Theorem 1 and Corollary 1] to V-harmonic maps.

4. Applications to holomorphic maps from almost Hermitian manifolds

In this section, we apply Theorem 3.3 to holomorphic maps from almost Hermitian manifolds to quasi-Kähler manifolds.

Let (M^{2m}, g, J) and (N^{2n}, h, J') be two almost Hermitian manifolds. A smooth map $u : M \to N$ is holomorphic if $du \circ J = J' \circ du$. It is easy to see that $du^\dagger \circ J' = J \circ du^\dagger$ from the definition of the transpose, so $du^\dagger \circ du \circ J = J \circ du^\dagger \circ du$ if u is holomorphic. Accordingly, the first two eigenvalues λ_1 and λ_2 are equal when u is a holomorphic map between almost Hermitian manifolds. Consequently, any holomorphic map between almost Hermitian manifolds is a bounded dilatation of order 1 (see also [8]). It is also a generalised dilatation of order 1.

DEFINITION 4.1 [7]. An almost Hermitian manifold (M, g, J) is a quasi-Kähler manifold if $(\nabla_X J)Y + (\nabla_{JX} J)JY = 0$ for all $X, Y \in \Gamma(TM)$.

LEMMA 4.2. *Let* $u : (M^{2m}, g, J) \to (N^{2n}, h, J')$ *be a holomorphic map between almost Hermitian manifolds. If* N *is a quasi-Kähler manifold, then* u *is a* V*-harmonic map, where* $V = -J\delta J$ *and* δ *is the co-differential operator on* M.

PROOF. Let $\{e_A\}_{A=1}^{2m} = \{e_i; Je_i\}_{i=1}^{m}$ be a locally orthonormal frame of TM. Omitting summation, by a standard computation,

$$\begin{aligned}
(\nabla_{Je_i} du)(Je_i) &= \nabla_{Je_i}(J'(du(e_i))) - du(\nabla_{Je_i}(Je_i)) \\
&= (\nabla_{Je_i} J')(du(e_i)) + J'(\nabla_{Je_i} du)(e_i) - du((\nabla_{Je_i} J)e_i) \\
&= J'(\nabla_{Je_i} J')(J'(du(e_i))) + J'(\nabla_{e_i} du)(Je_i) - du((\nabla_{Je_i} J)e_i) \\
&= J'(\nabla_{Je_i} J')(du(Je_i)) + J'\nabla_{e_i}(du(Je_i)) \\
&\quad - J'(du((\nabla_{e_i} J)e_i)) - J'(du(J\nabla_{e_i} e_i)) - du((\nabla_{Je_i} J)e_i) \\
&= J'(\nabla_{Je_i} J')(du(Je_i)) + J'\nabla_{e_i}(J'(du(e_i))) \\
&\quad - J'(du((\nabla_{e_i} J)e_i)) + du(\nabla_{e_i} e_i) - du((\nabla_{Je_i} J)e_i) \\
&= J'(\nabla_{Je_i} J')(du(Je_i)) + J'(\nabla_{e_i} J')(du(e_i)) \\
&\quad - (\nabla_{e_i} du)(e_i) - J'(du((\nabla_{e_i} J)e_i)) - du((\nabla_{Je_i} J)e_i).
\end{aligned}$$

By the definition of the tension field and since N is a quasi-Kähler manifold,

$$\begin{aligned}
\tau(u) &= (\nabla_{e_i} du)(e_i) + (\nabla_{Je_i} du)(Je_i) = -J'(du((\nabla_{e_i} J)e_i)) - du((\nabla_{Je_i} J)e_i) \\
&= du(-J(\nabla_{e_i} J)e_i - J(\nabla_{Je_i} J)(Je_i)) = du(J\delta J),
\end{aligned}$$

which establishes the lemma. □

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The next result follows from Definition 4.1, Theorem 3.3 and Lemma 4.2.

THEOREM 4.3. *Let (M^{2m}, g, J) be a complete almost Hermitian manifold such that $\mathrm{Ric}^M + \frac{1}{2}L_{J\delta J}g \geq -A$, where A is a constant. Let (N^{2n}, h, J') be a quasi-Kähler manifold with sectional curvature bounded above by a negative constant $-B$. Let $u : M \to N$ be a nonconstant holomorphic map. Then $A > 0$ and*

$$u^*h \leq \frac{A\ell^2}{4B} g,$$

where $\ell = \min\{2m, 2n\}$. In particular, if $A \leq 0$, then there is no nonconstant holomorphic map from M to N.

As in [5], replacing the sectional curvature by the holomorphic bisectional curvature when the target manifold is an almost Kähler manifold, we have the following distance decreasing theorem up to a constant depending only on the curvature bounds.

THEOREM 4.4. *Let (M^{2m}, g, J) be as in Theorem 4.3. Let (N^{2n}, h, J') be an almost Kähler manifold with holomorphic bisectional curvature bounded above by a negative constant $-B$. Let $u : M \to N$ be a nonconstant holomorphic map. Then $A > 0$ and*

$$u^*h \leq \frac{A}{B} g.$$

In particular, if $A \leq 0$, then there is no nonconstant holomorphic map from M to N.

Yau [15] proved that there is no bounded holomorphic function on a complete Kähler manifold with nonnegative Ricci curvature. Similarly, we have also the following corollary from Theorem 4.3.

COROLLARY 4.5. *Let (M, g, J) be an almost Hermitian manifold with $\mathrm{Ric}^M + \frac{1}{2}L_{J\delta J}g \geq 0$. Then there is no nonconstant bounded holomorphic function $u : M \to \mathbb{C}$.*

Similar to Corollary 3.5, we have also the following result for intermediate volume elements for a holomorphic map in the almost Hermitian case.

COROLLARY 4.6. *Under the assumptions in Theorem 4.3,*

$$|\wedge^r du|^2 \leq \binom{\ell}{r}\left(\frac{A\ell\beta^2}{2B}\right)^r,$$

where $r = 1, \ldots, \ell = \min\{2m, 2n\}$.

REMARK 4.7. Theorems 4.3 and 4.4 generalise [5, Corollary 2 and Theorem 2]. Corollary 4.5 improves [8, Proposition 8] to the case where the domain manifold is an almost Hermitian manifold without the condition of nonpositive sectional curvature.

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