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Geometric Weil representation: local field case

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Abstract

Let k be an algebraically closed field of characteristic greater than 2, and let $F = k((t))$ and $G = \mathbb{S}\mathrm{p}_{2d}$. In this paper we propose a geometric analog of the Weil representation of the metaplectic group $\widetilde{G}(F)$. This is a category of certain perverse sheaves on some stack, on which $\widetilde{G}(F)$ acts by functors. This construction will be used by Lysenko (in [*Geometric theta-lifting for the dual pair* $\mathbb{S}\mathbb{O}_{2m}$, $\mathbb{S}\mathrm{p}_{2n}$, math.RT/0701170] and subsequent publications) for the proof of the geometric Langlands functoriality for some dual reductive pairs.

1. Introduction

1.1 This paper and the following paper [Lys07] form a series, where we prove the geometric Langlands functoriality for the dual reductive pair $\mathbb{S}\mathrm{p}_{2n}, \mathbb{S}\mathbb{O}_{2m}$ (in the everywhere non-ramified case).

Let $k = \mathbb{F}_q$ with q odd and set $\mathcal{O} = k[[t]] \subset F = k((t))$. Write Ω for the completed module of relative differentials of $\mathcal{O}$ over k. Let M be a free $\mathcal{O}$-module of rank $2d$ with symplectic form $\wedge^2 M \to \Omega$ and set $G = \mathbb{S}\mathrm{p}(M)$. The group $G(F)$ admits a non-trivial metaplectic extension

$$1 \to \{\pm 1\} \to \tilde{G}(F) \to G(F) \to 1$$

(defined up to a unique isomorphism). Let $\psi : k \to \bar{\mathbb{Q}}_\ell^*$ be a non-trivial additive character and let $\chi : \Omega(F) \to \bar{\mathbb{Q}}_\ell^*$ be given by $\chi(\omega) = \psi(\operatorname{Res} \omega)$. Write $H = M \oplus \Omega$ for the Heisenberg group of M with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2, a_1 + a_2 + \tfrac{1}{2}\omega\langle m_1, m_2\rangle), \quad m_i \in M, a_i \in \Omega.$$

Denote by $\mathcal{S}_\psi$ the Weil representation of $H(M)(F)$ with central character χ. As a representation of $\tilde{G}(F)$, it decomposes $\mathcal{S}_\psi \widetilde{\to} \mathcal{S}_{\psi,\mathrm{odd}} \oplus \mathcal{S}_{\psi,\mathrm{even}}$ into a direct sum of two irreducible smooth representations, where the even (respectively, the odd) part is unramified (respectively, ramified).

The discovery of this representation by Weil in [Wei64] had a major influence on the theory of automorphic forms. Among numerous developments and applications are Howe duality for reductive dual pairs, particular cases of classical Langlands functoriality, Siegel–Weil formulas, the relation with L-functions, the representation-theoretic approach to the theory of theta-series. We refer the reader to [Ger77, How79, LV80, MVW87, Pra98] for the history and further details.

In this paper we introduce a geometric analog of the Weil representation $\mathcal{S}_\psi$. The pioneering work in this direction is due to Deligne [Del82], where a geometric approach to the Weil representation of a symplectic group over a finite field was set up. It was further extended by Gurevich and Hadani in [GH05, GH04]. The point of this paper is to develop the geometric theory in the case when a finite field is replaced by a local non-archimedean field.

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First, we introduce a k-scheme $\mathcal{L}_d(M(F))$ of discrete Lagrangian lattices in $M(F)$ and a certain μ_2-gerb $\widetilde{\mathcal{L}}_d(M(F))$ over it. We view the metaplectic group $\tilde{G}(F)$ as a group stack over k. We construct a category $W(\widetilde{\mathcal{L}}_d(M(F)))$ of certain perverse sheaves on $\widetilde{\mathcal{L}}_d(M(F))$, which provides a geometric analog of $\mathcal{S}_{\psi,\mathrm{even}}$. The metaplectic group $\tilde{G}(F)$ acts on the category $W(\widetilde{\mathcal{L}}_d(M(F)))$ by functors. This action is *geometric* in the sense that it comes from a natural action of $\tilde{G}(F)$ on $\widetilde{\mathcal{L}}_d(M(F))$ (cf. Theorem 2).

The category $W(\widetilde{\mathcal{L}}_d(M(F)))$ has a distinguished object $S_{M(F)}$ corresponding to the unique non-ramified vector of $\mathcal{S}_{\psi,\mathrm{even}}$.

Our category $W(\widetilde{\mathcal{L}}_d(M(F)))$ is obtained from Weil representations of symplectic groups $\mathbb{S}\mathrm{p}_{2r}(k)$ by some limit procedure. This uses a construction of geometric canonical intertwining operators for such representations. A similar result has been announced by Gurevich and Hadani in [GH05] and proved for $d=1$ in [GH04]. We give a proof for any d (cf. Theorem 1). After this paper had been written we learned about a new preprint [GH07], where a result similar to our Theorem 1 is proved for all d. However, the sheaves of canonical intertwining operators constructed in *loc. cit.* and in this paper live on different bases.

Finally, in § 7 we give a global application. Let X be a smooth projective curve. Write Ω_X for the canonical line bundle on X. Let G denote the sheaf of automorphisms of $\mathcal{O}_X^d \oplus \Omega_X^d$ preserving the natural symplectic form $\wedge^2(\mathcal{O}_X^d \oplus \Omega_X^d) \to \Omega_X$.

Our Theorem 3 relates $S_{M(F)}$ with the theta-sheaf Aut on the moduli stack $\widetilde{\mathrm{Bun}}_G$ of metaplectic bundles on X introduced in [Lys06]. This result will play an important role in [Lys07].

1.2 Notation

In § 2 we let $k=\mathbb{F}_q$ of characteristic $p>2$. Starting from § 3 we assume that k is either finite as above or algebraically closed with a fixed inclusion $\mathbb{F}_q \hookrightarrow k$. All the schemes (or stacks) we consider are defined over k.

Fix a prime $\ell \neq p$. For a scheme (or stack) S write $\mathrm{D}(S)$ for the bounded derived category of ℓ-adic étale sheaves on S, and $\mathrm{P}(S) \subset \mathrm{D}(S)$ for the category of perverse sheaves.

Fix a non-trivial character $\psi : \mathbb{F}_p \to \bar{\mathbb{Q}}_\ell^*$, and write $\mathcal{L}_\psi$ for the corresponding Artin–Shreier sheaf on $\mathbb{A}^1$. Fix a square root $\bar{\mathbb{Q}}_\ell(\frac{1}{2})$ of the sheaf $\bar{\mathbb{Q}}_\ell(1)$ on $\mathrm{Spec}\,\mathbb{F}_q$. Isomorphism classes of such sheaves correspond to square roots of q in $\bar{\mathbb{Q}}_\ell$.

If $V \to S$ and $V^* \to S$ are dual rank n vector bundles over a stack S, we normalize the Fourier transform $\mathrm{Four}_\psi : \mathrm{D}(V) \to \mathrm{D}(V^*)$ by $\mathrm{Four}_\psi(K) = (p_{V^*})_!(\xi^*\mathcal{L}_\psi \otimes p_V^* K)[n](n/2)$, where p_V, p_{V^*} are the projections, and $\xi : V \times_S V^* \to \mathbb{A}^1$ is the pairing.

Our conventions about $\mathbb{Z}/2\mathbb{Z}$-gradings are those of [Lys06].

2. Canonical intertwining operators: the finite field case

2.1 Let M be a symplectic k-vector space of dimension $2d$. The symplectic form on M is denoted by $\omega\langle\cdot,\cdot\rangle$. The Heisenberg group $H = M \times \mathbb{A}^1$ with operation

$$(m_1, a_1)(m_2, a_2) = (m_1 + m_2,\ a_1 + a_2 + \tfrac{1}{2}\omega\langle m_1, m_2\rangle), \quad m_i \in M, a_i \in \mathbb{A}^1$$

is algebraic over k. Set $G = \mathbb{S}\mathrm{p}(M)$. Write $\mathcal{L}(M)$ for the variety of Lagrangian subspaces in M. Fix a one-dimensional k-vector space $\mathcal{J}$ (purely of degree $d \bmod 2$ as $\mathbb{Z}/2\mathbb{Z}$-graded). Let $\mathcal{A}$ be the (purely of degree zero as $\mathbb{Z}/2\mathbb{Z}$-graded) line bundle over $\mathcal{L}(M)$ with fibre $\mathcal{J} \otimes \det L$ at $L \in \mathcal{L}(M)$. Write $\widetilde{\mathcal{L}}(M)$ for the gerb of square roots of $\mathcal{A}$. The line bundle $\mathcal{A}$ is G-equivariant, so G acts naturally on $\widetilde{\mathcal{L}}(M)$.

For a k-point $L \in \mathcal{L}(M)$ write L^0 for a k-point of $\widetilde{\mathcal{L}}(M)$ over L. Write

$$\bar{L} = L \oplus k.$$

This is a subgroup of $H(k)$ equipped with the character $\chi_L : \bar{L} \to \bar{\mathbb{Q}}_\ell^*$ given by $\chi_L(l, a) = \psi(a)$, $l \in L$, $a \in k$. Write

$$\mathcal{H}_L = \{f : H(k) \to \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_L(\bar{l}) f(h), \text{ for } \bar{l} \in \bar{L}, h \in H\}.$$

This is a representation of $H(k)$ by right translations. Write $\mathcal{S}(H)$ for the space of all $\bar{\mathbb{Q}}_\ell$-valued functions on $H(k)$. The group G acts naturally in $\mathcal{S}(H)$. For $L \in \mathcal{L}(M)$, $g \in G$ we have an isomorphism $\mathcal{H}_L \to \mathcal{H}_{gL}$ sending f to gf.

The purpose of §§ 2 and 3 is to study the canonical intertwining operators (and their geometric analogs) between various models $\mathcal{H}_L$ of the Weil representation. The corresponding results for a finite field were formulated by Gurevich and Hadani [GH05] without a proof (we give all proofs for the sake of completeness). Besides, our setting is a bit different from that in *loc. cit.*, as we work with gerbs instead of the total space of the corresponding line bundles.

2.2 For k-points $L^0, N^0 \in \widetilde{\mathcal{L}}(M)$ we will define canonical intertwining operators

$$F_{N^0,L^0} : \mathcal{H}_L \to \mathcal{H}_N.$$

They will satisfy the following properties:

- $F_{L^0,L^0} = \mathrm{id}$;
- $F_{R^0,N^0} \circ F_{N^0,L^0} = F_{R^0,L^0}$ for any $R^0, N^0, L^0 \in \widetilde{\mathcal{L}}(M)$;
- for any $g \in G$ we have $g \circ F_{N^0,L^0} \circ g^{-1} = F_{gN^0,gL^0}$.

In Remark 2, § 3.1, we will define a function F^{cl} on the set of k-points of $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$, which we denote by $F_{N^0,L^0}(h)$ for $h \in H$. This function will realize the operator F_{N^0,L^0} by

$$(F_{N^0,L^0} f)(h_1) = \int_{h_2 \in H} F_{N^0,L^0}(h_1 h_2^{-1}) f(h_2)\, dh_2.$$

All our measures on finite sets are normalized by requiring the volume of a point to be one. Given two functions $f_1, f_2 : H \to \bar{\mathbb{Q}}_\ell$, their convolution $f_1 * f_2 : H \to \bar{\mathbb{Q}}_\ell$ is defined by

$$(f_1 * f_2)(h) = \int_{v \in H} f_1(hv^{-1}) f_2(v)\, dv, \quad h \in H.$$

The function F_{N^0,L^0} will satisfy the following properties:

- $F_{N^0,L^0}(\bar{n}h\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l}) F_{N^0,L^0}(h)$ for $\bar{l} \in \bar{L}$, $\bar{n} \in \bar{N}$, $h \in H$;
- $F_{gN^0,gL^0}(gh) = F_{N^0,L^0}(h)$ for $g \in G$, $h \in H$;
- the convolution property: $F_{R^0,L^0} = F_{R^0,N^0} * F_{N^0,L^0}$ for any $R^0, N^0, L^0 \in \widetilde{\mathcal{L}}(M)$;
- under the natural action of μ_2 on the set $\widetilde{\mathcal{L}}(M)(k)$ of (isomorphism classes of) k-points, F_{N^0,L^0} is odd as a function of N^0 and of L^0.

2.3 First, we define the non-normalized function $\tilde{F}_{N,L} : H \to \bar{\mathbb{Q}}_\ell$. It will depend only on $N, L \in \mathcal{L}(M)$, not on their enhanced structure.

Given $N, L \in \mathcal{L}(M)$, let $\chi_{NL} : \bar{N}\bar{L} \to \bar{\mathbb{Q}}_\ell$ be the function given by

$$\chi_{NL}(\bar{n}\bar{l}) = \chi_N(\bar{n})\chi_L(\bar{l}),$$

which is correctly defined in the sense that $\chi_{NL}(\bar{n}\bar{l})$ depends not on the pair $(\bar{n}, \bar{l})$ but only on their product in the Heisenberg group. Note that $\bar{N}\bar{L} = \bar{L}\bar{N}$ but $\chi_{NL} \neq \chi_{LN}$ in general. Set

$$\tilde{F}_{N,L}(h) = \begin{cases} \chi_{NL}(h) & \text{if } h \in \bar{N}\bar{L}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\chi_{LL} = \chi_L$.

Given $L, R, N \in \mathcal{L}(M)$ with $N \cap L = N \cap R = 0$, define $\theta(R, N, L) \in \bar{\mathbb{Q}}_\ell$ as follows. There is a unique map $b : L \to N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. Set

$$\theta(R, N, L) = \int_{l \in L} \psi\left(\frac{1}{2}\omega\langle l, b(l)\rangle\right) dl.$$

This expression has been considered in [Lys06, Appendix B].

LEMMA 1.

(a) *Let $L, N \in \mathcal{L}(M)$. If $L \cap N = 0$ then $\tilde{F}_{L,N} * \tilde{F}_{N,L} = q^{2d+1}\tilde{F}_{L,L}$.*

(b) *Let $L, R, N \in \mathcal{L}(M)$ with $N \cap L = N \cap R = 0$. Then $\tilde{F}_{R,N} * \tilde{F}_{N,L} = q^{d+1}\theta(R, N, L)\tilde{F}_{R,L}$.*

Proof. (b) Using $L \oplus N = N \oplus R = M$, for $h \in H$ we get

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1}\int_{v \in \bar{N}\backslash H} \chi_{RN}(hv^{-1})\chi_{NL}(v)\, dv = q^{d+1}\int_{r \in R} \chi_{RN}(h(-r, 0))\chi_{NL}(r, 0)\, dr.$$

Because of the equivariance property of $\tilde{F}_{R,N} * \tilde{F}_{N,L}$, we may assume that $h = (n, 0)$, $n \in N$. We get

$$\begin{aligned}(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1}\int_{r \in R} \chi_{RN}((n, 0)(-r, 0))\chi_{NL}(r, 0)\, dr \\ &= q^{d+1}\int_{r \in R} \psi(\omega\langle r, n\rangle)\chi_{NL}(r, 0)\, dr. \qquad (1)\end{aligned}$$

The latter formula essentially says that the resulting function on N is the Fourier transform of some local system on R (the symplectic form on M induces an isomorphism $R \xrightarrow{\sim} N^*$). This will be used for geometrization in Lemma 2.

There is a unique map $b : L \to N$ such that $R = \{l + b(l) \in L \oplus N \mid l \in L\}$. So, the above integral rewrites

$$\begin{aligned}(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) &= q^{d+1}\int_{l \in L} \psi(\omega\langle l, n\rangle)\chi_{NL}(l + b(l), 0)\, dl \\ &= q^{d+1}\int_{l \in L} \psi(\omega\langle l, n\rangle)\chi_{NL}\left(\left(b(l), \frac{1}{2}\omega\langle l, b(l)\rangle\right)(l, 0)\right) dl \\ &= q^{d+1}\int_{l \in L} \psi\left(\omega\langle l, n\rangle + \frac{1}{2}\omega\langle l, b(l)\rangle\right) dl. \qquad (2)\end{aligned}$$

Note that if $R = L$ then $b = 0$ and the latter formula yields item (a).

Let us identify $N \xrightarrow{\sim} L^*$ via the map sending $n \in N$ to the linear functional $l \mapsto \omega\langle l, n\rangle$. Denote by $\langle \cdot, \cdot \rangle$ the symmetric pairing between L and L^*. By Sublemma 1 below, the value (2) vanishes unless $n \in (R+L) \cap N = \operatorname{Im} b$. In the latter case pick $l_1 \in L$ with $b(l_1) = n$. Then

$$\chi_{RL}(n, 0) = \psi(-\tfrac{1}{2}\omega\langle l_1, b(l_1)\rangle).$$

So, we get for $L' = \operatorname{Ker} b$,

$$(\tilde{F}_{R,N} * \tilde{F}_{N,L})(h) = q^{d+1+\dim L'} \chi_{RL}(h) \int_{l \in L/L'} \psi\left(\frac{1}{2}\omega\langle l, b(l)\rangle\right) dl.$$

We are done. □

Sublemma 1. *Let L be a d-dimensional k-vector space, $b \in \operatorname{Sym}^2 L^*$ and $u \in L^*$. View b as a map $b : L \to L^*$, and let L' be the kernel of b. Then*

$$\int_{l \in L} \psi\left(\langle l, u\rangle + \frac{1}{2}\langle l, b(l)\rangle\right) dl \tag{3}$$

is supported at $u \in (L/L')^*$ and there equals

$$q^{\dim L'} \psi\left(-\frac{1}{2}\langle b^{-1}u, u\rangle\right) \int_{L/L'} \psi\left(\frac{1}{2}\langle l, b(l)\rangle\right) dl,$$

where $b : L/L' \xrightarrow{\sim} (L/L')^*$, so that $b^{-1}u \in L/L'$. (Here the scalar product is between L and L^*, so is symmetric.)

Proof. Let $L' \subset L$ denote the kernel of $b : L \to L^*$. Integrating first along the fibres of the projection $L \to L/L'$ we will get a zero result unless $u \in (L/L')^*$. For any $l_0 \in L$ the integral (3) equals

$$\begin{aligned} &\int_{l \in L} \psi\left(\langle l + l_0, u\rangle + \frac{1}{2}\langle l + l_0, b(l) + b(l_0)\rangle\right) dl \\ &\quad = \psi\left(\langle l_0, u\rangle + \frac{1}{2}\langle l_0, b(l_0)\rangle\right) \int_{l \in L} \psi\left(\langle l, u + b(l_0)\rangle + \frac{1}{2}\langle l, b(l)\rangle\right) dl. \end{aligned}$$

Assuming that $u \in (L/L')^*$, take l_0 such that $u = -b(l_0)$. Then (3) becomes

$$\psi\left(\frac{1}{2}\langle l_0, u\rangle\right) \int_{l \in L} \psi\left(\frac{1}{2}\langle l, b(l)\rangle\right) dl.$$

We are done. □

Remark 1. The expression (3) is the Fourier transform from L to L^*. In the geometric setting we will use item (b) of Lemma 1 only under the additional assumption $R \cap L = 0$.

3. Geometrization

3.1 Let M, H, $\mathcal{L}(M)$ and $\widetilde{\mathcal{L}}(M)$ be as in § 2.1. Recall that $G = \mathbb{S}\mathrm{p}(M)$. For each $L \in \mathcal{L}(M)$ we have a rank-one local system χ_L on $\bar{L} = L \times \mathbb{A}^1$ defined by $\chi_L = \mathrm{pr}^* \mathcal{L}_\psi$, where $\mathrm{pr} : L \times \mathbb{A}^1 \to \mathbb{A}^1$ is the projection. Let $\mathcal{H}_L$ denote the category of perverse sheaves on H which are $(\bar{L}, \chi_L)$-equivariant under the left multiplication; this is a full subcategory in $\mathrm{P}(H)$. Write $\mathrm{D}\mathcal{H}_L \subset \mathrm{D}(H)$ for the full subcategory of objects whose all perverse cohomologies lie in $\mathcal{H}_L$.

Denote by $C \to \mathcal{L}(M)$ (respectively, $\bar{C} \to \mathcal{L}(M)$) the vector bundle whose fibre over $L \in \mathcal{L}(M)$ is L (respectively, $\bar{L} = L \times \mathbb{A}^1$). Its inverse image to $\widetilde{\mathcal{L}}(M)$ is denoted by the same symbol.

Write $\chi_{\bar{C}}$ for the local system $p^*\mathcal{L}_\psi$ on $\bar{C}$, where $p : \bar{C} \to \mathbb{A}^1$ is the projection on the center sending $(L \in \mathcal{L}(M), (l, a) \in \bar{L})$ to a. Consider the maps

$$\mathrm{pr}, \mathrm{act}_{lr} : \bar{C} \times \bar{C} \times H \to \mathcal{L}(M) \times \mathcal{L}(M) \times H \times H,$$

where act_{lr} sends $(\bar{n} \in \bar{N}, \bar{l} \in \bar{L}, h)$ to $(N, L, \bar{n}h\bar{l})$, and pr sends the above point to (N, L, h). We say that a perverse sheaf K on $\mathcal{L}(M) \times \mathcal{L}(M) \times H$ is act_{lr}*-equivariant* if it admits an isomorphism

$$\mathrm{act}_{lr}^* K \xrightarrow{\sim} \mathrm{pr}^* K \otimes \mathrm{pr}_1^* \chi_{\bar{C}} \otimes \mathrm{pr}_2^* \chi_{\bar{C}}$$

satisfying the usual associativity condition and whose restriction to the unit section is the identity (such an isomorphism is unique if it exists). One has a similar definition for $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$.

Let

$$\mathrm{act}_G : G \times \widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H \to \widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$$

be the action map sending (g, N^0, L^0, h) to (gN^0, gL^0, gh). For this map we have the usual notion of a *G-equivariant perverse sheaf* on $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$. As G is connected, a perverse sheaf on $\widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$ admits at most one G-equivariant structure.

If S is a stack then for $K, F \in \mathrm{D}(S \times H)$ define their convolution $K * F \in \mathrm{D}(S \times H)$ by

$$K * F = \mathrm{mult}_!(\mathrm{pr}_1^* K \otimes \mathrm{pr}_2^* F) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{d+1-2\dim \mathcal{L}(M)},$$

where $\mathrm{pr}_i : S \times H \times H \to S \times H$ is the projection to the ith component in the pair $H \times H$ (and the identity on S). The multiplication map $\mathrm{mult} : H \times H \to H$ sends (h_1, h_2) to $h_1 h_2$.

Let

$$(\mathcal{L}(M) \times H)_\triangle \hookrightarrow \mathcal{L}(M) \times H \tag{4}$$

be the closed subscheme of those $(L \in \mathcal{L}(M), h \in H)$ for which $h \in \bar{L}$. Let

$$\alpha_\triangle : (\mathcal{L}(M) \times H)_\triangle \to \mathbb{A}^1$$

be the map sending (L, h) to a, where $h = (l, a)$, $l \in L$, $a \in \mathbb{A}^1$. Define a perverse sheaf

$$\tilde{F}_\triangle = \alpha_\triangle^* \mathcal{L}_\psi \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{d+1+\dim \mathcal{L}(M)},$$

which we extend by zero under (4).

Since $\widetilde{\mathcal{L}}(M) \to \mathcal{L}(M)$ is a μ_2-gerb, μ_2 acts on each $K \in \mathrm{D}(\widetilde{\mathcal{L}}(M))$, and we say that K is *genuine* if $-1 \in \mu_2$ acts on K as -1.

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Theorem 1. *There exists an irreducible perverse sheaf F on $\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times H$ (pure of weight zero) with the following properties:*

- *for the diagonal map $i:\widetilde{\mathcal{L}}(M)\times H\to\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times H$ the complex i^*F identifies canonically with the inverse image of*
$$\tilde{F}_{\triangle}\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim\mathcal{L}(M)}$$
under the projection $\widetilde{\mathcal{L}}(M)\times \mathrm{H}\to\mathcal{L}(M)\times H$;
- *F is act_{lr}-equivariant;*
- *F is G-equivariant;*
- *F is genuine in the first and the second variable;*
- *the convolution property for F holds, namely for the ijth projections*
$$q_{ij}:\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times H\to\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times H$$
inside the triple $\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)$ we have $(q_{12}^*F)*(q_{23}^*F)\widetilde{\to}q_{13}^*F$ canonically.

The proof of Theorem 1 is given in §§ 3.2–3.4.

Remark 2. In the case when $k=\mathbb{F}_q$ define F^{cl} as the trace of the geometric Frobenius on F.

3.2 Let $U\subset\mathcal{L}(M)\times\mathcal{L}(M)$ be the open subset of pairs $(N,L)\in\mathcal{L}(M)\times\mathcal{L}(M)$ such that $N\cap L=0$. Define a perverse sheaf $\tilde{F}_U$ on $U\times H$ as follows. Let
$$\alpha_U:U\times H\to\mathbb{A}^1$$
be the map sending (N,L,h) to $a+\frac{1}{2}\omega\langle l,n\rangle$, where $l\in L$, $n\in N$, $a\in\mathbb{A}^1$ are uniquely defined by $h=(n+l,a)$. Set
$$\tilde{F}_U=\alpha_U^*\mathcal{L}_\psi\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim H+2\dim\mathcal{L}(M)}. \tag{5}$$
Write $U\times_{\mathcal{L}(M)}U\subset\mathcal{L}(M)\times\mathcal{L}(M)\times\mathcal{L}(M)$ for the open subscheme classifying (R,N,L) with $N\cap L=N\cap R=0$. Let
$$q_i:U\times_{\mathcal{L}(M)}U\to U$$
be the projection on the ith factor, so q_1 (respectively, q_2) sends (R,N,L) to (R,N) (respectively, to (N,L)). Let $q:U\times_{\mathcal{L}(M)}U\to\mathcal{L}(M)\times\mathcal{L}(M)$ be the map sending (R,N,L) to (R,L). Write
$$(U\times_{\mathcal{L}(M)}U)_0=q^{-1}(U).$$

The geometric analog of $\theta(R,N,L)$ is the following (shifted) perverse sheaf Θ on $U\times_{\mathcal{L}(M)}U$. Let $\pi_C:C_3\to U\times_{\mathcal{L}(M)}U$ be the vector bundle whose fibre over (R,N,L) is L. We have a map $\beta:C_3\to\mathbb{A}^1$ defined as follows. Given a point $(R,N,L)\in U\times_{\mathcal{L}(M)}U$, there is a unique map $b:L\to N$ such that $R=\{l+b(l)\in L\oplus N=M\mid l\in L\}$. Set $\beta(R,N,L,l)=\frac{1}{2}\omega\langle l,b(l)\rangle$. Set
$$\Theta=(\pi_C)_!\beta^*\mathcal{L}_\psi\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^d.$$

Write $Y=\mathcal{L}(M)\times\mathcal{L}(M)$, and let $\mathcal{A}_Y$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) line bundle on Y whose fibre at (R,L) is $\det R\otimes\det L$. Write $\tilde{Y}$ for the gerb of square roots of $\mathcal{A}_Y$. Note that $\mathcal{A}_Y$ is G-equivariant, so G acts on $\tilde{Y}$ naturally.

The following perverse sheaf S_M on $\tilde{Y}$ was introduced in [Lys06, Definition 2]. Let $Y_i\subset Y$ be the locally closed subscheme given by $\dim(R\cap L)=i$ for $(R,L)\in Y_i$. The restriction of $\mathcal{A}_Y$ to each Y_i admits the following G-equivariant square root. For a point $(R,L)\in Y_i$ we have an

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isomorphism $L/(R\cap L)\widetilde{\to}(R/(R\cap L))^*$ sending l to the functional $r\mapsto\omega\langle r,l\rangle$. It induces a $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\det R\otimes\det L\widetilde{\to}\det(R\cap L)^2$.

So, for the restriction $\widetilde{Y}_i$ of the gerb $\tilde{Y}\to Y$ to Y_i we get a trivialization

$$\widetilde{Y}_i\widetilde{\to}Y_i\times B(\mu_2). \tag{6}$$

Write W for the non-trivial local system of rank one on $B(\mu_2)$ corresponding to the covering $\operatorname{Spec} k\to B(\mu_2)$.

Definition 1. Let $S_{M,g}$ (respectively, $S_{M,s}$) denote the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell\boxtimes W)\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim Y}$$

from $\widetilde{Y}_0$ to $\tilde{Y}$ (respectively, of $(\bar{\mathbb{Q}}_\ell\boxtimes W)\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim Y-1}$ from $\widetilde{Y}_1$ to $\tilde{Y}$). Set $S_M=S_{M,g}\oplus S_{M,s}$.

Let

$$\pi_Y:U\times_{\mathcal{L}(M)}U\to\tilde{Y}$$

be the map sending (R,N,L) to

$$(R,L,\mathcal{B},\epsilon:\mathcal{B}^2\widetilde{\to}\det R\otimes\det L),$$

where $\mathcal{B}=\det L$ and ϵ is the isomorphism induced by ϵ_0. Here $\epsilon_0:L\widetilde{\to}R$ is the isomorphism sending $l\in L$ to $l+b(l)\in R$. In other words, ϵ_0 sends l to the unique $r\in R$ such that $r=l \bmod N\in M/N$. Write also $\tilde{U}=\widetilde{Y}_0$.

Define $\mathcal{E}\in\mathrm{D}(\operatorname{Spec} k)$ by

$$\mathcal{E}=\mathrm{R}\Gamma_c(\mathbb{A}^1,\beta_0^*\mathcal{L}_\psi)\otimes\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}),$$

where $\beta_0:\mathbb{A}^1\to\mathbb{A}^1$ sends x to x^2. Then $\mathcal{E}$ is a one-dimensional vector space placed in cohomological degree zero. The geometric Frobenius $\mathrm{Fr}_{\mathbb{F}_q}$ acts on $\mathcal{E}^2$ by 1 if $-1\in(\mathbb{F}_q^*)^2$ and by -1 otherwise. A choice of $\sqrt{-1}\in k$ yields an isomorphism $\mathcal{E}^2\widetilde{\to}\bar{\mathbb{Q}}_\ell$, so $\mathcal{E}^4\widetilde{\to}\bar{\mathbb{Q}}_\ell$ canonically.

As in [Lys06, Proposition 5], one gets a canonical isomorphism

$$\pi_Y^*(S_{M,g}\otimes\mathcal{E}^d\oplus S_{M,s}\otimes\mathcal{E}^{d-1})\widetilde{\to}\Theta\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{2\dim\mathcal{L}(M)}. \tag{7}$$

Since $d\geq 1$, the restriction $\pi_Y:(U\times_{\mathcal{L}(M)}U)_0\to\tilde{U}$ is smooth of relative dimension $\dim\mathcal{L}(M)$, with geometrically connected fibres. It is convenient to introduce a rank-one local system Θ_U on $\tilde{U}$ equipped with a canonical isomorphism

$$\Theta\widetilde{\to}\pi_Y^*\Theta_U \tag{8}$$

over $(U\times_{\mathcal{L}(M)}U)_0$. The local system Θ_U is defined up to a unique isomorphism.

Let $i_U:U\to U\times_{\mathcal{L}(M)}U$ be the map sending (L,N) to (L,N,L). Let $p_1:U\to\mathcal{L}(M)$ be the projection sending (L,N) to L.

Lemma 2.

(a) *The complex*

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{L}(M)}$$

is an irreducible perverse sheaf on $U \times_{\mathcal{L}(M)} U \times H$ *pure of weight zero. We have canonically that*

$$i_U^*((q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U)) \widetilde{\to} p_1^* \tilde{F}_\Delta \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{L}(M)}$$

over $U \times H$.

(b) *There is a canonical isomorphism*

$$(q_1^* \tilde{F}_U) * (q_2^* \tilde{F}_U) \widetilde{\to} q^* \tilde{F}_U \otimes \Theta$$

over $(U \times_{\mathcal{L}(M)} U)_0 \times H$.

Proof. (a) This follows from the properties of the Fourier transform as in Lemma 1, formula (1).

(b) The proof of Lemma 1 goes through in the geometric setting. Our additional assumption that $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$ means that $b : L \to N$ is an isomorphism (it simplifies the argument a little). □

Remark 3. Let $i_\Delta : \mathcal{L}(M) \to \tilde{Y}$ be the map sending L to $(L, L, \mathcal{B} = \det L)$ equipped with the isomorphism $\mathrm{id} : \mathcal{B}^2 \widetilde{\to} \det L \otimes \det L$. The commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{i_U} & U \times_{\mathcal{L}(M)} U \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_Y} \\ \mathcal{L}(M) & \xrightarrow{i_\Delta} & \tilde{Y} \end{array} \tag{9}$$

together with (7) yield a canonical isomorphism

$$i_\Delta^* S_M \widetilde{\to} \begin{cases} \mathcal{E}^{-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2\dim \mathcal{L}(M) - d}, & d \text{ is even}, \\ \mathcal{E}^{1-d} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{2\dim \mathcal{L}(M) - d}, & d \text{ is odd}. \end{cases}$$

3.3 Consider the following diagram.

$$\begin{array}{ccccc} \tilde{U} & \xleftarrow{\tilde{q}_1} & (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_2} & \tilde{U} \\ & & \downarrow{\scriptstyle \tilde{q}} & & \\ & & \tilde{U} & & \end{array}$$

Here $\tilde{q}$ is the restriction of π_Y, and the map $\tilde{q}_i$ is the lifting of q_i defined as follows. We set $\tilde{q}_1(R, N, L) = \tilde{q}(R, L, N)$ and $\tilde{q}_2(R, N, L) = \tilde{q}(N, R, L)$.

The following property is a geometric counterpart of the way the Maslov index of (R, N, L) changes under permutations of three Lagrangian subspaces.

Lemma 3.

(a) *For* $i = 1, 2$ *we have canonically that* $\tilde{q}_i^* \Theta_U \otimes \tilde{q}^* \Theta_U \widetilde{\to} \bar{\mathbb{Q}}_\ell$ *over* $(U \times_{\mathcal{L}(M)} U)_0$.

(b) *We have* $\Theta_U^2 \widetilde{\to} \mathcal{E}^{2d}$ *canonically, so* $\Theta_U^4 \widetilde{\to} \bar{\mathbb{Q}}_\ell$ *canonically.*

Proof. (a) The two isomorphisms are obtained similarly, we consider only the case $i = 2$. For a point $(R, N, L) \in (U \times_{\mathcal{L}(M)} U)_0$ we have isomorphisms $b : L \widetilde{\to} N$ and $b_0 : L \widetilde{\to} R$ such

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that $R=\{l+b(l)\mid l\in L\}$ and $N=\{l+b_0(l)\mid l\in L\}$. Clearly, $b_0(-l)=l+b(l)$ for $l\in L$. Let $\beta_2: L\times L\to\mathbb{A}^1$ be the map sending (l,l_0) to $\frac{1}{2}\omega\langle l,b(l)\rangle+\frac{1}{2}\omega\langle l,b_0(l)\rangle$. We must show that

$$\mathrm{R}\Gamma_c(L\times L,\beta_2^*\mathcal{L}_\psi)\,\tilde{\to}\,\bar{\mathbb{Q}}_\ell[2d](d).$$

The quadratic form $(l,l_0)\mapsto\omega\langle l,b(l)\rangle-\omega\langle l_0,b(l_0)\rangle$ is hyperbolic on $L\oplus L$. Consider the isotopic subspace $Q=\{(l,l)\in L\times L\mid l\in L\}$. Integrating first along the fibres of the projection $L\times L\to(L\times L)/Q$ and then over $(L\times L)/Q$, one gets the desired isomorphism.

(b) This follows from (7). □

Define a perverse sheaf F_U on $\tilde{U}\times H$ by

$$F_U=\mathrm{pr}_1^*\,\Theta_U\otimes\tilde{F}_U.$$

It is understood that we take the inverse image of $\tilde{F}_U$ under the projection $\tilde{U}\times H\to U\times H$ in the above formula. Let F be the intermediate extension of F_U under the open immersion $\tilde{U}\times H\subset\tilde{Y}\times H$.

Remark 4. In the case when $d=0$ we have $H=\mathbb{A}^1$ and $\tilde{Y}=B(\mu_2)$. In this case by definition $F=W\boxtimes\mathcal{L}_\psi\otimes\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})$ over $\tilde{Y}\times H=B(\mu_2)\times\mathbb{A}^1$.

Combining Lemma 3 and item (b) of Lemma 2, we get the following.

Lemma 4. *We have canonically that* $(\tilde{q}_1^*F_U)*(\tilde{q}_2^*F_U)\,\tilde{\to}\,\tilde{q}^*F_U\otimes\mathcal{E}^{2d}$ *over* $(U\times_{\mathcal{L}(M)}U)_0\times H$.

We have a map $\xi:\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\to\tilde{Y}$ sending $(\mathcal{B}_1,N,\mathcal{B}_1^2\,\tilde{\to}\,\mathcal{J}\otimes\det N;\mathcal{B}_2,L,\mathcal{B}_2^2\,\tilde{\to}\,\mathcal{J}\otimes\det L)$ to $(N,L,\mathcal{B})$, where $\mathcal{B}=\mathcal{B}_1\otimes\mathcal{B}_2\otimes\mathcal{J}^{-1}$ is equipped with the natural isomorphism $\mathcal{B}^2\,\tilde{\to}\,\det N\otimes\det L$. The restriction of F under

$$\xi\times\mathrm{id}:\widetilde{\mathcal{L}}(M)\times\widetilde{\mathcal{L}}(M)\times H\to\tilde{Y}\times H$$

is also denoted by F. Clearly, F is an irreducible perverse sheaf of weight zero.

Consider the cartesian square as follows.

$$\begin{array}{ccc}(U\times_{\mathcal{L}(M)}U)_0\times H & \hookrightarrow & (U\times_{\mathcal{L}(M)}U)\times H\\ \downarrow{\scriptstyle \pi_Y\times\mathrm{id}} & & \downarrow{\scriptstyle \pi_Y\times\mathrm{id}}\\ \tilde{U}\times H & \hookrightarrow & \tilde{Y}\times H\end{array}$$

This diagram together with Lemma 2 yield a canonical isomorphism over $(U\times_{\mathcal{L}(M)}U)\times H$,

$$(\pi_Y\times\mathrm{id})^*F\,\tilde{\to}\,(q_1^*\tilde{F}_U)*(q_2^*\tilde{F}_U),\tag{10}$$

by intermediate extension from $(U\times_{\mathcal{L}(M)}U)_0\times H$. This gives an explicit formula for F.

Consider the following diagram.

$$\begin{array}{ccc}U\times H & \xrightarrow{i_U\times\mathrm{id}} & U\times_{\mathcal{L}(M)}U\times H\\ \downarrow{\scriptstyle p_1\times\mathrm{id}} & & \downarrow{\scriptstyle \pi_Y\times\mathrm{id}}\\ \mathcal{L}(M)\times H & \xrightarrow{i_\triangle\times\mathrm{id}} & \tilde{Y}\times H\end{array}$$

This is obtained from (9) by multiplication with H. By Lemma 2 and (10), we get canonically that

$$(p_1\times\mathrm{id})^*(i_\triangle\times\mathrm{id})^*F\,\tilde{\to}\,(p_1\times\mathrm{id})^*\tilde{F}_\triangle\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim\mathcal{L}(M)}.$$

Since $\tilde{F}_\triangle$ is perverse and p_1 has connected fibres, this isomorphism descends to a uniquely defined isomorphism

$$(i_\triangle \times \mathrm{id})^* F \xrightarrow{\sim} \tilde{F}_\triangle \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{L}(M)}.$$

By construction, F is act_{lr}-equivariant and G-equivariant (this holds for F_U and this property is preserved by the intermediate extension).

3.4 To finish the proof of Theorem 1, it remains to establish the convolution property of F. We actually prove it in the following form.

Write $\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$ for the stack classifying $R, N, L \in \mathcal{L}(M)$ together with one-dimensional k-vector spaces $\mathcal{B}_1, \mathcal{B}_2$ and isomorphisms $\mathcal{B}_1^2 \xrightarrow{\sim} \det R \otimes \det N$ and $\mathcal{B}_2^2 \xrightarrow{\sim} \det N \otimes \det L$. We have a diagram

$$\begin{array}{ccccc} \tilde{Y} & \xleftarrow{\tau_1} & \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y} & \xrightarrow{\tau_2} & \tilde{Y} \\ & & \downarrow{\scriptstyle \tau} & & \\ & & \tilde{Y} & & \end{array}$$

where τ_1 (respectively, τ_2) sends the above collection to $(R, N, \mathcal{B}_1) \in \tilde{Y}$ (respectively, $(N, L, \mathcal{B}_2) \in \tilde{Y}$). The map τ sends the above collection to $(R, L, \mathcal{B})$, where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2 \otimes (\det N)^{-1}$ is equipped with $\mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L$.

PROPOSITION 1. *There is a canonical isomorphism over* $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H$,

$$(\tau_1^* F) * (\tau_2^* F) \xrightarrow{\sim} \tau^* F. \qquad (11)$$

Proof.

Step 1. Consider the following diagram.

$$\begin{array}{ccc} (U \times_{\mathcal{L}(M)} U)_0 & \xrightarrow{\tilde{q}_1 \times \tilde{q}_2} & (\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \\ & \searrow^{\tilde{q}} & \downarrow{\scriptstyle \tau} \\ & & \tilde{U} \end{array}$$

It becomes 2-commutative over $\mathrm{Spec}\, \mathbb{F}_q(\sqrt{-1})$. More precisely, for $K \in \mathrm{D}(\tilde{U})$ we have a canonical isomorphism functorial in K,

$$\tilde{q}^* K \otimes \mathcal{E}^{2d} \xrightarrow{\sim} (\tilde{q}_1 \times \tilde{q}_2)^* \tau^* K.$$

Indeed, let (R, N, L) be a k-point of $(U \times_{\mathcal{L}(M)} U)_0$, and let $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$ be its image under $\tilde{q}_1 \times \tilde{q}_2$. So, $\mathcal{B}_1 = \det N$ and $\pi_Y(R, L, N) = (R, N, \mathcal{B}_1)$, $\mathcal{B}_2 = \det L$ and $\pi_Y(N, R, L) = (N, L, \mathcal{B}_2)$. Write

$$\tau(R, N, L, \mathcal{B}_1, \mathcal{B}_2) = (R, L, \mathcal{B}, \delta : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L).$$

Write $\tilde{q}(R, N, L) = (R, L, \mathcal{B}, \delta_0 : \mathcal{B}^2 \xrightarrow{\sim} \det R \otimes \det L)$. It suffices to show that $\delta_0 = (-1)^d \delta$.

Let $\epsilon_1 : N \xrightarrow{\sim} R$ be the isomorphism sending $n \in N$ to $r \in R$ such that $r = n \bmod L$. Write $\epsilon_2 : L \xrightarrow{\sim} N$ for the isomorphism sending $l \in L$ to $n \in N$ such that $l = n \bmod R$. Let $\epsilon_0 : L \xrightarrow{\sim} R$ be the isomorphism sending $l \in L$ to $r \in R$ such that $r = l \bmod N$. We get two isomorphisms

$$\mathrm{id} \otimes \det \epsilon_0,\ \det \epsilon_1 \otimes \det \epsilon_2 :\ \det N \otimes \det L \xrightarrow{\sim} \det R \otimes \det N.$$

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We must show that $\mathrm{id} \otimes \det \epsilon_0 = (-1)^d \det \epsilon_1 \otimes \det \epsilon_2$. Pick a base $\{n_1, \dots, n_d\}$ in N. Define $r_i \in R$, $l_i \in L$ by $n_i = r_i + l_i$. Then

$$\epsilon_1(n_i) = r_i, \quad \epsilon_2(l_i) = n_i, \quad \epsilon_0(l_i) = -r_i.$$

So, $\epsilon_0(l_1 \wedge \dots \wedge l_d) = (-1)^d r_1 \wedge \dots \wedge r_d$. On the other hand, $\det \epsilon_1 \otimes \det \epsilon_2$ sends

$$(n_1 \wedge \dots \wedge n_d) \otimes (l_1 \wedge \dots \wedge l_d)$$

to $(r_1 \wedge \dots \wedge r_d) \otimes (n_1 \wedge \dots \wedge n_d)$.

Step 2. The isomorphism (6) for $i = 0$ yields $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \widetilde{\to} (U \times_{\mathcal{L}(M)} U)_0 \times B(\mu_2) \times B(\mu_2)$. The corresponding 2-automorphisms $\mu_2 \times \mu_2$ of $(\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y})$ act in the same way on both sides of (11). Now from Step 1 it follows that the isomorphism of Lemma 4 descends under $\tilde{q}_1 \times \tilde{q}_2$ to the desired isomorphism (11) over $(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H$.

Step 3. To finish the proof it suffices to show that $(\tau_1^* F) * (\tau_2^* F)$ is perverse, the intermediate extension under the open immersion

$$(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0 \times H \subset (\tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}) \times H.$$

Let us first explain the idea informally, at the level of functions. In this step for $(N, R, \mathcal{B}) \in \tilde{Y}$ we denote by $F_{N,R,\mathcal{B}} : H \to \bar{\mathbb{Q}}_\ell$ the function trace of Frobenius of the sheaf F.

Given $(R, N, \mathcal{B}_1) \in \tilde{Y}$ and $(N, L, \mathcal{B}_2) \in \tilde{Y}$ pick any $S, T \in \mathcal{L}(M)$ such that $(R, S, N) \in U \times_{\mathcal{L}(M)} U$, $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ and $S \cap T = S \cap L = 0$. Assuming that

$$(R, N, \mathcal{B}_1) = \pi_Y(R, S, N) \quad \text{and} \quad (N, L, \mathcal{B}_2) = \pi_Y(N, T, L),$$

by (10) we get

$$\begin{aligned} F_{R,N,\mathcal{B}_1} * F_{N,L,\mathcal{B}_2} &= (\tilde{F}_{R,S} * \tilde{F}_{S,N}) * (\tilde{F}_{N,T} * \tilde{F}_{T,L}) = q^{d+1}\theta(S, N, T)\tilde{F}_{R,S} * \tilde{F}_{S,T} * \tilde{F}_{T,L} \\ &= q^{2d+2}\theta(S, N, T)\theta(S, T, L)\tilde{F}_{R,S} * \tilde{F}_{S,L} = q^{2d+2}\theta(S, N, T)\theta(S, T, L)F_{R,L,\mathcal{B}}, \end{aligned}$$

where $(R, L, \mathcal{B}) = \pi_Y(R, S, L)$. Now we turn back to the geometric setting.

Step 4. Consider the scheme $\mathcal{W}$ classifying $(R, S, N) \in U \times_{\mathcal{L}(M)} U$ and $(N, T, L) \in U \times_{\mathcal{L}(M)} U$ such that $S \cap T = S \cap L = 0$. Let

$$\kappa : \mathcal{W} \to \tilde{Y} \times_{\mathcal{L}(M)} \tilde{Y}$$

be the map sending the above point to $(R, N, L, \mathcal{B}_1, \mathcal{B}_2)$, where $(R, N, \mathcal{B}_1) = \pi_Y(R, S, N)$ and $(N, L, \mathcal{B}_2) = \pi_Y(N, T, L)$. The map κ is smooth and surjective. It suffices to show that

$$\kappa^*((\tau_1^* F) * (\tau_2^* F))$$

is a shifted perverse sheaf, the intermediate extension from $\kappa^{-1}(\tilde{U} \times_{\mathcal{L}(M)} \tilde{U})_0$.

Let $\mu : \mathcal{W} \to U \times_{\mathcal{L}(M)} U$ be the map sending a point of $\mathcal{W}$ to (R, S, L). Applying (10) several times as in Step 3, we learn that there is a local system of rank one and order two, say $\mathcal{I}$ on $\mathcal{W}$ such that

$$\kappa^*((\tau_1^* F) * (\tau_2^* F)) \widetilde{\to} \mathcal{I} \otimes \mu^* \pi_Y^* F.$$

Since F is an irreducible perverse sheaf, our assertion follows. □

Thus, Theorem 1 is proved.

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3.5 Now given k-points $N^0, L^0 \in \widetilde{\mathcal{L}}(M)$, let $F_{N^0,L^0} \in \mathrm{D}(H)$ be the $*$-restriction of F under $(N^0, L^0) \times \mathrm{id} : H \hookrightarrow \tilde{Y} \times H$. Define the functor $\mathcal{F}_{N^0,L^0} : \mathrm{D}\mathcal{H}_L \to \mathrm{D}\mathcal{H}_N$ by

$$\mathcal{F}_{N^0,L^0}(K) = F_{N^0,L^0} * K.$$

To see that it preserves perversity we can pick $S^0 \in \widetilde{\mathcal{L}}(M)$ with $N \cap S = L \cap S = 0$ and use $\mathcal{F}_{N^0,L^0} = \mathcal{F}_{N^0,S^0} \circ \mathcal{F}_{S^0,L^0}$. This reduces the question to the case $N \cap L = 0$; in the latter case $\mathcal{F}_{N^0,L^0}$ is nothing but the Fourier transform.

By Theorem 1, for $N^0, L^0, R^0 \in \widetilde{\mathcal{L}}(M)$ the diagram is canonically 2-commutative.

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{R^0,L^0}} & \mathrm{D}\mathcal{H}_R \\ & \searrow^{\mathcal{F}_{N^0,L^0}} & \downarrow{\scriptstyle \mathcal{F}_{N^0,R^0}} \\ & & \mathrm{D}\mathcal{H}_N \end{array}$$

3.6 Non-ramified Weil category

For a k-point $L^0 \in \widetilde{\mathcal{L}}(M)$ let $i_{L^0} : \widetilde{\mathcal{L}}(M) \to \widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H$ be the map sending N^0 to $(N^0, L^0, 0)$. We get a functor $\mathcal{F}_{L^0} : \mathrm{D}\mathcal{H}_L \to \mathrm{D}(\widetilde{\mathcal{L}}(M))$ sending K to the complex

$$i^*_{L^0}(F * \mathrm{pr}_3^* K) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{L}(M) - 2d - 1}.$$

For any k-points $L^0, N^0 \in \widetilde{\mathcal{L}}(M)$ the diagram commutes.

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathrm{D}(\widetilde{\mathcal{L}}(M)) \\ & \searrow^{\mathcal{F}_{L^0,N^0}} & \uparrow{\scriptstyle \mathcal{F}_{N^0}} \\ & & \mathrm{D}\mathcal{H}_N \end{array} \tag{12}$$

One checks that $\mathcal{F}_{L^0}$ is exact for the perverse t-structure.

Definition 2. The *non-ramified Weil category* $W(\widetilde{\mathcal{L}}(M))$ is the essential image of $\mathcal{F}_{L_0} : \mathcal{H}_L \to \mathrm{P}(\widetilde{\mathcal{L}}(M))$. This is a full subcategory in $\mathrm{P}(\widetilde{\mathcal{L}}(M))$ independent of L^0, because (12) is commutative.

The group G acts naturally on $\widetilde{\mathcal{L}}(M)$, and hence also on $\mathrm{P}(\widetilde{\mathcal{L}}(M))$. This action preserves the full subcategory $W(\widetilde{\mathcal{L}}(M))$.

At the classical level, for $L \in \mathcal{L}(M)$ the G-representation $\mathcal{H}_L \widetilde{\to} \mathcal{H}_{L,\mathrm{odd}} \oplus \mathcal{H}_{L,\mathrm{even}}$ is a direct sum of two irreducible ones consisting of odd and even functions, respectively. The category $W(\widetilde{\mathcal{L}}(M))$ is a geometric analog of the space $\mathcal{H}_{L,\mathrm{even}}$. The geometric analog of the whole Weil representation $\mathcal{H}_L$ is as follows.

Definition 3. Let $\mathrm{act}_l : \bar{C} \times H \to \widetilde{\mathcal{L}}(M) \times H$ be the map sending $(L^0, h, \bar{l} \in \bar{L})$ to $(L^0, \bar{l}h)$. A perverse sheaf $K \in \mathrm{P}(\widetilde{\mathcal{L}}(M) \times H)$ is $(\bar{C}, \chi_{\bar{C}})$-*equivariant* if it is equipped with an isomorphism

$$\mathrm{act}_l^* K \widetilde{\to} \mathrm{pr}^* K \otimes \mathrm{pr}_1^* \chi_{\bar{C}}$$

satisfying the usual associativity property, and whose restriction to the unit section is the identity.

The complete Weil category $W(M)$ is the category of pairs (K, σ), where $K \in \mathrm{P}(\widetilde{\mathcal{L}}(M) \times H)$ is a $(\bar{C}, \chi_{\bar{C}})$-equivariant perverse sheaf, and

$$\sigma : F * \mathrm{pr}_{23}^* K \widetilde{\to} \mathrm{pr}_{13}^* K$$

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is an isomorphism for the projections $\mathrm{pr}_{13}, \mathrm{pr}_{23} : \widetilde{\mathcal{L}}(M) \times \widetilde{\mathcal{L}}(M) \times H \to \widetilde{\mathcal{L}}(M) \times H$. The map σ must be compatible with the associativity constraint and the unit section constraint of F.

The group G acts on $\widetilde{\mathcal{L}}(M) \times H$ sending $(g \in G, L^0, h)$ to (gL^0, gh). This action extends to an action of G on the category $W(M)$.

4. Compatibility property

4.1 In this section we establish the following additional property of the canonical intertwining operators. Let $V \subset M$ be an isotropic subspace, and $V^\perp \subset M$ its orthogonal complement. Let $\mathcal{L}(M)_V \subset \mathcal{L}(M)$ be the open subscheme of $L \in \mathcal{L}(M)$ such that $L \cap V = 0$. Set $M_0 = V^\perp/V$. We have a map $p_V : \mathcal{L}(M)_V \to \mathcal{L}(M_0)$ sending L to $L_V := L \cap V^\perp$.

Write $Y = \mathcal{L}(M) \times \mathcal{L}(M)$ and $Y_V = \mathcal{L}(M)_V \times \mathcal{L}(M)_V$. The gerb $\widetilde{Y}$ is defined as in § 3.2; write $\widetilde{Y}_V$ for its restriction to Y_V. Set $Y_0 = \mathcal{L}(M_0) \times \mathcal{L}(M_0)$. We have the corresponding gerb $\widetilde{Y}_0$ defined as in § 3.2. We extend the map $p_V \times p_V$ to a map

$$\pi_V : \widetilde{Y}_V \to \widetilde{Y}_0$$

sending $(L_1, L_2, \mathcal{B}, \mathcal{B}^2 \widetilde{\to} \det L_1 \otimes \det L_2)$ to

$$(L_{1,V}, L_{2,V}, \mathcal{B}_0,\ \mathcal{B}_0^2 \widetilde{\to} \det L_{1,V} \otimes \det L_{2,V}).$$

Here $L_{i,V} = L_i \cap V^\perp$ and $\mathcal{B}_0 = \mathcal{B} \otimes \det V$. We have used the exact sequences

$$0 \to L_{i,V} \to L_i \to M/V^\perp \to 0$$

yielding canonical ($\mathbb{Z}/2\mathbb{Z}$-graded) isomorphisms $\det L_{i,V} \otimes \det V^* \widetilde{\to} \det L_i$.

Write $H_0 = M_0 \oplus k$ for the Heisenberg group of M_0. For $L \in \mathcal{L}(M)_V$ we have the categories $\mathcal{H}_L$ and $\mathcal{H}_{L_V}$ of certain perverse sheaves on H and H_0, respectively. To such L we associate a transition functor $T^L : \mathcal{H}_{L_V} \to \mathcal{H}_L$ which will be fully faithful and exact for the perverse t-structures.

For brevity write $H^V = V^\perp \times \mathbb{A}^1$. First, at the level of functions, given $f \in \mathcal{H}_{L_V}$ consider it as a function on H^V via the composition $H^V \xrightarrow{\alpha_V} H_0 \xrightarrow{f} \bar{\mathbb{Q}}_\ell$, where α_V sends (v, a) to $(v \bmod V, a)$. Then there is a unique $f_1 \in \mathcal{H}_L$ such that $f_1(m) = q^{\dim V} f(m)$ for all $m \in H^V$. We use the property $V^\perp + L = M$. We set

$$(T^L)(f) = f_1. \tag{13}$$

The image of T^L is

$$\{f_1 \in \mathcal{H}_L \mid f(h(v, 0)) = f(h), h \in H, v \in V\}.$$

Note that $H^V \subset H$ is a subgroup, and $V = \{(v, 0) \in H^V \mid v \in V\} \subset H^V$ is a normal subgroup lying in the center of H^V. The operator $T^L : \mathcal{H}_{L_V} \to \mathcal{H}_L$ commutes with the action of H^V. It is understood that on $\mathcal{H}_{L_V}$ this group acts via its quotient $H^V \xrightarrow{\alpha_V} H_0$.

On the geometric level, consider the map $s : L \times H^V \to H$ sending $(l, (v, a))$ to the product in the Heisenberg group $(l, 0)(v, a) \in H$. Note that s is smooth and surjective (it is an affine fibration of rank $\dim L_V$). Given $K \in \mathcal{H}_{L_V}$ there is a (defined up to a unique isomorphism) perverse sheaf $T^L K \in \mathcal{H}_L$ equipped with

$$s^*(T^L K) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim L_V} \widetilde{\to} \bar{\mathbb{Q}}_\ell \boxtimes \alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim V + \dim L}.$$

The *compatibility property* of the canonical intertwining operators is as follows.

Proposition 2. *Let $(L, N, \mathcal{B}) \in \widetilde{Y}_V$, and write $(L_V, N_V, \mathcal{B}_0)$ for the image of $(L, N, \mathcal{B})$ under π_V. Write $\mathcal{F}_{N^0,L^0} : \mathcal{H}_L \to \mathcal{H}_N$ and $\mathcal{F}_{N^0_V,L^0_V} : \mathcal{H}_{L_V} \to \mathcal{H}_{N_V}$ for the corresponding functors defined as in § 3.5. Then the diagram of categories is canonically 2-commutative.*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow{\scriptstyle \mathcal{F}_{N^0_V,L^0_V}} & & \downarrow{\scriptstyle \mathcal{F}_{N^0,L^0}} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace $\mathcal{H}$ by $\mathrm{D}\mathcal{H}$ in the above diagram.

4.2 First, we realize the functors T^L by a universal kernel, namely, we define a perverse sheaf T on $\mathcal{L}(M)_V \times H \times H_0$ as follows.

Recall the vector bundle $\bar{C} \to \mathcal{L}(M)$; its fibre over L is $\bar{L} = L \times \mathbb{A}^1$. Write $\bar{C}_V$ for the restriction of $\bar{C}$ to the open subscheme $\mathcal{L}(M)_V$. We have a closed immersion

$$i_0 : \bar{C}_V \times H^V \to \mathcal{L}(M)_V \times H \times H_0$$

sending $(\bar{l} \in \bar{L}, u \in H^V)$ to $(L, \bar{l}u, \alpha_V(u))$, where the product $\bar{l}u$ is taken in H. The perverse sheaf T is defined by

$$T = (i_0)_! \, \mathrm{pr}_1^* \, \chi_{\bar{C}} \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \bar{C} + \dim V + \dim H_0},$$

where $\mathrm{pr}_1 : \bar{C}_V \times H^V \to \bar{C}_V$ is the projection, and $\chi_{\bar{C}}$ was defined in § 3.1.

For $L \in \mathcal{L}(M)_V$ let T_L be the $*$-restriction of T under $(L, \mathrm{id}) : H \times H_0 \to \mathcal{L}(M)_V \times H \times H_0$. Define $T^L : \mathrm{D}\mathcal{H}_{L_V} \to \mathrm{D}\mathcal{H}_L$ by

$$T^L(K) \widetilde{\to} \mathrm{pr}_{1!}(T_L \otimes \mathrm{pr}_2^* \, K) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim V - d - \dim \mathcal{L}(M)} \tag{14}$$

for the diagram of projections $H \overset{\mathrm{pr}_1}{\leftarrow} H \times H_0 \overset{\mathrm{pr}_2}{\to} H_0$. It is exact for the perverse t-structures.

The sheaf T has the following properties. At the level of functions, the corresponding function $T_L : H \times H_0 \to \bar{\mathbb{Q}}_\ell$ satisfies

$$T_L(\bar{l}h, \bar{l}_0 h_0) = \chi_L(\bar{l}) \chi_{L_V}(\bar{l}_0)^{-1} T_L(h, h_0), \quad \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V.$$

The geometric analog is as follows. Let ${}^0\bar{C} \to \mathcal{L}(M)_V$ be the vector bundle whose fibre over $L \in \mathcal{L}(M)_V$ is $\bar{L} \times \bar{L}_V$. Consider the diagram

$$\mathcal{L}(M)_V \times H \times H_0 \xleftarrow{\mathrm{pr}^V} {}^0\bar{C} \times H \times H_0 \xrightarrow{\mathrm{act}^V_{lr}} \mathcal{L}(M)_V \times H \times H_0,$$

where pr^V is the projection, and act^V_{lr} sends

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V, h \in H, h_0 \in H_0)$$

to $(L, \bar{l}h, \bar{l}_0 h_0)$. Let ${}^0p : {}^0\bar{C} \to \mathbb{A}^1$ be the map sending

$$(L \in \mathcal{L}(M)_V, \bar{l} \in \bar{L}, \bar{l}_0 \in \bar{L}_V)$$

to $p(\bar{l}) - p(\bar{l}_0)$. Here $p : \bar{L} \to \mathbb{A}^1$ and $p : \bar{L}_V \to \mathbb{A}^1$ are the projections on the center. Set ${}^0\chi = ({}^0p)^* \mathcal{L}_\psi$. Then T is act^V_{lr}*-equivariant*, that is, it admits an isomorphism

$$(\mathrm{act}^V_{lr})^* T \widetilde{\to} (\mathrm{pr}^V)^* T \otimes \mathrm{pr}_1^*({}^0\chi),$$

satisfying the usual associativity property, and its restriction to the unit section is the identity.

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4.3 We will prove a geometric version of the equality (up to an explicit power of q)

$$\int_{u\in H} F_{N^0,L^0}(hu^{-1})T_L(u,h_0)\,du = \int_{v\in H_0} T_N(h,v)F_{N^0_V,L^0_V}(vh_0^{-1})\,dv$$

for $h\in H$, $h_0\in H_0$. Here $(N^0,L^0)\in \widetilde{Y}_V$ and

$$(N^0_V,L^0_V)=\pi_V(N^0,L^0).$$

Write $\mathrm{inv}: H \xrightarrow{\sim} H$ for the map sending h to h^{-1}. Set ${}^{\mathrm{inv}}F=(\mathrm{id}\times\mathrm{inv})^*F$ for $\mathrm{id}\times\mathrm{inv}: \widetilde{Y}\times H\to\widetilde{Y}\times H$. For $i=1,2$ write $p_i:\widetilde{Y}_V\to\mathcal{L}(M)_V$ for the projection on the ith factor. Let q_0 denote the composition

$$\widetilde{Y}_V\times H\times H_0 \xrightarrow{\mathrm{pr}_{13}} \widetilde{Y}_V\times H_0 \xrightarrow{\pi_V\times\mathrm{id}} \widetilde{Y}_0\times H_0.$$

Proposition 2 is an immediate consequence of the following.

Lemma 5. *There is a canonical isomorphism over* $\widetilde{Y}_V\times H\times H_0$,

$$(\mathrm{pr}_{12}^*F)*_H(p_2\times\mathrm{id})^*T \xrightarrow{\sim} (q_0^*({}^{\mathrm{inv}}F))*_{H_0}(p_1\times\mathrm{id})^*T,$$

where $\mathrm{pr}_{12}:\widetilde{Y}_V\times H\times H_0\to\widetilde{Y}_V\times H$ *and* $p_1\times\mathrm{id}, p_2\times\mathrm{id}:\widetilde{Y}_V\times H\times H_0\to\mathcal{L}(M)_V\times H\times H_0$.

Let $i_V: H^V\hookrightarrow H$ be the natural closed immersion. It is elementary to check that Lemma 5 is equivalent to the following.

Lemma 6. *There is a canonical isomorphism of (shifted) perverse sheaves*

$$(\mathrm{id}\times\alpha_V)_!i_V^*F \xrightarrow{\sim} (\pi_V\times\mathrm{id})^*F\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim.\mathrm{rel}(\pi_V)+\dim V} \qquad (15)$$

for the following diagram.

$$\begin{array}{ccccc} & & \widetilde{Y}_V\times H^V & \xrightarrow{i_V} & \widetilde{Y}_V\times H \\ & & \downarrow{\scriptstyle \mathrm{id}\times\alpha_V} & & \\ \widetilde{Y}_0\times H_0 & \xleftarrow{\pi_V\times\mathrm{id}} & \widetilde{Y}_V\times H_0 & & \end{array}$$

Proof. Write $U(M_0)$ for the scheme U constructed out of the symplectic space M_0. It classifies pairs of Lagrangian subspaces in M_0 that do not intersect. We have a 2-commutative diagram

$$\begin{array}{ccccc} U(M_0)\times_{\mathcal{L}(M_0)}U(M_0) & \xleftarrow{\pi_W} & W_V & \xhookrightarrow{i_W} & U\times_{\mathcal{L}(M)}U \\ \downarrow{\scriptstyle \pi_{Y_0}} & & \downarrow{\scriptstyle \pi_{Y,V}} & \swarrow{\scriptstyle \pi_Y} & \\ \widetilde{Y}_0 & \xleftarrow{\pi_V} & \widetilde{Y}_V & & \end{array}$$

where the square is cartesian thus defining W_V, π_W, and $\pi_{Y,V}$. The map i_W is a locally closed immersion. Write a point of W_V as a triple $(N,R,L)\in\mathcal{L}(M)$ such that $N,L\in\mathcal{L}(M)_V$, $V\subset R\subset V^\perp$, and $N\cap R=R\cap L=0$. The map π_W sends (N,R,L) to (N_V,R_V,L_V) with $R_V=R/V$.

Let us establish the isomorphism (15) after restriction under $\pi_{Y,V}\times\alpha_V: W_V\times H^V\to\tilde{Y}_V\times H_0$. We first give the argument at the level of functions and then check that it holds in the geometric setting.

Consider a point of W_V given by a triple $(N,R,L)\in\mathcal{L}(M)$, so $N,L\in\mathcal{L}(M)_V$, $V\subset R\subset V^\perp$, and $N\cap R=R\cap L=0$. We have $V^\perp=R\oplus L_V$. Let $h\in H^V$; write $h=(r,a)(l_1,0)$ for uniquely

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defined $r \in R, l_1 \in L_V, a \in k$. Write $(N^0, L^0) \in \widetilde{Y}_V$ for the image of (N, R, L) under $\pi_{Y,V}$. Using (10), we get

$$\begin{aligned}\int_{v\in V} F_{N^0,L^0}(h(v,0))\,dv &= q^{\dim \mathcal{L}(M)-(d+1)/2}\int_{v\in V, u\in H}\tilde{F}_{N,R}(u)\tilde{F}_{R,L}(u^{-1}h(v,0))\,dv\,du\\ &= q^{\dim \mathcal{L}(M)+(d+1)/2}\int_{v\in V, u\in H/\tilde{R}}\tilde{F}_{N,R}(u)\tilde{F}_{R,L}(u^{-1}(r,a)(v,0))\,dv\,du\\ &= q^{\dim \mathcal{L}(M)+(d+1)/2}\int_{v\in V, l\in L}\tilde{F}_{N,R}(l,0)\tilde{F}_{R,L}((-l,0)(r,a)(v,0))\,dv\,dl.\end{aligned}$$

Since $(-l,0)(r+v,a) = (r+v, a+\omega\langle r+v, l\rangle)(-l,0)$, the latter expression equals

$$q^{-d/2}\int_{v\in V, l\in L}\tilde{F}_{N,R}(l,0)\psi(a+\omega\langle r+v,l\rangle)\,dv\,dl = q^{\dim V-d/2}\int_{l\in L_V}\tilde{F}_{N,R}(l,0)\psi(a+\omega\langle r,l\rangle)\,dl.$$

For $l \in L_V$ we get $\tilde{F}_{N,R}(l,0) = q^{\dim \mathcal{L}(M_0)-\dim \mathcal{L}(M)-\dim V}\tilde{F}_{N_V,R_V}(l,0)$. Indeed, since $V^\perp = R \oplus N_V$, there are unique $r_1 \in R, n_1 \in N_V$ such that $l = n_1 + r_1$. For $\bar{r}_1 = r_1 \bmod V \in M_0$ we get

$$\begin{aligned}\tilde{F}_{N,R}(l,0) &= q^{-\dim \mathcal{L}(M)-(2d+1)/2}\chi_{NR}(l,0) = q^{-\dim \mathcal{L}(M)-(2d+1)/2}\psi(\tfrac{1}{2}\omega\langle r_1, n_1\rangle)\\ &= q^{-\dim \mathcal{L}(M)-(2d+1)/2}\chi_{N_VR_V}(\bar{r}_1+n_1,0) = q^{\dim \mathcal{L}(M_0)-\dim \mathcal{L}(M)-\dim V}\tilde{F}_{N_V,R_V}(l,0).\end{aligned}$$

Further, we claim that

$$\tilde{F}_{R_V,L_V}((-l,0)\alpha_V(h)) = q^{-\dim \mathcal{L}(M_0)-\dim H_0/2}\psi(a+\omega\langle r,l\rangle).$$

This follows from definition (5) of $\tilde{F}_U$ and the formula $(-l,0)(r,a) = (r, a+\omega\langle r,l\rangle)(-l,0)$.

Combining the above we get

$$\begin{aligned}\int_{v\in V} F_{N^0,L^0}(h(v,0))\,dv &= q^{c}\int_{l\in L_V}\tilde{F}_{N_V,R_V}(l,0)\tilde{F}_{R_V,L_V}((-l,0)\alpha_V(h))\,dl\\ &= q^{c+\dim V-d-1}\int_{u\in H_0}\tilde{F}_{N_V,R_V}(u)\tilde{F}_{R_V,L_V}(u^{-1}\alpha_V(h))\,du\end{aligned}$$

with $c = (\dim H_0 - d)/2 + 2\dim \mathcal{L}(M_0) - \dim \mathcal{L}(M)$. By (10), the latter expression identifies with $F_{N_V^0, L_V^0}(h)$ up to an explicit power of q.

The argument holds in the geometric setting yielding the desired isomorphism γ over $W_V \times H^V$. For any point $(N_V, L_V\mathcal{B}_0) \in \widetilde{Y}_0$ such that $N_V \neq L_V$ the fibre of π_{Y_0} over this point is geometrically connected. So, for $\dim V < d$, the isomorphism γ descends to a uniquely defined isomorphism (15). The case $\dim V = d$ is easier and is left to the reader. □

Remark 5. Let $i_H : \operatorname{Spec} k \hookrightarrow H$ denote the zero section. Arguing as in Lemma 6, for the map $\operatorname{id} \times i_H : \widetilde{Y} \to \widetilde{Y} \times H$ one gets a canonical isomorphism

$$(\operatorname{id} \times i_H)^* F \,\tilde{\to}\, (S_{M,g}\otimes \mathcal{E}^d \oplus S_{M,s}\otimes\mathcal{E}^{d-1}) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim H}.$$

However, it will not be used in this paper.

4.4 The functors T^L satisfy the following transitivity property. Assume that $V_1 \subset V$ is another isotropic subspace in M. Let $M_1 = V_1^\perp/V_1$ and $H_1 = M_1 \times \mathbb{A}^1$ be the corresponding Heisenberg group. Then for $L \in \mathcal{L}(M)_V$ we also have $L_{V_1} := L \cap V_1^\perp$ and the category $\mathcal{H}_{L_{V_1}}$ of certain

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perverse sheaves on H_1. Then the following diagram is canonically 2-commutative.

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^{L_{V_1}}} & \mathcal{H}_{L_{V_1}} \\ & \searrow^{T^L} & \downarrow{\scriptstyle T^L} \\ & & \mathcal{H}_L \end{array}$$

4.5 We will also need one more compatibility property of the canonical intertwining operators. Let $V \subset V^{\perp} \subset M$ be as in § 4.1. Write $i_{0,V} : \mathcal{L}(M_0) \to \mathcal{L}(M)$ for the closed immersion sending L_0 to the preimage of L_0 under $V^{\perp} \to V^{\perp}/V$.

For $L \in \mathcal{L}(M)$ with $V \subset L$ set $L_V = L/V \in \mathcal{L}(M_0)$. Let $(\mathcal{L}(M_0) \times \mathcal{L}(M)_V)\tilde{}$ denote the restriction of the gerb $\tilde{Y}$ under

$$\mathcal{L}(M_0) \times \mathcal{L}(M)_V \xrightarrow{i_{0,V} \times \mathrm{id}} \mathcal{L}(M) \times \mathcal{L}(M)_V \subset Y.$$

Define $\pi_{0,V} : (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)\tilde{} \to \widetilde{Y}_0$ as the map sending $(L, N, \mathcal{B}, \mathcal{B}^2 \widetilde{\to} \det L \otimes \det N)$ to

$$(L_V, N_V, \mathcal{B}, \mathcal{B}^2 \widetilde{\to} \det L_V \otimes \det N_V).$$

Here $L \in \mathcal{L}(M)$ with $V \subset L$. We have used the canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\det L \otimes \det N \widetilde{\to} \det L_V \otimes \det N_V$.

Recall the closed immersion $i_V : H^V \hookrightarrow H$. For $L \in \mathcal{L}(M)$ with $V \subset L$ define the transition functor $T^L : \mathcal{H}_{L_V} \to \mathcal{H}_L$ by

$$T^L(K) = i_{V!}\alpha_V^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim V}.$$

The proof of the following proposition is similar to that of Proposition 2 and is left to the reader.

Proposition 3. *Let $(L, N, \mathcal{B}) \in (\mathcal{L}(M_0) \times \mathcal{L}(M)_V)\tilde{}$, and let $(L_V, N_V, \mathcal{B})$ denote its image under $\pi_{0,V}$. Write $\mathcal{F}_{N^0,L^0} : \mathcal{H}_L \to \mathcal{H}_N$ and $\mathcal{F}_{N_V^0,L_V^0}$ for the corresponding functors defined as in § 3.5. Then the diagram of categories is canonically 2-commutative.*

$$\begin{array}{ccc} \mathcal{H}_{L_V} & \xrightarrow{T^L} & \mathcal{H}_L \\ \downarrow{\scriptstyle \mathcal{F}_{N_V^0,L_V^0}} & & \downarrow{\scriptstyle \mathcal{F}_{N^0,L^0}} \\ \mathcal{H}_{N_V} & \xrightarrow{T^N} & \mathcal{H}_N \end{array}$$

One may also replace $\mathcal{H}$ by $\mathrm{D}\mathcal{H}$ in the above diagram. □

5. Discrete Lagrangian lattices and the metaplectic group

5.1 Set $\mathcal{O} = k[[t]] \subset F = k((t))$. Denote by Ω the completed module of relative differentials of $\mathcal{O}$ over k. Let M be a free $\mathcal{O}$-module of rank $2d$ with symplectic form $\wedge^2 M \to \Omega$. Write G for the group scheme over $\operatorname{Spec} \mathcal{O}$ of automorphisms of M preserving the symplectic form. Consider the Tate space $M(F)$ (cf. [Dri02, 4.2.13 for the definition]); it is equipped with the symplectic form $(m_1, m_2) \mapsto \operatorname{Res} \omega\langle m_1, m_2\rangle$.

For a k-subspace $L \subset M(F)$ write

$$L^{\perp} = \{m \in M(F) \mid \operatorname{Res} \omega\langle m, l\rangle = 0 \text{ for all } l \in L\}.$$

For two k-subspaces $L_1, L_2 \subset M$ we get $(L_1 + L_2)^\perp = L_1^\perp \cap L_2^\perp$. For a finite-dimensional symplectic k-vector space U write $\mathcal{L}(U)$ for the variety of Lagrangian subspaces in U.

As in *loc. cit.*, we say that an $\mathcal{O}$-submodule $R \subset M(F)$ is a *c-lattice* if $M(-N) \subset R \subset M(N)$ for some integer N. A *Lagrangian d-lattice* in $M(F)$ is a k-vector subspace $L \subset M(F)$ such that $L^\perp = L$ and there exists a c-lattice R with $R \cap L = 0$. Note that the condition $R \cap L = 0$ implies that $R^\perp + L = M(F)$. Let $\mathcal{L}_d(M(F))$ denote the set of Lagrangian d-lattices in $M(F)$.

For a given c-lattice $R \subset M(F)$ write

$$\mathcal{L}_d(M(F))_R = \{L \in \mathcal{L}_d(M(F)) \mid L \cap R = 0\}.$$

If R is a c-lattice in $M(F)$ with $R \subset R^\perp$ then $\mathcal{L}_d(M(F))_R$ is a naturally a k-scheme (not of finite type over k). Indeed, for each c-lattice $R_1 \subset R$ we have the variety

$$\mathcal{L}(R_1^\perp/R_1)_R := \{L_1 \in \mathcal{L}(R_1^\perp/R_1) \mid L_1 \cap R/R_1 = 0\}.$$

For $R_2 \subset R_1 \subset R$ we get a map $p_{R_2,R_1} : \mathcal{L}(R_2^\perp/R_2)_R \to \mathcal{L}(R_1^\perp/R_1)_R$ sending L_2 to

$$L_1 = (L_2 \cap (R_1^\perp/R_2)) + R_1.$$

The map p_{R_2,R_1} is a composition of two affine fibrations of constant rank. Then $\mathcal{L}_d(M(F))_R$ is the inverse limit of $\mathcal{L}(R_1^\perp/R_1)_R$ over the partially ordered set of c-lattices $R_1 \subset R$.

If $R' \subset R$ is another c-lattice then $\mathcal{L}_d(M(F))_R \subset \mathcal{L}_d(M(F))_{R'}$ is an open immersion (as it is an open immersion on each term of the projective system). So, $\mathcal{L}_d(M(F))$ is a k-scheme that can be seen as the inductive limit of $\mathcal{L}_d(M(F))_R$.

Let us define the categories $\mathrm{P}(\mathcal{L}_d(M(F)))$ and $\mathrm{P}_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ of perverse sheaves and $G(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{L}_d(M(F))$.

For $r \geq 0$ set

$${}_r\mathcal{L}_d(M(F)) = \mathcal{L}_d(M(F))_{M(-r)},$$

where the group $G(\mathcal{O})$ acts on ${}_r\mathcal{L}_d(M(F))$ naturally. First, define the category $\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$ as follows.

For $N + r \geq 0$ set ${}_{N,r}M = t^{-N}M/t^rM$. For $N \geq r \geq 0$ the action of $G(\mathcal{O})$ on ${}_r\mathcal{L}({}_{N,N}M) := \mathcal{L}({}_{N,N}M)_{M(-r)}$ factors through $G(\mathcal{O}/t^{2N})$. For $r_1 \geq 2N$ the kernel

$$\mathrm{Ker}(G(\mathcal{O}/t^{r_1})) \to G(\mathcal{O}/t^{2N}))$$

is unipotent, so that we have an equivalence (exact for the perverse t-structures)

$$\mathrm{D}_{G(\mathcal{O}/t^{2N})}({}_r\mathcal{L}({}_{N,N}M)) \xrightarrow{\sim} \mathrm{D}_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}({}_{N,N}M)).$$

Define $\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}({}_{N,N}M))$ as $\mathrm{D}_{G(\mathcal{O}/t^{r_1})}({}_r\mathcal{L}({}_{N,N}M))$ for any $r_1 \geq 2N$. It is equipped with the perverse t-structure.

For $N_1 \geq N \geq r \geq 0$ the fibres of the above projection

$$p : {}_r\mathcal{L}({}_{N_1,N_1}M) \to {}_r\mathcal{L}({}_{N,N}M)$$

are isomorphic to affine spaces of fixed dimension, and p is smooth and surjective. Hence, this map yields transition functors (exact for the perverse t-structures and fully faithful embeddings)

$$\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}({}_{N,N}M)) \to \mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}({}_{N_1,N_1}M))$$

and

$$\mathrm{D}({}_r\mathcal{L}({}_{N,N}M)) \to \mathrm{D}({}_r\mathcal{L}({}_{N_1,N_1}M)).$$

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We define $\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$ as the inductive 2-limit of $\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}({}_{N,N}M))$ as N goes to plus infinity. The category $\mathrm{D}({}_r\mathcal{L}_d(M(F)))$ is defined similarly. Both are equipped with perverse t-structures.

If $N_1 \geq N \geq r_1 \geq r \geq 0$ we have the diagram

$$\begin{array}{ccc} {}_r\mathcal{L}({}_{N_1,N_1}M) & \xrightarrow{p} & {}_r\mathcal{L}({}_{N,N}M) \\ \downarrow j & & \downarrow j \\ {}_{r_1}\mathcal{L}({}_{N_1,N_1}M) & \xrightarrow{p} & {}_{r_1}\mathcal{L}({}_{N,N}M) \end{array}$$

where j are natural open immersions. The restriction functors $j^*: \mathrm{D}_{G(\mathcal{O})}({}_{r_1}\mathcal{L}({}_{N,N}M)) \to \mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}({}_{N,N}M))$ yield (in the limit as N goes to plus infinity) the functors

$$j^*_{r_1,r}: \mathrm{D}_{G(\mathcal{O})}({}_{r_1}\mathcal{L}_d(M(F))) \to \mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

of restriction with respect to the open immersion $j_{r_1,r}: {}_r\mathcal{L}_d(M(F)) \hookrightarrow {}_{r_1}\mathcal{L}_d(M(F))$. Define $\mathrm{D}_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ as the projective 2-limit of

$$\mathrm{D}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$$

as r goes to plus infinity. Similarly, $\mathrm{P}_{G(\mathcal{O})}(\mathcal{L}_d(M(F)))$ is defined as the projective 2-limit of $\mathrm{P}_{G(\mathcal{O})}({}_r\mathcal{L}_d(M(F)))$. Along the same lines, one defines the categories $\mathrm{P}(\mathcal{L}_d(M(F)))$ and $\mathrm{D}(\mathcal{L}_d(M(F)))$.

5.2 Relative determinant

For a pair of c-lattices M_1, M_2 in $M(F)$ define the relative determinant $\det(M_1 : M_2)$ as the following $\mathbb{Z}/2\mathbb{Z}$-graded one-dimensional k-vector space. If R is a c-lattice in $M(F)$ such that $R \subset M_1 \cap M_2$ then

$$\det(M_1 : M_2) \widetilde{\to} \det(M_1/R) \otimes \det(M_2/R)^{-1}.$$

It is defined up to a unique isomorphism.

Write Gr_G for the affine grassmanian $G(F)/G(\mathcal{O})$ of G (cf. [Dri02, § 4.5]). For $R \in \mathrm{Gr}_G$, $L \in \mathcal{L}_d(M(F))$ define the relative determinant $\det(R : L)$ as the following ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) one-dimensional vector space. Pick a c-lattice $R_1 \subset R$ such that $R_1 \cap L = 0$. Then in $R_1^\perp/R_1$ one gets two Lagrangian subspaces R/R_1 and $L_{R_1} := L \cap R_1^\perp$. Set

$$\det(R : L) = \det(R/R_1) \otimes \det(L_{R_1}).$$

If $R_2 \subset R_1$ is another c-lattice then the exact sequence

$$0 \to L_{R_1} \to L \cap R_2^\perp \to R_2^\perp/R_1^\perp \to 0$$

yields a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\begin{aligned} \det(R/R_2) \otimes \det(L_{R_2}) &\widetilde{\to} \det(R_1/R_2) \otimes \det(R/R_1) \otimes \det(L_{R_1}) \otimes \det(R_2^\perp/R_1^\perp) \\ &\widetilde{\to} \det(R/R_1) \otimes \det(L_{R_1}). \end{aligned}$$

So, $\det(R : L)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded line defined up to a unique isomorphism. Another way to say this is as follows. Consider the complex $R \oplus L \xrightarrow{s} M(F)$ placed in cohomological degrees 0 and 1, where $s(r, l) = r + l$. It has finite-dimensional cohomologies and

$$\det(R : L) = \det(R \oplus L \xrightarrow{s} M(F)).$$

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For $g \in G(F)$ we have canonically that

$$\det(gR : gL) \widetilde{\to} \det(R : L).$$

For $R_1, R_2 \in \mathrm{Gr}_G$, $L \in \mathcal{L}_d(M(F))$ we have canonically that

$$\det(R_1 : L) \widetilde{\to} \det(R_1 : R_2) \otimes \det(R_2 : L).$$

5.3 Write $\mathcal{A}_d$ for the line bundle on $\mathcal{L}_d(M(F))$ with fibre $\det(M : L)$ at $L \in \mathcal{L}_d(M(F))$. Clearly, $\mathcal{A}_d$ is $G(\mathcal{O})$-equivariant, so we may see $\mathcal{A}_d$ as the line bundle on the stack quotient $\mathcal{L}_d(M(F))/G(\mathcal{O})$. Let $\tilde{\mathcal{L}}_d(M(F))$ denote the μ_2-gerb of square roots of $\mathcal{A}_d$.

The categories of the corresponding perverse sheaves $\mathrm{P}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F))$ and $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F))$ are defined as above. Namely, first for $r \geq 0$ define $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}_d(M(F)))$ as follows. For $N \geq r$ take $r_1 \geq 2N$ and consider the stack quotient ${}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1})$. We have the line bundle, say $\mathcal{A}_N$, on this stack whose fibre at L is $\det(M/M(-N)) \otimes \det L$. Here $L \subset {}_{N,N}M$ is a Lagrangian subspace such that $L \cap (M(-r)/M(-N)) = 0$. Write $({}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{}$ for the gerb of square roots of this line bundle. Let $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N,N}M))$ denote the category $\mathrm{D}(({}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{})$ for any $r_1 \geq 2N$ (we have canonical equivalences exact for the perverse t-structures between such categories for various r_1).

Assume that $N_1 \geq N \geq r$ and $r_1 \geq 2N_1$. For the projection

$$p : {}_r\mathcal{L}({}_{N_1,N_1}M)/G(\mathcal{O}/t^{r_1}) \to {}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1})$$

we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $p^*\mathcal{A}_N \widetilde{\to} \mathcal{A}_{N_1}$. This yields a transition map

$$({}_r\mathcal{L}({}_{N_1,N_1}M)/G(\mathcal{O}/t^{r_1}))\tilde{} \to ({}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{}.$$

The corresponding inverse image yields a transition functor

$$\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N,N}M)) \to \mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N_1,N_1}M)) \tag{16}$$

exact for the perverse t-structures (and a fully faithful embedding). We define $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}_d(M(F)))$ as the inductive 2-limit of $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N,N}M))$ as N goes to plus infinity.

For $N \geq r' \geq r$ and $r_1 \geq 2N$ we have an open immersion

$$\tilde{j} : ({}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{} \subset ({}_{r'}\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{}$$

and hence the $*$-restriction functors

$$\tilde{j}^* : \mathrm{D}_{G(\mathcal{O})}({}_{r'}\tilde{\mathcal{L}}({}_{N,N}M)) \to \mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N,N}M))$$

compatible with the transition functors (16). Passing to the limit as N goes to plus infinity, we get the functors

$$\tilde{j}^*_{r',r} : \mathrm{D}_{G(\mathcal{O})}({}_{r'}\tilde{\mathcal{L}}_d(M(F))) \to \mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}_d(M(F))).$$

Define $\mathrm{D}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$ as the projective 2-limit of $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}_d(M(F)))$ as r goes to plus infinity, and similarly for $\mathrm{P}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F)))$.

Along the same lines one defines the categories $\mathrm{P}(\tilde{\mathcal{L}}_d(M(F)))$ and $\mathrm{D}(\tilde{\mathcal{L}}_d(M(F)))$.

5.4 Metaplectic group

Let $\mathcal{A}_G$ be the line bundle on the ind-scheme $G(F)$ whose fibre at g is $\det(M : gM)$. Write $\widetilde{G}(F) \to G(F)$ for the gerb of square roots of $\mathcal{A}_G$. The stack $\widetilde{G}(F)$ has a structure of a group stack.

The product map $m : \widetilde{G}(F) \times \widetilde{G}(F) \to \widetilde{G}(F)$ sends

$$(g_1, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \widetilde{\to} \det(M : g_1 M)), (g_2, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \widetilde{\to} \det(M : g_2 M))$$

to the collection $(g_1 g_2, \mathcal{B}, \sigma : \mathcal{B}^2 \widetilde{\to} \det(M : g_1 g_2 M)$, where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ and σ is the composition

$$(\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : g_1 M) \otimes \det(M : g_2 M)$$
$$\xrightarrow{\mathrm{id} \otimes g_1} \det(M : g_1 M) \otimes \det(g_1 M : g_1 g_2 M) \widetilde{\to} \det(M : g_1 g_2 M).$$

Informally speaking, one may think of the exact sequence of group stacks

$$1 \to B(\mu_2) \to \widetilde{G}(F) \to G(F) \to 1.$$

We also have a canonical section $G(\mathcal{O}) \to \widetilde{G}(F)$ sending g to

$$(g, \mathcal{B} = k, \mathrm{id} : \mathcal{B}^2 \widetilde{\to} \det(M : M)).$$

The group stack $\widetilde{G}(F)$ acts naturally on $\widetilde{\mathcal{L}}_d(M(F))$, and the action map $\widetilde{G}(F) \times \widetilde{\mathcal{L}}_d(M(F)) \to \widetilde{\mathcal{L}}_d(M(F))$ sends

$$(g, \mathcal{B}_1, \sigma_1 : \mathcal{B}_1^2 \widetilde{\to} \det(M : gM)), (L, \mathcal{B}_2, \sigma_2 : \mathcal{B}_2^2 \widetilde{\to} \det(M : L))$$

to the collection $(gL, \mathcal{B})$, where $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ is equipped with the isomorphism

$$(\mathcal{B}_1 \otimes \mathcal{B}_2)^2 \xrightarrow{\sigma_1 \otimes \sigma_2} \det(M : gM) \otimes \det(M : L) \xrightarrow{\mathrm{id} \otimes g} \det(M : gM) \otimes \det(gM : gL) \widetilde{\to} \det(M : gL).$$

5.5 For $g \in G(F)$ and a c-lattice $R \subset R^\perp$ in $M(F)$ we have an isomorphism of symplectic spaces $g : R^\perp / R \widetilde{\to} (gR)^\perp / gR$. For each c-lattice $R_1 \subset R$ we have the following diagram.

$$\begin{array}{ccc} \mathcal{L}(R_1^\perp / R_1)_R & \xrightarrow{\widetilde{g}} & \mathcal{L}(gR_1^\perp / gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \mathcal{L}(R^\perp / R) & \xrightarrow{\widetilde{g}} & \mathcal{L}(gR^\perp / gR) \end{array}$$

Let $\mathcal{A}_{R_1}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) line bundle on $\mathcal{L}(R_1^\perp / R_1)_R$ whose fibre at L is $\det L \otimes \det(M : R_1)$. Assume that $\tilde{g} = (g, \mathcal{B}, \mathcal{B}^2 \widetilde{\to} \det(M : gM))$ is a k-point of $\tilde{G}(F)$ over g. It yields the following diagram.

$$\begin{array}{ccc} \widetilde{\mathcal{L}}(R_1^\perp / R_1)_R & \xrightarrow{\widetilde{\tilde{g}}} & \widetilde{\mathcal{L}}(gR_1^\perp / gR_1)_{gR} \\ \downarrow p & & \downarrow p \\ \widetilde{\mathcal{L}}(R^\perp / R) & \xrightarrow{\widetilde{\tilde{g}}} & \widetilde{\mathcal{L}}(gR^\perp / gR) \end{array}$$

Here the top horizontal arrow sends $(L, \mathcal{B}_1, \mathcal{B}_1^2 \widetilde{\to} \det L \otimes \det(M : R_1))$ to

$$(gL, \mathcal{B}_2, \sigma : \mathcal{B}_2^2 \widetilde{\to} \det(gL) \otimes \det(M : gR_1)),$$

where $\mathcal{B}_2 = \mathcal{B}_1 \otimes \mathcal{B}$ and σ is the composition

$$(\mathcal{B}_1 \otimes \mathcal{B})^2 \widetilde{\to} \det L \otimes \det(M : R_1) \otimes \det(M : gM)$$
$$\xrightarrow{g \otimes \mathrm{id}} \det(gL) \otimes \det(gM : gR_1) \otimes \det(M : gM) \widetilde{\to} \det(gL) \otimes \det(M : gR_1).$$

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In the limit by R_1 the corresponding functors $\tilde{g}^*: \mathrm{P}(\widetilde{\mathcal{L}}(gR_1^\perp/gR_1)_{gR}) \widetilde{\to} \mathrm{P}(\widetilde{\mathcal{L}}(R_1^\perp/R_1)_R)$ yield an equivalence

$$\tilde{g}^*: \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))_{gR}) \widetilde{\to} \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))_R).$$

Taking one more limit by the partially ordered set of c-lattices R, one gets an equivalence

$$\tilde{g}^*: \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))) \widetilde{\to} \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))).$$

In this sense $\tilde{G}(F)$ acts on $\mathrm{P}(\widetilde{\mathcal{L}}_d(M(F)))$.

6. Canonical intertwining operators: the local field case

6.1 Keep the notation of § 5. Write $H = M \oplus \Omega$ for the Heisenberg group defined as in § 2.1; this is a group scheme over Spec $\mathcal{O}$.

For $L \in \mathcal{L}_d(M(F))$ we have the subgroup $\bar{L} = L \oplus \Omega(F) \subset H(F)$ and the character $\chi_L: \bar{L} \to \bar{\mathbb{Q}}_\ell^*$ given by $\chi_L(l, a) = \chi(a)$. Here $\chi: \Omega(F) \to \bar{\mathbb{Q}}_\ell^*$ sends a to $\psi(\mathrm{Res}\, a)$. In the classical setting we let $\mathcal{H}_L$ denote the space of functions $f: H(F) \to \bar{\mathbb{Q}}_\ell$ satisfying:

(C1) $f(\bar{l}h) = \chi_L(\bar{l}) f(h)$, for $h \in H, \bar{l} \in \bar{L}$;

(C2) there exists a c-lattice $R \subset M(F)$ such that $f(h(r, 0)) = f(h)$ for $r \in R, h \in H$.

Note that such an f has automatically compact support modulo $\bar{L}$. The group $H(F)$ acts on $\mathcal{H}_L$ by right translations; this is a model of the Weil representation. Let us introduce a geometric analog of $\mathcal{H}_L$.

Given a c-lattice $R \subset M(F)$ such that $R \subset R^\perp$ write $H_R = (R^\perp/R) \oplus k$ for the Heisenberg group corresponding to the symplectic space $R^\perp/R$. If $L \in \mathcal{L}_d(M(F))_R$ then $L_R := L \cap R^\perp \subset R^\perp/R$ is Lagrangian. Set $\bar{L}_R = L_R \oplus k \subset H_R$. Let $\chi_{L,R}: \bar{L}_R \to \bar{\mathbb{Q}}_\ell^*$ be the character sending (l, a) to $\psi(a)$. Set

$$\mathcal{H}_{L_R} = \{f: H_R \to \bar{\mathbb{Q}}_\ell \mid f(\bar{l}h) = \chi_{L,R}(\bar{l}) f(h),\ h \in H_R, \bar{l} \in \bar{L}_R\}.$$

Lemma 7. *There is a canonical embedding $T_R^L: \mathcal{H}_{L_R} \hookrightarrow \mathcal{H}_L$ whose image is the subspace of those $f \in \mathcal{H}_L$ which satisfy*

$$f(h(r, 0)) = f(h) \quad \text{for } r \in R, h \in H. \tag{17}$$

Proof. Set

$${}'\mathcal{H}_{L_R} = \{\phi: R^\perp/R \to \bar{\mathbb{Q}}_\ell \mid \phi(r + l) = \chi(\tfrac{1}{2}\omega\langle r, l\rangle)\phi(r), r \in R^\perp/R, l \in L_R\}.$$

We have an isomorphism $\mathcal{H}_{L_R} \widetilde{\to} {}'\mathcal{H}_{L_R}$ sending f to ϕ given by $\phi(r) = f(r, 0)$. Given $f \in \mathcal{H}_L$ satisfying (17), we associate to f a function $\phi \in {}'\mathcal{H}_{L_R}$ given by

$$\phi(r) = q^{\frac{1}{2}\dim R^\perp/R} f(r, 0)$$

for $r \in R^\perp$. This defines the map T_R^L. □

Assume that $S \subset R \subset M(F)$ are c-lattices and $R \cap L = 0$. Recall the operator $\mathcal{H}_{L_R} \overset{T^{L_S}}{\to} \mathcal{H}_{L_S}$ given by (13), which corresponds to the isotropic subspace $R/S \subset S^\perp/S$. The composition $\mathcal{H}_{L_R} \overset{T^{L_S}}{\to} \mathcal{H}_{L_S} \overset{T_S^L}{\to} \mathcal{H}_L$ equals T_R^L.

The geometric analog of $\mathcal{H}_L$ is as follows. For a c-lattice R such that $R \cap L = 0$ and $R \subset R^\perp$ we have the category $\mathcal{H}_{L_R}$ of perverse sheaves on H_R which are $(\bar{L}_R, \chi_{L,R})$-equivariant, and the corresponding category $\mathrm{D}\mathcal{H}_{L_R}$. For $S \subset R$ as above we have an (exact for the perverse structure and fully faithful) transition functor (14), which we now denote by

$$T^L_{S,R} : \mathrm{D}\mathcal{H}_{L_R} \to \mathrm{D}\mathcal{H}_{L_S}.$$

Define $\mathcal{H}_L$ (respectively, $\mathrm{D}\mathcal{H}_L$) as the inductive 2-limit of $\mathcal{H}_{L_R}$ (respectively, of $\mathrm{D}\mathcal{H}_{L_R}$) over the partially ordered set of c-lattices R such that $R \cap L = 0$ and $R \subset R^\perp$. So, $\mathcal{H}_L$ is abelian and $\mathrm{D}\mathcal{H}_L$ is a triangulated category.

6.2 Let $R \subset R^\perp$ be a c-lattice in $M(F)$. We have a projection

$$\mathcal{L}_d(M(F))_R \to \mathcal{L}(R^\perp/R)$$

sending L to L_R. Let $\mathcal{A}_R$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree-zero line bundle on $\mathcal{L}(R^\perp/R)$ whose fibre at L_1 is $\det L_1 \otimes \det(M : R)$. Write $\widetilde{\mathcal{L}}(R^\perp/R)$ for the gerb of square roots of $\mathcal{A}_R$. The restriction of $\mathcal{A}_R$ to $\mathcal{L}_d(M(F))_R$ identifies canonically with $\mathcal{A}_d$. The above projection lifts naturally to a morphism of gerbs

$$\widetilde{\mathcal{L}}_d(M(F))_R \to \widetilde{\mathcal{L}}(R^\perp/R). \tag{18}$$

Given k-points $N^0, L^0 \in \widetilde{\mathcal{L}}_d(M(F))$ we are going to associate to them in a canonical way a functor

$$\mathcal{F}_{N^0,L^0} : \mathrm{D}\mathcal{H}_L \to \mathrm{D}\mathcal{H}_N \tag{19}$$

sending $\mathcal{H}_L$ to $\mathcal{H}_N$. To do so, consider a c-lattice $R \subset R^\perp$ in $M(F)$ such that $L, N \in \mathcal{L}_d(M(F))_R$. Write $N^0_R, L^0_R \in \widetilde{\mathcal{L}}(R^\perp/R)$ for the images of N^0 and L^0 under (18). By definition, the enhanced structure on L_R and N_R is given by one-dimensional vector spaces $\mathcal{B}_L, \mathcal{B}_N$ equipped with

$$\mathcal{B}_L^2 \xrightarrow{\sim} \det L_R \otimes \det(M : R), \quad \mathcal{B}_N^2 \xrightarrow{\sim} \det N_R \otimes \det(M : R),$$

and hence an isomorphism $\mathcal{B}^2 \xrightarrow{\sim} \det L_R \otimes \det N_R$ for $\mathcal{B} := \mathcal{B}_L \otimes \mathcal{B}_N \otimes \det(M : R)^{-1}$. We denote by

$$\mathcal{F}_{N^0_R, L^0_R} : \mathrm{D}\mathcal{H}_{L_R} \to \mathrm{D}\mathcal{H}_{N_R}$$

the canonical intertwining functor defined in § 3.5 corresponding to $(N_R, L_R, \mathcal{B}) \in \tilde{Y}$, where $Y = \mathcal{L}(R^\perp/R) \times \mathcal{L}(R^\perp/R)$. The following is an immediate consequence of Proposition 2.

Proposition 4. *Let $S \subset R \subset R^\perp \subset S^\perp$ be c-lattices such that $L^0, N^0 \in \widetilde{\mathcal{L}}_d(M(F))_R$. Then the following diagram of categories is canonically 2-commutative.*

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{T^L_{S,R}} & \mathrm{D}\mathcal{H}_{L_S} \\ \downarrow{\scriptstyle \mathcal{F}_{N^0_R,L^0_R}} & & \downarrow{\scriptstyle \mathcal{F}_{N^0_S,L^0_S}} \\ \mathrm{D}\mathcal{H}_{N_R} & \xrightarrow{T^N_{S,R}} & \mathrm{D}\mathcal{H}_{N_S} \end{array}$$

Define (19) as the limit of functors $\mathcal{F}_{N^0_R,L^0_R}$ over the partially ordered set of c-lattices $R \subset R^\perp$ such that $L, N \in \mathcal{L}_d(M(F))_R$. As in § 3.5, one shows that for $L^0, N^0, R^0 \in \widetilde{\mathcal{L}}_d(M(F))$ the diagram is canonically 2-commutative.

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{R^0,L^0}} & \mathrm{D}\mathcal{H}_R \\ & {\scriptstyle \mathcal{F}_{N^0,L^0}} \searrow & \downarrow{\scriptstyle \mathcal{F}_{N^0,R^0}} \\ & & \mathrm{D}\mathcal{H}_N \end{array}$$

Our main result in the local field case is as follows.

Theorem 2. *For each k-point $L^0 \in \widetilde{\mathcal{L}}_d(M(F))$ there is a canonical functor*

$$\mathcal{F}_{L^0} : \mathrm{D}\mathcal{H}_L \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))) \tag{20}$$

sending $\mathcal{H}_L$ to $\mathrm{P}(\widetilde{\mathcal{L}}_d(M(F)))$. For a pair of k-points (L^0, N^0) in $\widetilde{\mathcal{L}}_d(M(F))$ the diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_L & \xrightarrow{\mathcal{F}_{L^0}} & \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))) \\ \downarrow{\scriptstyle \mathcal{F}_{N^0,L^0}} & \nearrow{\scriptstyle \mathcal{F}_{N^0}} & \\ \mathrm{D}\mathcal{H}_N & & \end{array} \tag{21}$$

is canonically 2-commutative. Let $W(\widetilde{\mathcal{L}}_d(M(F)))$ be the essential image of

$$\mathcal{F}_{L^0} : \mathcal{H}_L \to \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F)));$$

this is a full subcategory independent of L^0. Besides, $W(\widetilde{\mathcal{L}}_d(M(F)))$ is preserved under the natural action of $\tilde{G}(F)$ on $\mathrm{P}(\widetilde{\mathcal{L}}_d(M(F)))$.

We will refer to $W(\widetilde{\mathcal{L}}_d(M(F))$ as the *non-ramified Weil category on* $\widetilde{\mathcal{L}}_d(M(F))$. Recall that in the classical setting

$$\mathcal{H}_L = \mathcal{H}_{L,\mathrm{odd}} \oplus \mathcal{H}_{L,\mathrm{even}}$$

is a direct sum of two irreducible representations of the metaplectic group (consisting of odd and even functions, respectively). The representation $\mathcal{H}_{L,\mathrm{odd}}$ is ramified, and hence $\mathcal{H}_{L,\mathrm{even}}$ is not. The category $W(\widetilde{\mathcal{L}}_d(M(F)))$ together with the action of $\tilde{G}(F)$ is a geometric counterpart of the representation $\mathcal{H}_{L,\mathrm{even}}$. The proof of Theorem 2 is given in §§ 6.3–6.4.

6.3 Let L^0 be a k-point of $\widetilde{\mathcal{L}}_d(M(F))$. Let $R \subset R^\perp$ be a c-lattice with $L \cap R = 0$. Write L^0_R for the image of L^0 under (18). Applying the construction of § 3.6 to the symplectic space $R^\perp/R$ with $L^0_R \in \tilde{\mathcal{L}}(R^\perp/R)$, one gets the functor

$$\mathcal{F}_{L^0_R} : \mathrm{D}\mathcal{H}_{L_R} \to \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)).$$

If N^0 is another k-point of $\widetilde{\mathcal{L}}_d(M(F))_R$ then, writing N^0_R for the image of N^0 in $\widetilde{\mathcal{L}}(R^\perp/R)$, we also get that the diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0_R}} & \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow{\scriptstyle \mathcal{F}_{N^0_R,L^0_R}} & \nearrow{\scriptstyle \mathcal{F}_{N^0_R}} & \\ \mathrm{D}\mathcal{H}_{N_R} & & \end{array} \tag{22}$$

is canonically 2-commutative.

Now let

$$_R\mathcal{F}_{L^0} : \mathrm{D}\mathcal{H}_{L_R} \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_R)$$

denote the composition of $\mathcal{F}_{L^0_R}$ with the (exact for the perverse t-structures) restriction functor $\mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_R)$ for the projection (18).

Let $S \subset R$ be another c-lattice. As in § 5.3, for the open immersion $j_{S,R} : \widetilde{\mathcal{L}}_d(M(F))_R \hookrightarrow \widetilde{\mathcal{L}}_d(M(F))_S$ we have the restriction functors $j^*_{S,R} : \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_S) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_R)$.

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Lemma 8. *The following diagram of functors is canonically* 2*-commutative.*

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{{}_R\mathcal{F}_{L^0}} & \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_R) \\ \downarrow{\scriptstyle T^L_{S,R}} & & \uparrow{\scriptstyle j^*_{S,R}} \\ \mathrm{D}\mathcal{H}_{L_S} & \xrightarrow{{}_S\mathcal{F}_{L^0}} & \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))_S) \end{array}$$

Proof. We have an open immersion $j:\widetilde{\mathcal{L}}(S^\perp/S)_R \hookrightarrow \widetilde{\mathcal{L}}(S^\perp/S)$ and a projection $p_{R/S}:\widetilde{\mathcal{L}}(S^\perp/S)_R \to \widetilde{\mathcal{L}}(R^\perp/R)$. Set $P_{R/S} = p^*_{R/S} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\mathrm{dim.rel}(p_{R/S})}$. It suffices to show that the following diagram is canonically 2-commutative.

$$\begin{array}{ccccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0_R}} & \mathrm{D}(\widetilde{\mathcal{L}}(R^\perp/R)) & \xrightarrow{P_{R/S}} & \mathrm{D}(\widetilde{\mathcal{L}}(S^\perp/S)_R)) \\ \downarrow{\scriptstyle T^L_{S,R}} & & & \nearrow{\scriptstyle j^*} & \\ \mathrm{D}\mathcal{H}_{L_S} & \xrightarrow{\mathcal{F}_{L^0_S}} & \mathrm{D}(\widetilde{\mathcal{L}}(S^\perp/S)) & & \end{array}$$

This follows from Lemma 5. □

Define $\mathcal{F}_{L^0,R}:\mathrm{D}\mathcal{H}_{L_R} \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F)))$ as the functor sending K_1 to the following object K_2. For a c-lattice $S \subset R$ we declare the restriction of K_2 to $\widetilde{\mathcal{L}}_d(M(F))_S$ to be

$$({}_S\mathcal{F}_{L^0} \circ T^L_{S,R})(K_1).$$

By Lemma 8, the corresponding projective system defines an object K_2 of $\mathrm{D}(\widetilde{\mathcal{L}}_d(M(F)))$.

Finally, for $S \subset R$ with $R \cap L = 0$ the diagram

$$\begin{array}{ccc} \mathrm{D}\mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0,R}} & \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))) \\ \downarrow{\scriptstyle T^L_{S,R}} & \nearrow{\scriptstyle \mathcal{F}_{L^0,S}} & \\ \mathrm{D}\mathcal{H}_{L_S} & & \end{array}$$

is canonically 2-commutative. We define (20) as the limit of the functors $\mathcal{F}_{L^0,R}$ over the partially ordered set of c-lattices $R \subset R^\perp$ such that $L \cap R = 0$. The commutativity of (21) follows from the commutativity of (22).

Definition 4. The non-ramified Weil category $W(\widetilde{\mathcal{L}}_d(M(F)))$ is the essential image of the functor $\mathcal{F}_{L^0}:\mathcal{H}_L \to \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F)))$. It does not depend on a choice of a k-point L^0 of $\widetilde{\mathcal{L}}_d(M(F))$.

6.4 Let $R \subset R^\perp$ be a c-lattice in $M(F)$, let $\tilde{g} \in \tilde{G}(F)$ be a k-point, and write g for its image in $G(F)$. As in § 5.5, we have an isomorphism $g: H_R \xrightarrow{\sim} H_{gR}$ of algebraic groups over k sending $(x, a) \in (R^\perp/R) \times \mathbb{A}^1$ to $(gx, a) \in (gR^\perp/gR) \times \mathbb{A}^1$. For $L \in \mathcal{L}_d(M(F))_R$ it induces an equivalence

$$g:\mathcal{H}_{L_R} \xrightarrow{\sim} \mathcal{H}_{gL_{gR}}.$$

If $L^0 \in \widetilde{\mathcal{L}}_d(M(F))_R$ is a k-point then the G-equivariance of F implies that the following diagram is canonically 2-commutative.

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0_R}} & \mathrm{P}(\widetilde{\mathcal{L}}(R^\perp/R)) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0_{gR}}} & \mathrm{P}(\widetilde{\mathcal{L}}(gR^\perp/gR)) \end{array}$$

This, in turn, implies that the following diagram is 2-commutative.

$$\begin{array}{ccc} \mathcal{H}_{L_R} & \xrightarrow{\mathcal{F}_{L^0,R}} & \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))) \\ \downarrow g & & \downarrow \tilde{g} \\ \mathcal{H}_{gL_{gR}} & \xrightarrow{\mathcal{F}_{\tilde{g}L^0,gR}} & \mathrm{P}(\widetilde{\mathcal{L}}_d(M(F))) \end{array}$$

Thus, Theorem 2 is proved.

6.5 Theta-sheaf

Let $L \in \mathcal{L}_d(M(F))_M$; this is equivalent to saying that $L \subset M(F)$ is a Lagrangian d-lattice such that $L \oplus M = M(F)$. Then the category $\mathcal{H}_{L_M}$ has a distinguished object $\mathcal{L}_\psi$ on $\mathbb{A}^1 = \mathrm{H}_M$. Write S_L for its image under $\mathcal{H}_{L_M} \to \mathcal{H}_L$. The line bundle $\mathcal{A}_d$ over $\mathcal{L}_d(M(F))_M$ is canonically trivialized, so L has a distinguished enhanced structure

$$(L, \mathcal{B}) = L^0 \in \widetilde{\mathcal{L}}_d(M(F))_M,$$

where $\mathcal{B} = k$ is equipped with $\mathrm{id} : \mathcal{B}^2 \xrightarrow{\sim} \det(M : L)$. The *theta-sheaf* $S_{M(F)}$ *over* $\widetilde{\mathcal{L}}_d(M(F))$ is defined as $\mathcal{F}_{L^0}(S_L)$. It does not depend on $L \in \mathcal{L}_d(M(F))_M$ in the sense that for another $N \in \mathcal{L}_d(M(F))_M$ the diagram (21) yields a canonical isomorphism $\mathcal{F}_{L^0}(S_L) \xrightarrow{\sim} \mathcal{F}_{N^0}(S_N)$. The perverse sheaf $S_{M(F)}$ has a natural $G(\mathcal{O})$-equivariant structure.

6.6 Relation with the Schrödinger model

Assume in addition that M is decomposed as $M \xrightarrow{\sim} U \oplus U^* \otimes \Omega$, where U is a free $\mathcal{O}$-module of rank d, both U and $U^* \otimes \Omega$ are isotropic, and the form $\omega : \wedge^2 M \to \Omega$ is given by $\omega\langle u, u^*\rangle = \langle u, u^*\rangle$ for $u \in U, u^* \in U^* \otimes \Omega$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between U and U^*. Let $\bar{U} = U(F) \oplus \Omega(F)$ viewed as a subgroup of $H(F)$; it is equipped with the character $\chi_U : \bar{U} \to \bar{\mathbb{Q}}_\ell^*$ given by $\chi_U(u, a) = \psi(\mathrm{Res}\, a)$, $a \in \Omega(F), u \in U(F)$. Write

$$\begin{aligned} \mathrm{Shr}_U = \{ f : H(F) \to \bar{\mathbb{Q}}_\ell \mid f(\bar{u}h) = \chi_U(\bar{u}) f(h), \bar{u} \in \bar{U}, h \in H(F), \\ f \text{ is smooth, of compact support modulo } \bar{U} \}, \end{aligned}$$

where $H(F)$ acts on it by right translations. This is the Schrödinger model of the Weil representation, and it identifies naturally with the Schwarz space $\mathcal{S}(U^* \otimes \Omega(F))$.

Recall the definition of the derived category $\mathrm{D}(U^* \otimes \Omega)$ and its subcategory of perverse sheaves $\mathrm{P}(U^* \otimes \Omega)$ given in [Lys07, § 4]. For $N, r \in \mathbb{Z}$ with $N + r \geq 0$ we write ${}_{N,r}U = t^{-N}U/t^r U$.

For $N_1 \geq N_2, r_1 \geq r_2$ we have a diagram

$${}_{N_2,r_2}(U^* \otimes \Omega) \xleftarrow{p} {}_{N_2,r_1}(U^* \otimes \Omega) \xrightarrow{i} {}_{N_1,r_1}(U^* \otimes \Omega),$$

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where p is the smooth projection and i is a closed immersion. We have a transition functor

$$\mathrm{D}({}_{N_2,r_2}(U^*\otimes\Omega))\to\mathrm{D}({}_{N_1,r_1}(U^*\otimes\Omega)) \tag{23}$$

sending K to $i_!p^*K\otimes(\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\mathrm{dim.rel}(p)}$, which is fully faithful and exact for the perverse t-structures. Then $\mathrm{D}(U^*\otimes\Omega(F))$ (respectively, $\mathrm{P}(U^*\otimes\Omega(F))$) is defined as the inductive 2-limit of $\mathrm{D}({}_{N,r}(U^*\otimes\Omega))$ (respectively, of $\mathrm{P}({}_{N,r}(U^*\otimes\Omega))$) as r, N go to infinity. The category $\mathrm{P}(U^*\otimes\Omega(F))$ is the geometric analog of the space Shr_U.

In this section we prove the following.

PROPOSITION 5. *For each k-point $L^0\in\widetilde{\mathcal{L}}_d(M(F))$ there is a canonical equivalence*

$$\mathcal{F}_{U(F),L^0}:\mathrm{D}(U^*\otimes\Omega(F))\to\mathrm{D}\mathcal{H}_L \tag{24}$$

which identifies $\mathrm{P}(U^*\otimes\Omega(F))$ with the category $\mathcal{H}_L$. For $L^0, N^0\in\widetilde{\mathcal{L}}_d(M(F))$ the following diagram is canonically 2-commutative.

$$\begin{array}{ccc}
\mathrm{D}(U^*\otimes\Omega(F)) & \xrightarrow{\mathcal{F}_{U(F),L^0}} & \mathrm{D}\mathcal{H}_L \\
\downarrow{\scriptstyle \mathcal{F}_{U(F),N^0}} & \nearrow{\scriptstyle \mathcal{F}_{L^0,N^0}} & \\
\mathrm{D}\mathcal{H}_N & &
\end{array}$$

For $N\geq 0$ consider the c-lattice $R=t^NM$ in $M(F)$ and the corresponding symplectic space $R^\perp/R={}_{N,N}M$. Set $U_R:={}_{N,N}U\in\mathcal{L}({}_{N,N}M)$. We have the line bundle $\mathcal{A}_N$ on $\mathcal{L}({}_{N,N}M)$ whose fibre at L is $\det({}_{0,N}M)\otimes\det L$. As above, $\widetilde{\mathcal{L}}({}_{N,N}M)$ is the gerb of square roots of $\mathcal{A}_N$. Let

$$U_R^0=(U_R,\det({}_{0,N}U))\in\widetilde{\mathcal{L}}({}_{N,N}M)$$

equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism $\det({}_{0,N}U)^2\,\widetilde{\to}\,\det U_R\otimes\det({}_{0,N}M)$.

Let $H_R={}_{N,N}M\times\mathbb{A}^1$ denote the corresponding Heisenberg group, which has the subgroup $\bar{U}_R=U_R\times\mathbb{A}^1$ equipped with the character $\chi_{U,R}:\bar{U}_R\to\bar{\mathbb{Q}}_\ell^*$ given by $\chi_{U,R}(u,a)=\psi(a)$, $a\in\mathbb{A}^1$. In the classical setting, $\mathcal{H}_{U_R}$ is the space of functions on H_R which are $(\bar{U}_R,\chi_{U,R})$-equivariant under the left multiplication. Set $\mathrm{Shr}_U^R=\{f\in\mathrm{Shr}_U \mid f(h(r,0))=f(h), r\in R, h\in H\}$.

LEMMA 9. *In the classical setting there is an isomorphism*

$$\mathrm{Shr}_U^R\,\widetilde{\to}\,\mathcal{H}_{U_R}. \tag{25}$$

Proof. Write $\mathcal{H}'_{U_R}=\{\phi':R^\perp/R\to\bar{\mathbb{Q}}_\ell \mid \phi'(m+u)=\psi(\frac{1}{2}\langle m,u\rangle)\phi'(m),\ u\in U_R\}$. We identify $\mathcal{H}_{U_R}\,\widetilde{\to}\,\mathcal{H}'_{U_R}$ via the map $\phi\mapsto\phi'$, where $\phi'(m)=\phi(m,0)$. Given $f\in\mathrm{Shr}_U^R$ for $m\in t^{-N}M$ the value $f(m,0)$ depends only on the image $\bar{m}$ of m under $t^{-N}M\to{}_{N,N}M$. The isomorphism (25) sends f to $\phi'\in\mathcal{H}'_{U_R}$ given by $\phi(\bar{m})=f(m,0)$. □

In the geometric setting $\mathcal{H}_{U_R}$ is the category of $(\bar{U}_R,\chi_{U,R})$-equivariant perverse sheaves on H_R. We identify it with $\mathrm{P}({}_{N,N}(U^*\otimes\Omega))$ as follows. Let $m_U:\bar{U}_R\times{}_{N,N}(U^*\otimes\Omega)\to H_R$ be the isomorphism sending $(\bar{u},h)$ to their product $\bar{u}h$ in H_R. The functor $\mathrm{D}({}_{N,N}(U^*\otimes\Omega))\to\mathrm{D}\mathcal{H}_{U_R}$ sending K to

$$(m_U)_!(\chi_{U,R}\boxtimes K)\otimes(\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim\bar{U}_R}$$

is an equivalence (exact for the perverse t-structures).

Let $N'\geq N$ and $S=t^{N'}M$. The corresponding transition functor (23) now yields a functor denoted by $T^U_{S,R}:\mathrm{D}\mathcal{H}_{U_R}\to\mathrm{D}\mathcal{H}_{U_S}$.

Let $L^0 \in \widetilde{\mathcal{L}}_d(M(F))$ be a k-point over $L \in \mathcal{L}_d(M(F))$. Assume that N is large enough so that $L \cap R = 0$. Let L^0_R denote the image of L^0 under (18). Define U^0_S, $L^0_S \in \widetilde{\mathcal{L}}(S^\perp/S)$ similarly.

Lemma 10. *The following diagram is canonically* 2*-commutative.*

$$\begin{CD} \mathrm{D}\mathcal{H}_{U_R} @>{T^U_{S,R}}>> \mathrm{D}\mathcal{H}_{U_S} \\ @V{\mathcal{F}_{L^0_R,U^0_R}}VV @VV{\mathcal{F}_{L^0_S,U^0_S}}V \\ \mathrm{D}\mathcal{H}_{L_R} @>{T^L_{S,R}}>> \mathrm{D}\mathcal{H}_{L_S} \end{CD}$$

Proof. Set $W = t^{N'}U \oplus t^N(U^* \otimes \Omega)$. The subspace $W/S \subset S^\perp/S$ is isotropic, and $U_S \cap (W/S) = L_S \cap (W/S) = 0$. Write $H_W = (W^\perp/W) \times \mathbb{A}^1$ for the corresponding Heisenberg group. Set $U_W = U_S \cap (W^\perp/S)$, $L_W = L_S \cap (W^\perp/S)$. Applying Proposition 2, we get the following 2-commutative diagram.

$$\begin{CD} \mathrm{D}\mathcal{H}_{U_W} @>{T^U_{S,W}}>> \mathrm{D}\mathcal{H}_{U_S} \\ @V{\mathcal{F}_{L^0_W,U^0_W}}VV @VV{\mathcal{F}_{L^0_S,U^0_S}}V \\ \mathrm{D}\mathcal{H}_{L_W} @>{T^L_{S,W}}>> \mathrm{D}\mathcal{H}_{L_S} \end{CD}$$

Now $R/W \subset W^\perp/W$ is an isotropic subspace, and $R/W \subset U_W$, $R/W \cap L_W = 0$. Note that $U_R = U_W/(R/W)$. Applying Proposition 3, we get the following 2-commutative diagram.

$$\begin{CD} \mathrm{D}\mathcal{H}_{U_R} @>{T^U_{W,R}}>> \mathrm{D}\mathcal{H}_{U_W} \\ @V{\mathcal{F}_{L^0_R,U^0_R}}VV @VV{\mathcal{F}_{L^0_W,U^0_W}}V \\ \mathrm{D}\mathcal{H}_{L_R} @>{T^L_{W,R}}>> \mathrm{D}\mathcal{H}_{L_W} \end{CD}$$

Our assertion easily follows. □

Proof of Proposition 5. Passing to the limit as N goes to infinity, the functors $\mathcal{F}_{L^0_R,U^0_R} : \mathrm{D}\mathcal{H}_{U_R} \to \mathrm{D}\mathcal{H}_{L_R}$ from Lemma 10 yield the desired functor (24). The second assertion follows by construction. □

Definition 5. Let $\mathcal{F}_{U(F)} : \mathrm{D}(U^* \otimes \Omega(F)) \to \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F)))$ denote the composition

$$\mathrm{D}(U^* \otimes \Omega(F)) \xrightarrow{\mathcal{F}_{U(F),L^0}} \mathrm{D}\mathcal{H}_L \xrightarrow{\mathcal{F}_{L^0}} \mathrm{D}(\widetilde{\mathcal{L}}_d(M(F))).$$

By Theorem 2 and Proposition 5, it does not depend on the choice of a k-point $L^0 \in \widetilde{\mathcal{L}}_d(M(F))$. By construction, $\mathcal{F}_{U(F)}$ is exact for the perverse t-structures.

We have a morphism of group stacks $\mathrm{GL}(U)(F) \to \tilde{G}(F)$ sending $g \in \mathrm{GL}(U)(F)$ to $(g, \mathcal{B} = \det(U : gU))$ equipped with a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism

$$\det(M : gM) \widetilde{\to} \det(U : gU) \otimes \det(U^* \otimes \Omega : g(U^* \otimes \Omega)) \widetilde{\to} \det(U : gU)^{\otimes 2}.$$

Let $\mathrm{GL}(U)(F)$ act on $\widetilde{\mathcal{L}}_d(M(F))$ via this homomorphism; let it also act naturally on $U^* \otimes \Omega(F)$. Then one may show that $\mathcal{F}_{U(F)}$ commutes with the action of $\mathrm{GL}(U)(F)$.

Note also that over $\mathrm{GL}(U)(\mathcal{O})$ the sections $\mathrm{GL}(U)(F) \to \tilde{G}(F)$ and $G(\mathcal{O}) \to \tilde{G}(F)$ are compatible.

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7. Global application

7.1 Assume k algebraically closed. Let X be a smooth connected projective curve. Let Ω be the canonical invertible sheaf on X. Let G be the group scheme over X of automorphisms of $\mathcal{O}_X^d \oplus \Omega^d$ preserving the symplectic form $\wedge^2(\mathcal{O}_X^d \oplus \Omega^d) \to \Omega$.

Write Bun_G for the stack of G-torsors on X, which classifies a rank $2d$-vector bundle $\mathcal{M}$ on X together with a symplectic form $\wedge^2\mathcal{M} \to \Omega$. Let $\mathcal{A}$ be the ($\mathbb{Z}/2\mathbb{Z}$-graded purely of degree zero) line bundle on Bun_G whose fibre at $\mathcal{M}$ is $\det \mathrm{R}\Gamma(X, \mathcal{M})$. Write $\widetilde{\mathrm{Bun}}_G$ for the gerb of square roots of $\mathcal{A}$ over Bun_G.

Recall the definition of the theta-sheaf Aut on $\widetilde{\mathrm{Bun}}_G$ [Lys06, Definition 1]. Let ${}_i\mathrm{Bun}_G \hookrightarrow \mathrm{Bun}_G$ be the locally closed substack given by $\dim \mathrm{H}^0(X, \mathcal{M}) = i$ for $\mathcal{M} \in \mathrm{Bun}_G$. Write ${}_i\widetilde{\mathrm{Bun}}_G$ for the restriction of $\widetilde{\mathrm{Bun}}_G$ to ${}_i\mathrm{Bun}_G$.

Let ${}_i\mathcal{B}$ be the line bundle on ${}_i\mathrm{Bun}_G$ whose fibre at $\mathcal{M} \in {}_i\mathrm{Bun}_G$ is $\det \mathrm{H}^0(X, \mathcal{M})$; we view it as $\mathbb{Z}/2\mathbb{Z}$-graded of degree $i \bmod 2$. For each i we have a canonical $\mathbb{Z}/2\mathbb{Z}$-graded isomorphism ${}_i\mathcal{B}^2 \xrightarrow{\sim} \mathcal{A}$, which yields a trivialization ${}_i\widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i\mathrm{Bun}_G \times B(\mu_2)$.

Define $\mathrm{Aut}_g \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G$ (respectively, $\mathrm{Aut}_s \in \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$) as the intermediate extension of

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathrm{Bun}_G}$$

(respectively, of $(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})^{\dim \mathrm{Bun}_G - 1})$ under ${}_i\widetilde{\mathrm{Bun}}_G \hookrightarrow \widetilde{\mathrm{Bun}}_G$. Set $\mathrm{Aut} = \mathrm{Aut}_g \oplus \mathrm{Aut}_s$.

7.2 Fix a closed point $x \in X$. Write $\mathcal{O}_x$ for the completed local ring of X at x, and F_x for its fraction field. Fix a G-torsor over $\mathrm{Spec}\, \mathcal{O}_x$; we think of it as a free $\mathcal{O}_x$-module M of rank $2d$ with symplectic form $\wedge^2 M \to \Omega(\mathcal{O}_x)$ and an action of $G(\mathcal{O}_x)$. We have a map

$$\xi_x : \mathrm{Bun}_G \to \mathcal{L}_d(M(F_x))/G(\mathcal{O}_x),$$

where $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$ is the stack quotient. It sends $\mathcal{M} \in \mathrm{Bun}_G$ to the Tate space $\mathcal{M}(F_x)$ with Lagrangian c-lattice $\mathcal{M}(\mathcal{O}_x)$ and Lagrangian d-lattice $\mathrm{H}^0(X - x, \mathcal{M})$.

The line bundle $\mathcal{A}_d$ on $\mathcal{L}_d(M(F_x))/G(\mathcal{O}_x)$ is that of § 5.3. Write $\widetilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x)$ for the gerb of square roots of $\mathcal{A}_d$.

We have canonically that $\xi_x^*\mathcal{A}_d \xrightarrow{\sim} \mathcal{A}$, so ξ lifts naturally to a map of gerbs

$$\tilde{\xi}_x : \widetilde{\mathrm{Bun}}_G \to \widetilde{\mathcal{L}}_d(M(F_x))/G(\mathcal{O}_x).$$

For $r \geq 0$ let ${}_{rx}\mathrm{Bun}_G \subset \mathrm{Bun}_G$ be the open substack given by $\mathrm{H}^0(X, \mathcal{M}(-rx)) = 0$. Write ${}_{rx}\widetilde{\mathrm{Bun}}_G$ for the restriction of the gerb $\widetilde{\mathrm{Bun}}_G$ to ${}_{rx}\mathrm{Bun}_G$. If $r' \geq r$ then ${}_{rx}\widetilde{\mathrm{Bun}}_G \subset {}_{r'x}\widetilde{\mathrm{Bun}}_G$ is an open substack, so we consider the projective 2-limit

$$\underset{r\to\infty}{\text{2-lim}}\, \mathrm{D}({}_{rx}\widetilde{\mathrm{Bun}}_G).$$

Note that $\text{2-lim}_{r\to\infty} \mathrm{P}({}_{rx}\widetilde{\mathrm{Bun}}_G) \xrightarrow{\sim} \mathrm{P}(\widetilde{\mathrm{Bun}}_G)$ is a full subcategory in the above limit. Let us define the restriction functor

$$\tilde{\xi}_x^* : \mathrm{D}_{G(\mathcal{O})}(\tilde{\mathcal{L}}_d(M(F))) \to \underset{r\to\infty}{\text{2-lim}}\, \mathrm{D}({}_{rx}\widetilde{\mathrm{Bun}}_G). \tag{26}$$

To do so, for $N \geq r \geq 0$ and $r_1 \geq 2N$ let

$$\xi_N : {}_{rx}\mathrm{Bun}_G \to {}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}) \tag{27}$$

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be the map sending $\mathcal{M}$ to the Lagrangian subspace $\mathrm{H}^0(X, \mathcal{M}(Nx)) \subset {}_{N,N}\mathcal{M}$. If $N_1 \geq N \geq r$ and $r_1 \geq 2N_1$ then the following diagram commutes.

$$\begin{array}{ccc} {}_{rx}\mathrm{Bun}_G & \xrightarrow{\xi_N} & {}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}) \\ & \searrow^{\xi_{N_1}} & \uparrow p \\ & & {}_r\mathcal{L}({}_{N_1,N_1}M)/G(\mathcal{O}/t^{r_1}) \end{array}$$

It induces a similar diagram between the gerbs (cf. § 5.3 for their definition).

$$\begin{array}{ccc} {}_{rx}\widetilde{\mathrm{Bun}}_G & \xrightarrow{\tilde{\xi}_N} & ({}_r\mathcal{L}({}_{N,N}M)/G(\mathcal{O}/t^{r_1}))\tilde{} \\ & \searrow^{\tilde{\xi}_{N_1}} & \uparrow \\ & & ({}_r\mathcal{L}({}_{N_1,N_1}M)/G(\mathcal{O}/t^{r_1}))\tilde{} \end{array}$$

The functors $K \mapsto \tilde{\xi}_N^* K \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\mathrm{dim.rel}(\xi_N)}$ from $\mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}({}_{N,N}M))$ to $\mathrm{D}({}_{rx}\widetilde{\mathrm{Bun}}_G)$ are compatible with the transition functors, so they yield a functor

$$ {}_r\xi_x^* : \mathrm{D}_{G(\mathcal{O})}({}_r\tilde{\mathcal{L}}_d(M(F))) \to \mathrm{D}({}_{rx}\widetilde{\mathrm{Bun}}_G).$$

Passing to the limit by r, one gets the desired functor (26).

Theorem 3. *The object $\tilde{\xi}_x^* S_{M(F_x)}$ lies in $\mathrm{P}(\widetilde{\mathrm{Bun}}_G)$, and there is an isomorphism of perverse sheaves*

$$\tilde{\xi}_x^* S_{M(F_x)} \xrightarrow{\sim} \mathrm{Aut}.$$

Proof. For $r \geq 0$ consider the map

$$\tilde{\xi}_r : {}_{rx}\widetilde{\mathrm{Bun}}_G \to (\mathcal{L}({}_{r,r}M)/G(\mathcal{O}/t^{2r}))\tilde{}.$$

Set $Y = \mathcal{L}({}_{r,r}M) \times \mathcal{L}({}_{r,r}M)$. Write $\mathcal{Y}$ for the stack quotient of Y by the diagonal action of $\mathbb{S}\mathrm{p}({}_{r,r}M)$. Let $\mathcal{A}_{\mathcal{Y}}$ be the $\mathbb{Z}/2\mathbb{Z}$-graded purely of degree-zero line bundle on $\mathcal{Y}$ with fibre $\det L_1 \otimes \det L_2$ at (L_1, L_2). Write $\tilde{\mathcal{Y}}$ for the gerb of square roots of $\mathcal{A}_{\mathcal{Y}}$ over $\mathcal{Y}$. The map $\mathcal{L}({}_{r,r}M) \to Y$ sending L_1 to $({}_{0,r}M, L_1) \in Y$ yields a morphism of stacks

$$\rho : (\mathcal{L}({}_{r,r}M)/G(\mathcal{O}/t^{2r}))\tilde{} \to \tilde{\mathcal{Y}}.$$

Write $S_{{}_{r,r}M}$ for the perverse sheaf on $\tilde{\mathcal{Y}}$ introduced in § 3.2, Definition 1. Set $\tau = \rho \circ \tilde{\xi}_r$. It suffices to establish for any $r \geq 0$ a canonical isomorphism

$$\tau^* S_{{}_{r,r}M} \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\mathrm{dim.rel}(\tau)} \xrightarrow{\sim} \mathrm{Aut} \tag{28}$$

over ${}_{rx}\widetilde{\mathrm{Bun}}_G$.

Recall that $Y_i \subset Y$ is the locally closed subscheme given by $\dim(L_1 \cap L_2) = i$ for $(L_1, L_2) \in Y$. Let $\mathcal{Y}_i$ be the stack quotient of Y_i by the diagonal action of $\mathbb{S}\mathrm{p}({}_{r,r}M)$, and set $\tilde{\mathcal{Y}}_i = \mathcal{Y}_i \times_{\mathcal{Y}} \tilde{\mathcal{Y}}$. Set

$${}_{rx,i}\widetilde{\mathrm{Bun}}_G = {}_{rx}\widetilde{\mathrm{Bun}}_G \cap {}_i\widetilde{\mathrm{Bun}}_G \quad \text{and} \quad {}_{rx,i}\mathrm{Bun}_G = {}_{rx}\mathrm{Bun}_G \cap {}_i\mathrm{Bun}_G.$$

For each i the map τ fits into a cartesian square as follows.

$$\begin{array}{ccc} {}_{rx,i}\widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau_i} & \tilde{\mathcal{Y}}_i \\ \downarrow & & \downarrow \\ {}_{rx}\widetilde{\mathrm{Bun}}_G & \xrightarrow{\tau} & \tilde{\mathcal{Y}} \end{array}$$

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Indeed, for $\mathcal{M} \in {}_{rx}\mathrm{Bun}_G$ the space $\mathrm{H}^0(X, \mathcal{M})$ equals the intersection of $\mathcal{M}/\mathcal{M}(-rx)$ and $\mathrm{H}^0(X, \mathcal{M}(rx))$ inside $\mathcal{M}(rx)/\mathcal{M}(-rx)$. By [Lys06, Theorem 1], the $*$-restriction of Aut to ${}_i\widetilde{\mathrm{Bun}}_G \xrightarrow{\sim} {}_i\mathrm{Bun}_G \times B(\mu_2)$ identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathrm{Bun}_G - i}.$$

Similarly, by [Lys06, Propositions 1 and 5], the $*$-restriction of S_M to $\tilde{\mathcal{Y}}_i \xrightarrow{\sim} \mathcal{Y}_i \times B(\mu_2)$ identifies with

$$(\bar{\mathbb{Q}}_\ell \boxtimes W) \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{Y} - i}.$$

Since the map τ_i is compatible with our trivializations of the corresponding gerbs, we get the isomorphism (28) over ${}_{rx,i}\widetilde{\mathrm{Bun}}_G$ for each i. Since Aut is perverse, this also shows that the left-hand side of (28) is placed in perverse degrees at most 0, and its $*$-restriction to ${}_{\leqslant 2}\widetilde{\mathrm{Bun}}_G$ is placed in perverse degrees less than 0.

The map τ is not smooth, but we overcome this difficulty as follows. Let us show that the left-hand side of (28) is placed in perverse degrees at most 0. Consider the stack $\mathcal{X}$ classifying $(\mathcal{M}, \mathcal{B}) \in {}_{rx}\widetilde{\mathrm{Bun}}_G$ and a trivialization

$$\mathcal{M}|_{\mathrm{Spec}\, \mathcal{O}_x/t_x^{2r}} \xrightarrow{\sim} M|_{\mathrm{Spec}\, \mathcal{O}_x/t_x^{2r}}$$

of the corresponding G-torsor. Let $\nu : \mathcal{X} \to \tilde{Y}$ be the map sending a point of $\mathcal{X}$ to the triple $(\mathcal{M}/\mathcal{M}(-rx), H^0(X, \mathcal{M}(rx)), \mathcal{B})$. Define $\mathcal{X}_1$ and $\mathcal{X}_3$ by the cartesian squares as follows.

$$\begin{array}{ccc}
\mathcal{X}_3 & \longrightarrow & C_3 \\
\downarrow{\scriptstyle \pi_{\mathcal{X}_3}} & & \downarrow{\scriptstyle \pi_C} \\
\mathcal{X}_1 & \longrightarrow & U \times_{\mathcal{L}(r,{}_rM)} U \\
\downarrow & & \downarrow{\scriptstyle \pi_Y} \\
\mathcal{X} & \xrightarrow{\nu} & \tilde{Y}
\end{array}$$

Using (7), we get an isomorphism

$$\mu^* \tau^* S_{r,{}_rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\mathrm{dim.rel}(\mu) + \mathrm{dim.rel}(\tau)} \xrightarrow{\sim} (\pi_{\mathcal{X}_3})_! \mathcal{E} \otimes (\bar{\mathbb{Q}}_\ell[1](\tfrac{1}{2}))^{\dim \mathcal{X}_3}$$

for some rank-one local system $\mathcal{E}$ on $\mathcal{X}_3$. Here $\mu : \mathcal{X}_1 \to {}_{rx}\widetilde{\mathrm{Bun}}_G$ is the projection, which is smooth. Since $\pi_{\mathcal{X}_3}$ is affine and $\mathcal{X}_3$ is smooth, the left-hand side of (28) is placed in perverse degrees at least 0.

Thus, there exists an exact sequence of perverse sheaves $0 \to K \to K_1 \to \mathrm{Aut} \to 0$ on ${}_{rx}\widetilde{\mathrm{Bun}}_G$, where $K_1 = \tau^* S_{r,{}_rM} \otimes (\bar{\mathbb{Q}}_\ell[1](\frac{1}{2}))^{\mathrm{dim.rel}(\tau)}$, and K is the extension by zero from ${}_{\leqslant 2}\widetilde{\mathrm{Bun}}_G$. But we know already that K_1 and Aut are isomorphic in the Grothendieck group of ${}_{rx}\widetilde{\mathrm{Bun}}_G$. So, K vanishes in this Grothendieck group, and hence $K = 0$. We are done. □

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Vincent Lafforgue vlafforg@math.jussieu.fr
Institut de Mathématiques, Université Paris 6, 175, rue du Chevaleret, 75013 Paris, France

Sergey Lysenko vlafforg@math.jussieu.fr
Institut de Mathématiques, Université Paris 6, 175, rue du Chevaleret, 75013 Paris, France

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