ON EXTENSIONS OF TOPOLOGIES

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1. Introduction. If (X, τ) is a topological space (with topology τ) and A is a subset of X, then the topology $\tau(A) = \{U \cup (V \cap A) | U, V \in \tau\}$ is said to be a *simple extension* of τ. It seems that N. Levine introduced this concept in **(4)** and he proved, among other results, the following:

(A) If (X, τ) is a regular (completely regular) space and A is a closed subset of X, then $(X, \tau(A))$ is a regular (completely regular) space.

(B) Let (X, τ) be a normal space, and A a closed subset of X. Then $(X, \tau(A))$ is normal if and only if $X - A$ is a normal subspace of (X, τ).

(C) Let (X, τ) be a countably compact (compact or Lindelöf) and $A \notin \tau$. Then $(X, \tau(A))$ is countably compact (compact or Lindelöf) if and only if $(X - A, \tau \cap (X - A))$ is, respectively, countably compact (compact or Lindelöf).

However, Levine failed to give necessary and sufficient conditions for $(X, \tau(A))$ to inherit the following topological properties from (X, τ): regularity, complete regularity, normality, perfect normality, collectionwise normality, paracompactness, stratifiability,[1] and metrizability. We shall settle this matter by proving the following:

(A′) $(X, \tau(A))$ inherits regularity from (X, τ) if and only if $cA - A$ (where cA denotes the closure of A in (X, τ)) is a closed subset of (X, τ).

(B′) $(X, \tau(A))$ inherits complete regularity (normality, collectionwise normality, paracompactness, stratifiability, or metrizability) from (X, τ) if and only if $(X, \tau(A))$ $(X - A, \tau(X - A))$ does, and is a regular space.

Furthermore we shall prove some results concerning the connectedness of $(X, \tau(A))$ and give some applications of our results.

Finally, we shall develop the concept of *infinite extensions* (see Definition 5.1) of topologies and prove that most results which are valid for simple extensions are also valid for countably infinite extensions.

Unless otherwise specified we adopt the terminology of Kelley **(3)**, except that all our spaces are T_1.

Received December 9, 1965, and in revised form, April 22, 1966. This research was supported by the NSF Grant GP-4770.

[1]A topological space (X, τ) is stratifiable if and only if to each open $U \subset X$ one can assign a sequence $\{U_1, U_2, \ldots\}$ of τ-open subsets of X such that $U_n^- \subset U$ for each n, $\bigcup_{n=1}^{\infty} U_n = U$, and $U_n \subset V_n$ (for each n) whenever $U \subset V$. The correspondence $U \to \{U_1, U_2, \ldots\}$ is called a *stratification* of (X, τ). Our stratifiable spaces are equivalent to the M_3-spaces of Ceder **(2)**. For the sake of perspective we point out that metrizable spaces are stratifiable, stratifiable spaces are perfectly paracompact, and all CW-complexes of Whitehead are stratifiable.

https://doi.org/10.4153/CJM-1967-040-9 Published online by Cambridge University Press

2. Notation and preliminary results. Let (X, τ) be a topological space, $\tau(A)$ a simple extension of τ, and N any subset of X. Then we make the following definitions:

(a) $\tau \cap B = \{U \cap B | U \in \tau\}$,

(b) N is τ-open (τ-closed) provided that $N \in \tau$ (N is a closed subset of (X, τ)),

(c) cN (Int N) denotes the closure of N (interior of N) with respect to τ,

(d) $c_A N$ denotes the closure of N with respect to $\tau(A)$,

(e) A family $\mathfrak{F}$ of subsets of X is τ-locally finite ($\tau\sigma$-locally finite) if it is locally finite (σ-locally finite) with respect to τ.

The same applies to any subset of Y of X and any topology μ on Y.

Finally we state some obvious, but vital, facts about $\tau(A)$, some of which already appear in Levine **(4)**.

LEMMA 2.1. *Let (X, τ) be a topological space and $\tau(A)$ a simple extension of τ. Then*

(a) $\tau \cap A = \tau(A) \cap A$,

(b) $\tau \cap (X - A) = \tau(A) \cap (X - A)$,

(c) *$X - A$ is $\tau(A)$-closed,*

(d) *If $x \notin A$ and B is any subset of X, then $x \in cB$ if and only if $x \in c_A B$.*

3. Basic properties of simple extensions. Before we embark on our main task, it seems appropriate to point out that whenever A is a τ-closed subset of X, then $(X, \tau(A))$ is the disjoint union of two $\tau(A)$-clopen[2] subsets A and $X - A$ such that $\tau(A) \cap A = \tau \cap A$ and

$$\tau(A) \cap (X - A) = \tau \cap (X - A).$$

Consequently, the following are easily seen to be true:

(a) If A is a τ-closed subset of X, then $(X, \tau(A))$ inherits regularity, complete regularity, normality, perfect normality, collectionwise normality, paracompactness, stratifiability, or metrizability from (X, τ) if and only if $(X - A, \tau(X - A))$ does.

(b) If A is a τ-closed subset of X, with $A \neq X$, then $(X, \tau(A))$ is never a connected or pathwise connected space (the same is not true otherwise, as will be demonstrated in Theorem 4.1).

We shall now prove, in the next two results, necessary and sufficient conditions for $(X, \tau(A))$ to inherit regularity from (X, τ). Our Theorem 3.2 will thus improve **(4**, Theorem 2**)**.

PROPOSITION 3.1. *Let (X, τ) be a regular topological space and let A be a subset of X. Then $(X, \tau(A))$ is not a regular space if and only if there exists*

[2]A subset A of any topological space (X, τ) is τ-clopen provided that A is τ-open and τ-closed.

an element x of A such that every τ-neighbourhood N of x contains some element of $cA - A$.

Proof. First we prove the "if" part: Let U be a $\tau(A)$-neighbourhood of x such that $U \subset A$ and assume that there exists a $\tau(A)$-closed $\tau(A)$-neighbourhood M of x such that $M \subset U$. Then $x \in M' \cap A \subset M$ for some $M' \in \tau$ and there exists $y \in M' \cap (cA - A)$, by hypothesis. Hence

$$y \in c(M' \cap A) \subset cM = c_A M,$$

due to (**4**, Lemma 4), a contradiction (since $c_A M = M \subset A$).

Now we prove the "only if" part: Assume that $(X, \tau(A))$ is not a regular space. Then there exists $x \in X$ at which $\tau(A)$ is not regular (i.e., some $\tau(A)$-neighbourhood U of x contains no $\tau(A)$-closed $\tau(A)$-neighbourhood of x). Clearly, $x \in A$ (assume that $x \notin A$ and $U = V \cup (\emptyset \cap A)$, where $\emptyset$ is the empty set, is a $\tau(A)$-neighbourhood of x and W is a τ-closed τ-neighbourhood of x such that $W \subset V$ (note that (X, τ) is a regular space). Since $c_A W \subset cW$, due to (**4**, Lemma 2), then $W \cup (\emptyset \cap A)$ is a $\tau(A)$-closed $\tau(A)$-neighbourhood of x which is contained in U and thus $(X, \tau(A))$ is regular at x). Furthermore if some τ-neighbourhood N of x does not contain any point of $cA - A$, then $(X, \tau(A))$ is certainly regular at x (let $U = \emptyset \cup (0 \cap A)$ be a $\tau(A)$-neighbourhood of x and let V be a τ-closed τ-neighbourhood of x such that $V \subset N \cap 0$. Then $c_A(V \cap A) = c(V \cap A) \subset V \cap cA$ due to (**4**, Lemma 4) and the fact that V is τ-closed. However $V \cap cA = V \cap A$ since $V \subset N \cap 0$ and N contains no points of $cA - A$. Consequently, we get that

$$c_A(V \cap A) \subset V \cap A \subset (N \cap 0) \cap A \subset U$$

and thus $(X, \tau(A))$ is regular at x, a contradiction which completes the proof.

THEOREM 3.2. *Let (X, τ) be a regular topological space and let A be a subset of X. Then $(X, \tau(A))$ is a regular space if and only if $cA - A$ is a τ-closed subset of X (i.e., $A \cap \operatorname{bdry} A \subset X - (cA - A) \in \tau$).*

Proof. It suffices to show that for each $x \in A \cap \operatorname{bdry} A$ there exists a τ-neighbourhood N of x such that $N \cap \operatorname{bdry} A \subset A \cap \operatorname{bdry} A$, a fact which follows immediately from Proposition 3.1.

We are now ready to improve (**4**, Theorem 4) as follows:

THEOREM 3.3. *Let (X, τ) be a completely regular topological space and let A be a subset of X. Then $(X, \tau(A))$ is a completely regular space if and only if it is a regular space.*

Proof. If $x \in X - A$ or $x \in \operatorname{Int} A$, then $\tau(A)$ is clearly completely regular at x (i.e., for every $\tau(A)$-neighbourhood N of x there exists a continuous function $f\colon X \to I$, where I is the closed unit interval, such that $f(x) = 0$ and $f(X - N) = 1$) since any τ-continuous function on X is $\tau(A)$-continuous and any τ-neighbourhood base for x is also a $\tau(A)$-neighbourhood base for x.

If $x \in A \cap \operatorname{bdry} A$ let $N \cap A$ be any $\tau(A)$-neighbourhood of x and consider a $\tau(A)$-neighbourhood $U \cap A$ of x, with $U \subset N$, such that

$$U \cap \operatorname{bdry} A \subset A \cap \operatorname{bdry} A.$$

Since $(A, \tau(A) \cap A)$ is completely regular by **(4**, Lemma 3**)**, there exists a $\tau(A)$-continuous function $f\colon A \to I$ such that $f(x) = 0$ and $f(A - U \cap A) = 1$. Now we define $f\colon X \to I$ as follows: $F(a) = f(a)$ for each $a \in A$ and $F(y) = 1$ for each $y \in X - A$. Clearly $F(X - N \cap A) = 1$ (since $U \subset N$), $F(x) = 0$, and it is easily seen that F is $\tau(A)$-continuous (since any $z \in U - A$ has a τ-neighbourhood which misses A).

Next we improve Theorem 5 in Levine **(4)**.

THEOREM 3.4. *Let (X, τ) be a normal space and A a subset of X. Then $(X, \tau(A))$ is a normal space if and only if it is a regular space and $X - A$ is a normal subspace of (X, τ).*

Proof. The "only if" part is obvious. We therefore prove the "if" part.

Let F and G be disjoint $\tau(A)$-closed subsets of X. Since $cA - A$ is a τ-closed subset of X, due to Theorem 3.2, then $F' = F \cap [cA - A]$ and $G' = G \cap [cA - A]$ are disjoint τ-closed subsets of X. Therefore there exist τ-open subsets U and V of X such that $U \cap V = \emptyset$, $F' \subset U$, and $G' \subset V$ (indeed, we shall choose U and V such that $cU \cap cV = \emptyset$ and

$$cU \cap cG = \emptyset = cV \cap cF,$$

which can be done because $F' \cap cG = \emptyset$ and $G' \cap cF = \emptyset$, and (X, τ) is normal). Then $F_* = (F - A) - U$ and $G_* = (G - A) - V$ are closed subsets of $(X - A, \tau \cap (X - A))$. Therefore there exist τ-open subsets U' and V' of X such that $(U' - A) \cap (V' - A) = \emptyset$, $U' - A \supset F_*$, and $V' - A \supset G_*$. However $F_*, G_* \subset X - cA$ and hence $U' - cA$ and $V' - cA$ are disjoint τ-open sets such that $U' - cA \supset F_*$ and $V' - cA \supset G_*$. Consequently, $U_* = U \cup [U' - (cA \cup cV)]$ and $V_* = V \cup [V' - (cA \cup cU)]$ are disjoint τ-open sets such that $F - A \subset U_*$ and $G - A \subset V_*$, since $cU \cap (G - A) = \emptyset = cV \cap (F - A)$.

Finally let $F_A = F - U_*$ and $G_A = G - V_*$. It is easily seen that F_A and G_A are disjoint τ-closed subsets of X and hence $cU \cup F_A$ and $cV \cup G_A$ are disjoint τ-closed subsets of X. Therefore there exist disjoint τ-open subsets M and N of X such that $M \supset cU \cup F_A$ and $N \supset cV \cup G_A$. Consequently, $U'' = U_* \cup (M \cap A)$ and $V'' = V_* \cup (N \cap A)$ are disjoint $\tau(A)$-open subsets of X (note that $M \cap V = \emptyset$ and $N \cap U = \emptyset$) such that $F \subset U''$ and $G \subset V''$, completing the proof.

Similarly we can prove the following:

THEOREM 3.5. *Let (X, τ) be an hereditarily normal space and A a subset of X. Then $(X, \tau(A))$ is an hereditarily normal space if and only if it is a regular space.*

Proof. This is immediate from the preceding result and the fact that, for each subset Y of X, $(Y, (\tau \cap Y)(A \cap Y)) = (Y, \tau(A) \cap Y)$.

Furthermore, the following is also true:

THEOREM 3.6. *Let (X, τ) be a perfectly normal space and A a subset of X. Then $(X, \tau(A))$ is a perfectly normal space if and only if it is a regular space.*

Proof. Since the "only if" part is clear, we proceed to prove the "if" part. Because of Theorem 3.4 $(X, \tau(A))$ is a normal space. We shall now show that each $W \in \tau(A)$ is the union of countably many $\tau(A)$-closed subsets of X: Clearly $W = U \cup (V \cap A)$ such that $U, V \in \tau$, and $V \cap (cA - A) = \emptyset$ (since $cA - A$ is τ-closed by Theorem 3.2). Then

$$U = \bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad V = \bigcup_{n=1}^{\infty} B_n$$

for some sequences $\{A_1, A_2, \ldots\}$ and $\{B_1, B_2, \ldots\}$ of τ-closed subsets of X. Hence

$$W = \bigcup_{n=1}^{\infty} [A_n \cup (B_n \cap A)]$$

and each $A_n \cup (B_n \cap A)$ is $\tau(A)$-closed (since each $B_n \cap A$ is $\tau(A)$-closed). Consequently $(X, \tau(A))$ is perfectly normal, which completes the proof.

Using the same basic method of proof of Theorem 3.4 we can also prove the following:

THEOREM 3.7. *Let (X, τ) be a collectionwise normal space and A a subset of X. Then $(X, \tau(A))$ is a collectionwise normal space if and only if it is a regular space and $X - A$ is a collectionwise normal subspace of (X, τ).*

Proof. Since the "only if" part is clear, we proceed with the proof of the "if" part. Let $\{A_\alpha\}_{\alpha \in F}$ be a $\tau(A)$-discrete collection of $\tau(A)$-closed subsets of X and let $A'_\alpha = A_\alpha \cap (cA - A)$ for each $\alpha \in F$. Then there exists a τ-discrete family $\{V_\alpha\}_{\alpha \in F}$ of τ-open subsets of X such that $A'_\alpha \subset U_\alpha$ for each $\alpha \in F$ (since $\{A'_\alpha\}_{\alpha \in F}$ is a τ-discrete family of τ-closed sets). Clearly, for each $\alpha \in F$,

$$A'_\alpha \cap c(\bigcup \{A_\beta \mid \beta \in F \text{ and } \beta \neq \alpha\}) = \emptyset$$

and thus we shall choose the sets U_α such that $cU_\alpha \cap A_\beta = \emptyset$ whenever $\alpha \neq \beta$. Letting $A_\alpha^* = (A_\alpha - A) - U_\alpha$, we get that $\{A_\alpha^*\}_{\alpha \in F}$ is a discrete family of closed subsets of $(X - A, \tau \cap (X - A))$ such that $A_\alpha^* \subset X - cA$ for each $\alpha \in F$. Consequently there exist pairwise disjoint τ-open subsets U'_α of X such that $U'_\alpha - cA \supset A_\alpha^*$. Letting $A_\alpha(A) = A_\alpha - U'_\alpha$ we get that

$\{cU_\alpha \cup A_\alpha(A)\}_{\alpha \in F}$ is a discrete family of τ-closed subsets of X. Therefore there exists pairwise disjoint τ-open subsets V_α of X such that

$$cU_\alpha \cup A_\alpha(A) \subset V_\alpha$$

for each $\alpha \in F$. Consequently $\{U_\alpha^* \cup (V_\alpha \cap A)\}_{\alpha \in F}$ is a pairwise disjoint family of $\tau(A)$-open subsets of X such that $A_\alpha \subset U_\alpha^* \cup (V_\alpha \cap A)$ for each $\alpha \in F$, and hence $(X, \tau(A))$ is collectionwise normal.

Finally we improve **(4**, Theorem 6**)** as follows:

THEOREM 3.8. *Let (X, τ) be a regular Lindelöf (compact; countably compact) space and A a subset of X. Then $(X, \tau(A))$ is a regular Lindelöf (compact; countably compact) space if and only if it is a regular space and $X - A$ is respectively a Lindelöf (compact; countably compact) subspace of (X, τ).*

Proof. Immediate from our Theorem 3.2 and **(4**, Theorem 6**)**.

We conclude this section with results concerning simple extensions of paracompact, stratifiable, and metrizable spaces. No similar results appear in **(4)**.

THEOREM 3.9. *Let (X, τ) be a paracompact regular space and A a subset of X. Then $(X, \tau(A))$ is a paracompact regular space if and only if it is a regular space and $X - A$ is a paracompact subspace of (X, τ).*

Proof. First we prove the "if" part. Let $\mathfrak{U}$ be a $\tau(A)$-open cover of X (without loss of generality we assume that for each $x \in cA$ there exists some $U \in \mathfrak{U} \cap \tau$ such that $x \in U$ and $U \cap A = \emptyset$, for each $a \in A$ there exists some $U \in \tau$ such that $a \in U \cap A \in \mathfrak{U}$, and for each $y \in cA - A$ there exists some $V \in \mathfrak{U} \cap \tau$ such that $y \in V$) and let $\mathfrak{W} = \{U \in \mathfrak{U} | \, U \in \tau$ and $U \cap [cA - A] \neq \emptyset\}$. Since $cA - A$ is a τ-closed subset of X, due to Theorem 3.2, there exists a τ-locally finite (and thus $\tau(A)$-locally finite) family $\mathfrak{W}'$ of τ-open subsets of X which covers $(\operatorname{bdry} A) - A$ and refines $\mathfrak{W}$ (we simply observe that $\mathfrak{W} \cup (X - (cA - A))$ is an open cover of the paracompact space (X, τ) and thus it has a τ-locally finite (τ-open) refinement $\mathfrak{W}_*$; therefore we let $\mathfrak{W}' = \{W \in \mathfrak{W}_* | \, W \cap [\operatorname{bdry} A) - A] \neq \emptyset\}$). We let $W' = \cup \mathfrak{W}'$. Since $A - W'$ is a τ-closed subset of X ($A - W' = (A \cup \operatorname{bdry} A) - W' = cA - W'$ since $W' \supset (\operatorname{bdry} A) - A$), we let

$$\mathfrak{V} = \{V \in \tau | \, V \cap A \in \mathfrak{U} \text{ and } V \cap (A - W') \neq \emptyset\}.$$

Then we can also find a τ-locally finite family $\mathfrak{V}_*$ of τ-open subsets of X which covers $A - W'$ and refines $\mathfrak{V}$. Hence $\mathfrak{V}' = \{V \cap A | \, V \in \mathfrak{V}_*\}$ is a τ-(A)locally finite family of $\tau(A)$-open subsets of X which refines

$$\{V \cap A \in \mathfrak{U} | \, V \cap (A - \cup \mathfrak{W}) \neq \emptyset\}.$$

https://doi.org/10.4153/CJM-1967-040-9 Published online by Cambridge University Press

Finally, let

$$\mathfrak{S} = \{U \in \mathfrak{U} | \ U \in \tau, U \cap (X - (W' \cup A)) \neq \emptyset \text{ and } U \cap A = \emptyset\}.$$

Clearly $\mathfrak{S}$ covers $X - (W' \cup A)$ since $W' \cup A \supset cA \cup W'$. Since $X - (W' \cup A)$ is a closed subset of $(X - A, \tau \cap (X - A))$, because $X - (W' \cup A) = (X - A) - W'$, then there exists a $\tau \cap (X - A)$-locally finite family $\mathfrak{S}'$ of τ-open subsets of X which covers $X - (W' \cup A)$ and refines $\mathfrak{S}$. Since $X - A$ is a $\tau(A)$-closed subset of X, $\mathfrak{S}'$ is also $\tau(A)$-locally finite.

It is now easily seen that $\mathfrak{U}' = \mathfrak{W}' \cup \mathfrak{V}' \cup \mathfrak{S}'$ is a $\tau(A)$-locally finite $\tau(A)$-open refinement of $\mathfrak{U}$.

To prove the "only if" part we need only observe that $X - A$ is a closed subspace of $(X, \tau(A))$, and closed subspaces of paracompact spaces are paracompact.

Similarly we can prove the following:

THEOREM 3.10. *Let (X, τ) be an hereditarily paracompact regular space and A a subset of X. Then $(X, \tau(A))$ is an hereditarily paracompact regular space if and only if it is a regular space.*

Proof. This is essentially the same as the proof of Theorem 3.5, due to Theorem 3.8.

THEOREM 3.11. *Let (X, τ) be a stratifiable space and A a subset of X. Then $(X, \tau(A))$ is a stratifiable space if and only if it is a regular space.*

Proof. Since the "only if" part is obvious, we proceed with the proof of the "if" part. Because of Theorem 3.2, if $N = X - (cA - A)$, then $n \in \tau$. Therefore for each $0 \cup (0' \cap A) \in \tau(A)$ we get that

$$0 \cup (0' \cap A) = 0 \cup (0' \cap N \cap A).$$

We now let $U \to \{U_1, U_2, \ldots\}$ be a stratification (see footnote 1) of (X, τ). Then

$$0 \cup (0' \cap A) \to \{0_n \cup [0' \cap N)_n \cap A]\}_{n=1}^{\infty}$$

is a stratification of $(X, \tau(A))$. We only need show that

$$c_A(0_n \cup [0' \cap N)_n \cap A) \subset 0 \cup (0' \cap A),$$

for which it suffices to show that $c_A((0' \cap N)_n \cap A) \subset 0' \cap A$. However, by **(4**, Lemma 4**)**,

$$c_A((0' \cap N)_n \cap A) = c((0' \cap N)_n \cap A) \subset c((0' \cap N)_n) \cap cA.$$

But $x \in c((0' \cap N)_n) \cap cA$ implies that $x \in 0' \cap N$ and thus $x \in A$ (clearly $x \notin (\text{bdry } A) - A$ since $x \in N$ and $x \in cA$). Hence

$$c_A((0' \cap N)_n \cap A) \subset c(0' \cap N)_n)) \cap A \subset 0' \cap A,$$

completing the proof.

THEOREM 3.12. *Let (X, τ) be a metrizable space and A a subset of X. Then $(X, \tau(A))$ is metrizable if and only if it is a regular space.*

Proof. Since (X, τ) is metrizable, it has an open base

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

such that each $\mathfrak{B}_n$ is discrete (i.e., $\mathfrak{B}$ is a σ-discrete base for (X, τ)). For each n let $\mathfrak{B}_n^* = \mathfrak{B}_n \cup \{\mathfrak{B} \cap A \mid B \in \mathfrak{B}_n \text{ and } B \cap A \neq \emptyset\}$. Then

$$\mathfrak{B}^* = \bigcup_{n=1}^{\infty} \mathfrak{B}_n^*$$

is an open base for $(X, \tau(A))$ such that each $\mathfrak{B}_n^*$ is $\tau(A)$-locally finite (i.e., $\mathfrak{B}^*$ is a σ-locally finite base for $(X, \tau(A))$. Consequently $(X, \tau(A))$ is metrizable (see, for example, (**3**, Theorem 18, p. 127).

We end this section by summarizing the preceding results in the following fashion:

THEOREM 3.13. *Let (X, τ) be a completely regular (hereditarily normal; perfectly normal; hereditarily paracompact; stratifiable or metrizable) space and A a subset of X. Then $(X, \tau(A))$ i,s respectively, completely regular (hereditarily normal; perfectly normal; hereditarily paracompact; stratifiable or metrizable) if and only if $cA - A$ is a τ-closed subset of X.*

THEOREM 3.14. *Let (X, τ) be a normal (paracompact; Lindelöf; compact; countably compact) space and A a subset of X. Then $(X, \tau(A))$ is, respectively, a normal (collectionwise normal; paracompact; Lindelöf; compact; countably compact) space if and only if $cA - A$ is a τ-closed subset of X and $X - A$ is respectively a normal (collectionwise normal; paracompact; Lindelöf; compact; countably compact) subspace of (X, τ).*

4. Connectivity and some applications.

THEOREM 4.1. *Let (X, τ) be a topological space and A any subset of X which is not τ-closed. If A and $X - A$ are connected subspaces of (X, τ), then $(X, \tau(A))$ is connected.*

Proof. Suppose U and V are $\tau(A)$-open disjoint sets which cover X. By (**4**, Corollary 1), $U \neq A$ and $V \neq A$. Therefore $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$ or $U - A \neq \emptyset$ and $V - A \neq \emptyset$, contradicting the fact that A and $X - A$ are connected (if $U \cap A \neq \emptyset$ and $V \cap A \neq \emptyset$, then they separate $(A, \tau \cap A)$; similarly, if $U - A \neq \emptyset$ and $V - A \neq \emptyset$, then they separate $X - A$).

The following two easily proved results are interesting consequences of Theorem 4.1.

COROLLARY 3.2. *Let (X, τ) be a topological space and A any subset of X which is not τ-closed. If A and $X - A$ are connected subspaces of (X, τ), then (X, τ) is connected.*

COROLLARY 4.3. *Let (X, τ) be a connected completely regular space which is not compact. Then the Stone–Čech compactification βX of X is connected whenever $\beta X - X$ is connected.*

THEOREM 4.4. *Let (X, τ) be a topological space and A a subset of X which is not τ-closed. If cA and $X - A$ are pathwise connected, then $(X, \tau(A))$ is pathwise connected.*

Proof. Assume we have $x \in A$ and $y \notin A$. To get a path from x to y simply get a path p_1: $[0, \frac{1}{2}] \to X$ from x to some point $w \in cA - A$ which is contained in cA; then get another path p_2: $[\frac{1}{2}, 1] \to X$ from w to y which is contained in $X - A$. Then the path $p_1 \cup p_2$ is a path in $(X, \tau(A))$ from x to y.

We end this section with an example which shows that the usual topology of the real line R is not maximal with respect to connectedness.

EXAMPLE 4.5. *Let (I, μ) be the closed unit interval with the usual topology μ. Then there exists a simple extension $\mu(A)$ of μ, with $A \notin \mu$, such that $(I, \mu(A))$ is a connected space.*

Proof. Let

$$A = \bigcup_{n=1}^{\infty} A_n,$$

where $A_1 = [\frac{3}{8}, \frac{5}{8}]$ and, for each $n > 1$, A_n is the union of the closed middle fourths of the maximal intervals contained in $[0, 1] - A_{n-1}$. Note that $[0, 1] - A \neq \emptyset$ since the Lebesgue measure of A is less than 1; furthermore $[0, 1] - A$ contains no intervals, and A is a dense subset of $]0, 1[$. We shall now show that $(R, \mu(A))$ is a connected space. Suppose it is not. Then $I = U \cup V$ such that $U, V \in \mu(A)$ and $U \cap V = \emptyset$. Since A is the union of closed (connected) intervals, we must have that, whenever u (or V) intersects one of the closed intervals contained in A, then U contains it.

Furthermore, no maximal closed subinterval of A is a $\mu(A)$-open subset of R. Hence we must have that, if U contains a maximal closed subinterval J of A, then U contains a μ-open set U' such that $J \subset V'$. Hence one can easily show that both $U, V \in \mu$ (since both must intersect closed intervals contained in A) and thus $U \cap V \neq \emptyset$ (since (R, μ) is connected), a contradiction.

It is quite easily seen (from Theorems 3.1, 3.12, and 4.1) that the usual topology of the cartesian plane is not a maximal pathwise connected separable metrizable topology.

5. Infinite extensions.

Definition 5.1. Let (X, τ) be a topological space and $\mathfrak{F} = \{\tau(A_\alpha)\}_{\alpha \in L}$ be a family of simple extensions of τ. Then Λ is the $\mathfrak{F}$-extension of τ if Λ is the smallest topology on X which contains $\tau(A_\alpha)$ for each $\alpha \in L$.

It is easily seen that (X, Λ) may not even be a normal space though $(X, \tau(A_\alpha))$ is a separable metrizable space for each $\alpha \in L$. Let (E_2, τ) be the plane with the half-open rectangle topology τ, see (**3**, Exercise L, p. 59) and for each $(a, b) \in E_2$ let $A_{ab} = \{(x, y) \in E_2 | \, a \leqslant x < a + 1, b \leqslant y < b + 1\}$. Also let μ be the usual topology of the plane. If $\mathfrak{F} = \{\mu(A_{ab}) | (a, b) \in E_2\}$, then it can be easily shown that τ is the $\mathfrak{F}$-extension of μ, and clearly each $(X, \mu(A_{ab}))$ is a separable metrizable space. However, (E_2, τ) is not a normal space, see (**3**, Exercise I, p. 133). Furthermore, infinite extensions of connected topologies are not necessarily connected: Let (E, μ) be the cartesian plane with the usual topology μ, $A = \{(x, y) \in E | \, y > 0 \text{ or } y = 0 \text{ and } x \geqslant 0\}$ and $B = E - A$. Then $\mu(A)$ and $\mu(B)$ are connected topologies, due to Theorem 4.1, but the $\mathfrak{F}$-extension of μ, where $\mathfrak{F} = \{\mu(A), \mu(B)\}$, is clearly not connected.

However, we shall prove the following results:

THEOREM 5.2. *Let (X, τ) be a metrizable (stratifiable) space and*

$$\mathfrak{F} = \{\tau(A_n)\}_{n=1}^{\infty}$$

be a countable family of simple extensions of τ such that each $(X, \tau(A_n))$ is a regular space. Then (X, Λ) is a metrizable (stratifiable) space, where Λ is the $\mathfrak{F}$-extension of τ.

THEOREM 5.3. *Let (X, τ) be a perfectly normal (perfectly paracompact[3]) space and $\mathfrak{F} = \{\tau(A_n)\}_{n=1}^{\infty}$ be a countable family of simple extensions of τ such that each $(X, \tau(A_n))$ is a regular space. Then (X, Λ) is perfectly normal (perfectly paracompact), where Λ is the $\mathfrak{F}$-extension of τ.*

THEOREM 5.4. *Let (X, τ) be a regular hereditarily Lindelöf space and*

$$\mathfrak{F} = \{\tau(A_n)\}_{n=1}^{\infty}$$

be a countable family of simple extensions of τ such that each $(X, \tau(A^n))$ is a regular space. Then (X, Λ) is hereditarily Lindelöf, where Λ is the $\mathfrak{F}$-extension of τ.

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[3]A topological space X is perfectly paracompact provided that X is paracompact and perfectly normal.

In order to prove the preceding theorems, we need to prove the following auxiliary results:

LEMMA 5.5. *Let* (X, τ) *be a topological space,* $\mathfrak{F} = \{\tau(A_n)\}_{n=1}^{\infty}$ *a countable family of simple extensions of* τ, $\mathfrak{F}_n = \{\tau(A_k)\}_{k=1}^{n}$, $\Lambda_1 = \tau(A_1)$, *and* $\Lambda_n = \Lambda_{n-1}(A_n)$ *for each* n. *Then the following are true:*

(a) Λ_n *is the* $\mathfrak{F}_n$*-extension of* τ.

(b) *If* $(X, \tau(A_k))$ *is a regular space for each* $k \leqslant n$, *then* (X, Λ_n) *is a regular space.*

(c) *If* $\mathfrak{B}_n$ *is an open base for* (X, Λ_n), *for each* n, *then*

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

is an open base for (X, Λ).

(d) *Each* (X, Λ_n) *inherits normality, perfect normality, paracompactness, stratifiability, and metrizability from* (X, τ).

Proof. The proof of (a) is straightforward.

(b) Let τ_1 and τ_2 be any regular topologies on a set Y and Λ the smallest topology on Y which contains τ_1 and τ_2. It is then easy to see that Λ is a regular topology on Y. Hence, by (a) and Definition 5.1, (X, Λ_n) is a regular space if $(X, \tau(A_k))$ is a regular space for each $k \leqslant n$.

(c) Straightforward, since $\Lambda_n \subset \Lambda_{n+1}$ for each n.

(d) Immediate from (b), the definition of Λ_n, and Theorems 3.4, 3.5, 3.6, 3.9, 3.11, and 3.12.

LEMMA 5.6. *Let* (X, τ) *be a paracompact space and* U *an open* F_σ*-subset of* X (*i.e.* U *is the union of countably many closed subsets of* X). *Then* U *is* τ*-paracompact* (*i.e. every* $\tau \cap U$)*-open cover* $\mathfrak{U}$ *of* U *has a* τ*-open refinement*

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

such that each $\mathfrak{B}_n$ *is* τ*-locally finite*).

Proof. Clearly

$$\bigcup_{n=1}^{\infty} A_n = U = \bigcup_{n=1}^{\infty} \operatorname{Int} A_n$$

for some countable family $\{A_1, A_2, \ldots\}$ of closed subsets of X. The rest of the proof is straightforward.

LEMMA 5.7. *Let* (X, τ) *be a topological space. Then* (X, τ) *is perfectly normal if and only if for each* $U \in \tau$ *there exists a sequence* $\{U_1, U_2, \ldots\}$ *of* τ*-open subsets of* X *such that*

$$\bigcup_{n=1}^{\infty} U_n = U = \bigcup_{n=1}^{\infty} cU_n.$$

Proof. Since the "only if" part is clear, we proceed with the proof of the "if" part, for which we assume, without loss of generality, that the sequence $\{U_1, U_2, \ldots\}$ is an increasing sequence. Let A and B be two disjoint τ-closed subsets of X, and let

$$U = \bigcup_{n=1}^{\infty} [(X - B)_n - c(X - A)_n] \quad \text{and} \quad V = \bigcup_{n=1}^{\infty} [(X - A)_n - c(X - B)_n].$$

Clearly $U, V \in \tau$, $A \subset U$, $B \subset V$. Thus we only need show that $U \cap V = \emptyset$ to complete the proof. Assume there exists $x \in U \cap V$. Then

$$x \in (X - B)_n - c(X - A)_n \quad \text{and} \quad x \in (X - A)_m - c(X - B_m)$$

for some n and m. Then

$$x \in (X - B)_n - c(X - B)_m \quad \text{and} \quad x \in (X - A)_m - c(X - A)_n.$$

Since either $n \leqslant m$ or $m \leqslant n$, we get a contradiction. Hence $U \cap V = \emptyset$.

We shall now prove Theorems 5.2, 5.3, and 5.4.

Proof of Theorem 5.2. (a) Assume (X, τ) is metrizable. Because of Lemma 5.5(d), let $\mathfrak{B}_n$ be a σ-locally finite open base for (X, σ_n). If

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n,$$

the $\mathfrak{B}$ is a σ-locally finite base for (X, Λ), by Lemma 5.5(c), and hence (X, Λ) is metrizable, by Lemma 5.5(b) and (**3**, Theorem 18, p. 127).

(b) Assume (X, τ) is stratifiable. Since our stratifiable spaces are equivalent to the M_3-spaces of Ceder (**2**; **1** footnote 2), we let $\mathfrak{B}_n$ be a σ-cushioned pair base (**2**, Definition 1.3) for each (X, Λ_n). If

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n,$$

then $\mathfrak{B}$ is a σ-cushioned pair base for (X, Λ), by Lemma 5.5(c), and hence (X, Λ) is stratifiable by Lemma 5.5(b).

Proof of Theorem 5.3. (a) Assume (X, τ) is perfectly normal and $U \in \Lambda$. Then

$$U = \bigcup_{n=1}^{F} U_n$$

such that $U_n \in \Lambda_n$, by Lemma 5.5(c). Since (X, Λ_m) is perfectly normal for each n, by Lemma 5.5(d), then

$$\bigcup_{m=1}^{\infty} W_{n,m} = U_n = \bigcup_{m=1}^{\infty} cW_{n,m},$$

https://doi.org/10.4153/CJM-1967-040-9 Published online by Cambridge University Press

where $cW_{n,m}$ denotes the closure of $W_{n,m}$ in (X, Λ_n). Note that $W_{n,m} \in \Lambda$ and $W_{n,m}^-$ is Λ-closed for each m. Hence

$$\bigcup_{n,m=1}^{\infty} W_{n,m} = U = \bigcup_{n,m=1}^{\infty} cW_{n,m},$$

where $cW_{n,m}$ denotes the closure of $W_{n,m}$ in (X, Λ), and thus (X, Λ) is perfectly normal, by Lemma 5.7.

(b) Assume (X, τ) is perfectly paracompact and let $\mathfrak{U}$ be a Λ-open cover of X. Without loss of generality, we assume that $\mathfrak{U} \subset \mathfrak{B}$, where

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

with $\mathfrak{B}_n$ a base for (X, Λ_n) for each n (see Lemma 5.5(c)). If

$$\mathfrak{U}_n = \{U \in \mathfrak{U} | \ U \in \mathfrak{B}_n\} \quad \text{and} \quad U_n = \cup \mathfrak{U}_n,$$

then U_n is Λ_n-paracompact for each n, by Lemma 5.6, and hence each U_n is Λ-paracompact. Consequently, it is easily seen that $\mathfrak{U}$ has a $\sigma\Lambda$-locally finite Λ-open refinement and thus (X, Λ) is paracompact. Clearly (X, Λ) is perfectly normal, by part (a), which completes the proof.

Proof of Theorem 5.4. Let $\mathfrak{U}$ be a Λ-open cover of X. By Lemma 5.5(c) there exists a Λ-open refinement

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

of $\mathfrak{U}$ such that each $\mathfrak{B}_n$ is a collection of Λ_n-open sets. Since each (X, Λ_n) is easily seen to be hereditarily Lindelöf, there exists a countable subfamily $\mathfrak{C}_n$ of $\mathfrak{B}_n$ such that $\mathfrak{C}_n$ covers $\cup \mathfrak{B}_n$. Consequently

$$\mathfrak{C} = \bigcup_{n=1}^{\infty} \mathfrak{C}_n$$

is a countable subcover of $\mathfrak{B}$ and it is now easily seen that $\mathfrak{U}$ has a countable subcover. Hence (X, Λ) is a Lindelöf space. Similarly we can show that (X, Λ) is hereditarily Lindelöf.

We must unfortunately point out that we cannot answer the following question: Is Theorem 5.3 valid for any normal or paracompact space (X, τ)? We conjecture that the answer to this question is negative. (We suggest that one should search for a completely regular space (X, τ) such that (X, τ) is not normal and

$$X = \bigcap_{n=1}^{\infty} U_n,$$

with U_n an open normal subset of the Stone–Čech compactification $(\beta X, \beta\tau)$ of (X, τ). It will then follow that $(\beta X, \Lambda_n)$, with $\Lambda_n = \beta\tau(\beta X - U_n)$, is a normal space for each n, but $(\beta X, \Lambda)$ is a regular non-normal space since X is a closed non-normal subset of $(\beta X, \Lambda)$.

We end our study of extensions of topologies with the following two easily proved results:

THEOREM 5.8. *Let* (X, τ) *be a separable (second countable) space and*

$$\mathfrak{F} = \{\tau(A_n)\}_{n=1}^{\infty}$$

be a sequence of simple extensions of τ *such that* $(A_n, \tau \cap A_n)$ *is separable for each* n. *Then* (X, Λ) *is separable (second countable) where* Λ *is the* $\mathfrak{F}$*-extension of* τ.

Proof. Because of Lemma 5.5(c), induction, and the definition of Λ_n, it suffices to prove that $(X, \tau(A_1))$ is separable (second countable). By (**4**, Theorem 8) the proof is thus completed.

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