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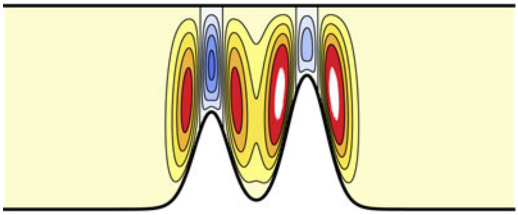
Focus on Fluids

A conundrum in conversion



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Maintaining the stratification of the ocean requires deep mixing. Part of the energy that provides this mixing is transferred over topography from the surface tides into the internal tides. Numerous theoretical, numerical and experimental studies have aimed at quantifying the energy transfer of this conversion mechanism. Maas (*J. Fluid Mech.*, this issue, vol. 684, 2011, pp. 5–24) constructs model topographies with localized response and no energy transfer into the propagating internal tide, a surprising result that raises questions about models of tidal conversion.

Key words: internal waves, ocean processes, topographic effects

1. Introduction

The Moon is receding from the Earth at a rate of 38 mm/year. This could be called the most accurately known constant in physical oceanography: the motion of the Moon leads to an energy loss of ∼3.7 TW from the Earth–Moon system through tidal dissipation that mostly takes place in the ocean. This is responsible for the maintenance of oceanic stratification, along with energy input from the winds (Munk 1997). Munk & Wunsch (1998) calculated that, to maintain the abyssal stratification, mechanical energy is needed to drive mixing and evaluated the origins of this mixing, bringing to the fore the role of tides. Estimates of the flux are 3.5 TW in the surface tide of which 2.6 TW goes to mixing the shallow bottom boundary layer and 0.9 TW goes into the internal tide. For comparison, the wind is estimated to produce 1.2 TW, combining with the internal tide to produce 2.1 TW of deep-ocean mixing.

The dissipation of tides by bottom drag in shallow regions was well known to Taylor (1919) and Jeffreys (1920). A simple estimate of the power input or energy flux is $\mathscr{P} \sim 0.0025\rho_0 u^3$ (W m^{-2}), where ρ_0 and u are representative values for the density of seawater and the tidal velocity respectively. The energy budget was closed by dissipation over continental shelves. The advent of satellite oceanography has transformed our understanding of tides in the ocean. Ray & Mitchum (1997) showed direct evidence of the coherent internal tide radiating away from the Hawaiian Ridge. Assimilating numerical tidal models have shown where energy loss from the global

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tide occurred: Egbert & Ray (2000) produced maps showing that in fact dissipation was occurring in the deep ocean over topographic features such as mid-ocean ridges.

This effect had previously been estimated to be small (e.g. Baines 1973). The underlying mechanism is that the usual surface tide moves deep stratified fluid over topography and in so doing generates internal waves, which are called the internal tide. The power dissipation scales like $\mathscr{P} \sim \rho_0 u^2 N \ell$ (W m^{-2}), where N is the buoyancy frequency at the ocean floor and ℓ is a length scale related to the topography. Since Munk & Wunsch (1998) there has been a great deal of interest in the tidal conversion problem: how the internal tide is produced by the interaction of the barotropic tide with topography. The Hawaii ocean mixing experiment (HOME), an oceanographic measurement programme, measured the mixing produced by the sharp bathymetry of the Hawaiian Island chain. Numerical calculations were carried out to obtain estimates of the conversion rates. Theoretical investigations probed the underlying physics of the problem. Laboratory experiments measured conversion rates in controlled conditions. The literature is now extensive; links and overviews are given in Garrett (2003) and Garrett & Kunze (2007). A review of internal tides in general can be found in Vlasenko, Stashchuk & Hutter (2005). The new study by Maas (2011) suggests that something is missing from the current understanding of the tidal conversion problem: Maas provides a constructive fashion of generating bathymetries that exhibit no conversion, yet clearly resemble the model bathymetries that have been employed in theoretical and laboratory studies (e.g. the idealized 'Luzon Strait' double-ridge shown in the figure by the title). Dai *et al.* (2011) have used the same method to obtain wave-like solutions.

2. Overview

Internal tides are internal gravity waves (IGWs). The dispersion relation of a plane internal gravity wave propagating in a Boussinesq, inviscid fluid with constant buoyancy frequency N can be written as $\omega/N = \cos\theta$, where θ is the angle the wavevector of the wave makes with the horizontal. Hence rays propagate at constant angle to the horizontal and the frequency is independent of wavenumber. In addition, the phase velocity and group velocity are perpendicular. These properties of the dispersion relation are responsible for some of the more unusual properties of these dispersive waves. In particular, reflection is not specular; rather, reflected waves make the same angle with the horizontal as the incoming wave.

We consider a two-dimensional problem: the barotropic tide 'sloshes' with velocity $U_0 \cos(\omega t)$ back and forth over the seafloor $z = H(x)$. The quantity $\alpha = \tan\theta$ can be thought of as the slope of the rays, and the ratio of α to the slope of the bathymetry, s, is critical in determining the nature of the conversion process. The limit $s \ll \alpha$ is the weak topography approximation (WTA) investigated by Bell (1975). This formulation has been used in conjunction with oceanic bathymetric surveys (St Laurent & Garrett 2002) and has been extended to three dimensions (Llewellyn Smith & Young 2002). Bathymetries with $s < \alpha$ are termed subcritical. When there are regions of the seafloor with $s \geqslant \alpha$, the bathymetry is supercritical. Llewellyn Smith & Young (2003) calculated the conversion from a knife-edge ridge and showed that there was no singularity in the conversion rate.

Maas considers the time-harmonic subcritical problem in a two-dimensional channel and rescales variables so that the ray slope is 45°. The underlying governing equation for the streamfunction, $\psi(x, z)$, is

$$\psi_{xx} - \psi_{zz} = 0, \tag{2.1}$$

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with boundary conditions $\psi = 0$ on $z = 0$ and $\psi = Q$ on $z = H(x)$. Equation (2.1) is the Poincaré equation, that is, the wave equation but in two spatial variables. This is unusual: similar boundary-value problems are usually elliptic rather than hyperbolic. Maas then introduces a mapping ξ, ζ, and points out that if $\xi(x, z)$ and $\zeta(x, z)$ satisfy the hyperbolic Cauchy–Riemann equations (Motter & Rosa 1998)

$$\xi_x = \zeta_z, \quad \xi_z = \zeta_x, \tag{2.2}$$

equation (2.1) is unchanged. Maas maps the flow domain into a channel with (constant) depth h in the (ξ, ζ) plane, for which the solution to (2.1) is

$$\psi = \frac{Q\zeta}{h} + \sum_{n=-\infty}^{\infty} \hat{\psi}_n \sin \frac{n\pi\zeta}{h} \mathrm{e}^{\mathrm{i}n\pi\xi/h}. \tag{2.3}$$

The first term $Q\zeta/h$ gives a non-radiating response. For the mapping to be single-valued, the Jacobian $J = \xi_x\zeta_z - \zeta_x\xi_z$ must not vanish, i.e. the topography is subcritical.

Using the boundary conditions, the problem reduces to the choice of a single function $f(q)$ with the bottom $z = H(x)$ determined by $-h = f(x + z) - f(x - z)$. Maas then constructs non-radiating solutions to the conversion problem. He recovers the wedge solution of Wunsch (1968) and then obtains mappings that, in the physical plane, resemble the usual hump shape as well as a smooth change in depth like a continental shelf. By construction, these solutions are bounded in space and do not radiate. For the shelf profile, Maas compares the theoretical results to a numerical solution of the underlying fluid equations: the numerical solution for his bathymetry is indeed non-radiating while a calculation with similar bathymetry radiates.

3. Future

The intuition that has been developed using theoretical models of internal conversion seems at odds with the non-conversion results of Maas. It is natural to inquire whether the approximations being used are at fault or whether this is a physical as opposed to mathematical result. Of the approximations used, the Boussinesq approximation and the use of Cartesian geometry do not raise any flags, and adding rotation would not change the underlying structure of the equations. The neglect of viscosity seems like an obvious candidate to explain the results, but the fact that the numerical simulations mentioned by Maas show very reduced conversion suggest this is not an artifact of the inviscid approximation. Finally, working in two dimensions is crucial for the transformation to work and for (2.2) to hold. It is well known that the wave equation has rather different properties in two and three dimensions, so one might wonder whether this is a two-dimensional result. Previous work in tidal conversion has not found significant differences between two- and three-dimensional situations. The search for time-harmonic solutions is undoubtedly a possible reason for the apparent paradox: a full initial-value calculation would show what wave field was actually produced over any given topographic feature. Theoretical studies have naturally considered the periodic problem for tides.

The related problem of internal wave attractors in an enclosed domain has been extensively studied by Maas and co-workers (see e.g. Lam & Maas 2008) although the problem goes back to John (1941). The peculiar nature of the Poincaré equation leads to the existence of wave attractors, which are seen in experiments (Maas *et al.* 1997; Hazewinkel *et al.* 2010). To quote the title of one such paper, these problems are linear yet nonlinear: the field equations are linear but the boundary conditions

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on finite topography lead to nonlinear problems. More work is needed to understand the relevance of non-converting topographies to the ocean. The fundamental question appears to be how one can physically pick the solution in the (ξ, ζ) variables to be non-propagating. What happens to perturbations of such a solution, either in terms of topography or when small amounts of propagating modes are added?

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