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Existence of valuations with smallest normalized volume

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ABSTRACT

Li introduced the normalized volume of a valuation due to its relation to \mathbf{K} -semistability. He conjectured that over a Kawamata log terminal (klt) singularity there exists a valuation with smallest normalized volume. We prove this conjecture and give an explicit example to show that such a valuation need not be divisorial.

1. Introduction

Fix a variety X of dimension n and $x \in X$ a closed point. Let $\text{Val}_{X,x}$ denote the set of real valuations on X with center equal to x . An element of $\text{Val}_{X,x}$ is an \mathbf{R} -valued valuation of the function field $K(X)$ that takes nonnegative values on $\mathcal{O}_{X,x} \subseteq K(X)$ and strictly positive values on the maximal ideal of $\mathcal{O}_{X,x}$. For examples, *divisorial valuations* centered at x form an important class inside $\text{Val}_{X,x}$. These valuations are determined by the order of vanishing along a prime divisor $E \subset Y$ where Y is normal and there is a proper birational morphism $f : Y \rightarrow X$ contracting E to x . We denote such a valuation by $\text{ord}_E \in \text{Val}_{X,x}$.

Li introduced the *normalized volume* function

$$\widehat{\text{vol}}_{X,x} : \text{Val}_{X,x} \longrightarrow \mathbf{R}_{>0} \cup \{+\infty\}$$

that sends a valuation v to its normalized volume, denoted $\widehat{\text{vol}}(v)$ [Li15]. To define the normalized volume, we recall the following. Given a valuation $v \in \text{Val}_{X,x}$, we have valuation ideals

$$\mathfrak{a}_m(v)_x := \{f \in \mathcal{O}_{X,x} \mid v(f) \geq m\} \subseteq \mathcal{O}_{X,x}$$

for all positive integers m . The volume of v is given by

$$\text{vol}(v) := \limsup_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X,x}/\mathfrak{a}_m(v)_x)}{m^n/n!}.$$

The normalized volume of v is

$$\widehat{\text{vol}}(v) := A_X(v)^n \text{vol}(v),$$

where $A_X(v)$ is the log discrepancy of v (see §2.5). When X has Kawamata log terminal (klt) singularities, $A_X(v) > 0$, and, thus, $\widehat{\text{vol}}(v) > 0$ for all $v \in \text{Val}_{X,x}$. Li conjectured the following.

CONJECTURE 1.1 [Li15]. If X has klt singularities at x , there exists a valuation $v^* \in \text{Val}_{X,x}$ that minimizes $\widehat{\text{vol}}_{X,x}$. Furthermore, such a minimizer v^* is unique (up to scaling) and quasimonomial.

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The above conjecture holds when $x \in X$ is a smooth point. As observed in [Li15], if x is a smooth point, then $\widehat{\text{vol}}_{X,x}$ is minimized at ord_x , the valuation that measures order of vanishing at x . Thus,

$$n^n = \widehat{\text{vol}}(\text{ord}_x) \leq \widehat{\text{vol}}(v)$$

for all $v \in \text{Val}_{X,x}$. The above observation follows from the work of de Fernex *et al.*

THEOREM 1.2 [FEM04]. *Let X be a variety of dimension n and $x \in X$ a smooth point. If $\mathfrak{a} \subseteq \mathcal{O}_{X,x}$ is an ideal that vanishes precisely at x , then*

$$n^n = \text{lct}(\mathfrak{m}_x)^n e(\mathfrak{m}) \leq \text{lct}(\mathfrak{a})^n e(\mathfrak{a})$$

where \mathfrak{m}_x is the maximal ideal of $\mathcal{O}_{X,x}$.

The authors of the previous theorem were motivated by their interest in singularity theory, as well as applications to birational rigidity [FEM03, FEM04, Fer13]. Li’s interest in volume minimization stems from questions concerning K-semistability of Fano varieties. Let V be a smooth Fano variety and $C(V, -K_V) := \text{Spec}(\bigoplus_{m \geq 0} H^0(V, -mK_V))$ the affine cone over V with cone point $0 \in C(V, -K_V)$. The blowup of $C(V, -K_V)$ at 0 has a unique exceptional divisor, which we denote by \tilde{V} .

THEOREM 1.3 [Li17, LL16, LX16]. *Let V be a smooth Fano variety. The following are equivalent.*

- (a) *The Fano variety V is K-semistable.*
- (b) *The function $\widehat{\text{vol}}_{C,0}$ is minimized at $\text{ord}_{\tilde{V}}$.*

Thus, if V is K-semistable, there exists a valuation centered at $0 \in C(V, -K_V)$ with smallest normalized volume. If V is not K-semistable, Conjecture 1.1 implies the existence of such a valuation. We prove the following.

MAIN THEOREM. *If $x \in X$ is a closed point on a klt variety, then there exists a valuation $v^* \in \text{Val}_{X,x}$ that is a minimizer of $\widehat{\text{vol}}_{X,x}$.*

In practice, it is rather difficult to pinpoint such a valuation v^* satisfying the conclusion of this theorem. For a good source of computable examples, we consider the toric setting.

THEOREM 1.4. *If X is a klt toric variety and $x \in X$ a torus invariant point, then*

$$\inf_{v \in \text{Val}_{X,x}^{\text{toric}}} \widehat{\text{vol}}(v) = \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v),$$

where $\text{Val}_{X,x}^{\text{toric}}$ denotes the set of toric valuations of X with center equal to x .

In § 8.3, we look at a concrete example, the cone over \mathbb{P}^2 blown up at a point. In this example, we find a quasimonomial valuation that minimizes the normalized volume function. Additionally, we show that there does not exist a divisorial volume minimizer. While this example is not new, our computation is unique in that it relies on purely algebraic methods. As explained in [LX16, Example 6.2], examples from Sasakian geometry with irregular Sasaki–Einstein metrics will provide similar examples. This example was looked at in [MS06, § 7].

Sketch of the proof of the main theorem

In order to prove the Main Theorem we first take a sequence of valuations $(v_i)_{i \in \mathbf{N}}$ such that

$$\lim_{i \rightarrow \infty} \widehat{\text{vol}}(v_i) = \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v).$$

Ideally, we would like to find a valuation v^* that is a limit point of the sequence $(v_i)_{i \in \mathbf{N}}$ and then argue that v^* is a minimizer of $\widehat{\text{vol}}_{X,x}$. To proceed with such an argument, one would likely need to show that $\widehat{\text{vol}}_{X,x}$ is a lower semicontinuous function on $\text{Val}_{X,x}$. It is unclear how to prove such a statement.¹

We proceed by shifting our focus. Instead of studying valuations $v \in \text{Val}_{X,x}$, we may consider ideals $\mathfrak{a} \subseteq \mathcal{O}_X$ that are \mathfrak{m}_x -primary. For an \mathfrak{m}_x -primary ideal, the *normalized multiplicity* of \mathfrak{a} is given by $\text{lct}(\mathfrak{a})^n e(\mathfrak{a})$, where

$$\text{lct}(\mathfrak{a}) := \min_{v \in \text{Val}_{X,x}} \frac{A_X(v)}{v(\mathfrak{a})} \quad \text{and} \quad e(\mathfrak{a}) := \lim_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_X/\mathfrak{a}^m)}{m^n/n!},$$

and the above invariants are the *log canonical threshold* and *Hilbert–Samuel multiplicity*.

We can also define similar invariants for graded sequences of \mathfrak{m}_x -primary ideals. Note that a graded sequence of ideals on X is a sequence of ideals $\mathfrak{a}_\bullet = \{\mathfrak{a}_m\}_{m \in \mathbf{N}}$ such that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$ for all $m, n \in \mathbf{N}$. The following proposition relates minimizing the normalized volume function to minimizing the normalized multiplicity.

PROPOSITION 4.3 [LIU16]. *If $x \in X$ is a closed point on a klt variety, then*

$$\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) = \inf_{\mathfrak{a}_\bullet \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) = \inf_{\mathfrak{a} \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a}). \tag{1.1}$$

While our goal is to find $v^* \in \text{Val}_{X,x}$ that achieves the first infimum of (1.1), we will instead find a graded sequence of \mathfrak{m}_x -primary ideals $\tilde{\mathfrak{a}}_\bullet$ that achieves the second infimum of the equation. After having constructed such a graded sequence $\tilde{\mathfrak{a}}_\bullet$, a valuation v^* that computes $\text{lct}(\tilde{\mathfrak{a}}_\bullet)$ (see § 2.9) will be a minimizer of $\widehat{\text{vol}}_{X,x}$.

To construct such a graded sequence, we will take our previously mentioned sequence of valuations $(v_i)_{i \in \mathbf{N}}$. This gives us a collection of graded sequences of ideals $(\mathfrak{a}_\bullet(v_i))_{i \in \mathbf{N}}$. Our goal will be to find a graded sequence $\tilde{\mathfrak{a}}_\bullet$ that is a ‘limit point’ of the previous collection.

We recall the work of de Fernex and Mustața [FM09], Kollár [Kol08], and de Fernex *et al.* [FEM10, FEM11] on *generic limits*. Given a collection of ideals $\{\mathfrak{a}_i\}_{i \in \mathbf{N}}$ where $\mathfrak{a}_i \subset k[x_1, \dots, x_r]$, there exists a field extension $k \subseteq K$ and an ideal $\tilde{\mathfrak{a}} \subset K[[x_1, \dots, x_r]]$ that encodes information on infinitely many members of $\{\mathfrak{a}_i\}_{i \in \mathbf{N}}$. We extend previous work on generic limits to find a ‘limit point’ of a collection of graded sequences of ideals.

Along the way, we will need a technical result on the rate of convergence of $(e(\mathfrak{a}_m(v))/m^n)_{m \in \mathbf{N}}$ for a valuation $v \in \text{Val}_{X,x}$. To perform this task, we extend the work of Ein *et al.* on approximation of valuation ideals [ELS03] and prove a technical, but also surprising, uniform convergence type result for the volume function.

¹ Li showed that if $\widehat{\text{vol}}_{X,x}$ is lower semicontinuous on $\text{Val}_{X,x}$, then there is a minimizer of $\widehat{\text{vol}}_{X,x}$ [Li15, Corollary 3.5]. Note that $\widehat{\text{vol}}_{X,x}(v) := A_X(v)^n \text{vol}(v)$ is a product of two functions. While A_X is lower semicontinuous on $\text{Val}_{X,x}$, vol fails to be lower semicontinuous in general [FJ04, Proposition 3.31].

PROPOSITION 3.7. *Let X be a klt variety of dimension n and $x \in X$ a closed point. For $\epsilon > 0$ and constants $B, E, r \in \mathbf{Z}_{>0}$, there exists $N = N(\epsilon, B, E, r) \in \mathbf{Z}_{>0}$ such that for every valuation $v \in \text{Val}_{X,x}$ with $\text{vol}(v) \leq B$, $A_X(v) \leq E$, and $v(\mathfrak{m}_x) \geq 1/r$, we have*

$$\text{vol}(v) \leq \frac{e(\mathfrak{a}_m(v))}{m^n} < \text{vol}(v) + \epsilon \quad \text{for all } m \geq N.$$

Structure of the paper

In §2 we provide preliminary information on valuations, graded sequences of ideals, and log canonical thresholds. Section 3 extends [ELS03] to klt varieties and gives a proof of the previous proposition on the volume of a valuation. Section 4 provides information on Li’s normalized volume function. Section 5 extends the theory of generic limits from ideals to graded sequences of ideals. Section 6 provides a proof of the Main Theorem. In §7, we explain that the arguments in this paper extend to the setting of log pairs. Lastly, §8 provides a proof of Theorem 1.4 and a computation of an example of a nondivisorial volume minimizer.

The paper also has two appendices that collect known statements that do not explicitly appear in the literature. Appendix A provides information on the behavior of the Hilbert–Samuel multiplicity and log canonical threshold in families. Appendix B provides a proof of the existence of valuations computing log canonical thresholds on klt varieties.

2. Preliminaries

Conventions

For the purpose of this paper, a variety is an irreducible, reduced, separated scheme of finite type over a field k . Furthermore, we will always assume that k is of characteristic zero, algebraically closed, and uncountable. We use the convention that $\mathbf{N} = \{1, 2, 3, \dots\}$.

2.1 Real valuations

Let X be a variety and $K(X)$ denote its function field. A *real valuation* of $K(X)$ is a group homomorphism

$$v : K(X)^\times \rightarrow \mathbf{R}$$

such that v is trivial on k (the base field) and $v(f + g) \geq \min\{v(f), v(g)\}$. We use the convention that $v(0) = +\infty$.

A valuation v gives rise to a valuation ring $\mathcal{O}_v \subset K(X)$, where $\mathcal{O}_v := \{f \in K(X) \mid v(f) \geq 0\}$. Note that if v is a valuation of $K(X)$ and $\lambda \in \mathbf{R}_{>0}$, scaling the outputs of v by λ gives a new valuation λv .

We say that v has a *center on X* if there exists a map $\pi : \text{Spec}(\mathcal{O}_v) \rightarrow X$ as below.

$$\begin{array}{ccc} \text{Spec}(K(X)) & \longrightarrow & X \\ \downarrow & \nearrow \pi & \downarrow \\ \text{Spec}(\mathcal{O}_v) & \longrightarrow & \text{Spec}(k) \end{array}$$

By [Har77, Theorem II.4.3], if such a map π exists, it is necessarily unique. Let ζ denote the unique closed point of $\text{Spec}(\mathcal{O}_v)$. If such a π exists, we define the *center* of v on X , denoted $c_X(v)$, to be $\pi(\zeta)$. We let Val_X (respectively, $\text{Val}_{X,x}$) denote the set of nontrivial real valuations of $K(X)$ with center on X (respectively, center equal to x).

Given a valuation $v \in \text{Val}_X$ and a nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}_X$, we may evaluate \mathfrak{a} along v by setting

$$v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a} \cdot \mathcal{O}_{X, c_X(v)}\}.$$

When X is affine,

$$v(\mathfrak{a}) = \min\{v(f) \mid f \in \mathfrak{a}(X)\}.$$

It follows from the above definition that if $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathcal{O}_X$ are nonzero ideals, then $v(\mathfrak{a}) \geq v(\mathfrak{b})$. In addition, $v(\mathfrak{a}) > 0$ if and only if $c_x(v) \in \text{Cosupp}(\mathfrak{a})$.²

We endow Val_X with the weakest topology such that, for every ideal \mathfrak{a} on X , the map $\text{Val}_X \rightarrow \mathbf{R} \cup \{+\infty\}$ defined by $v \mapsto v(\mathfrak{a})$ is continuous. For information on the space of valuations, see [JM12] and [BFFU15].

2.2 Divisorial valuations

Let $f : Y \rightarrow X$ be a proper birational morphism with Y normal and $E \subset Y$ a prime divisor. The discrete valuation ring $\mathcal{O}_{Y,E}$ gives rise to a valuation $\text{ord}_E \in \text{Val}_X$ that sends $a \in K(X)^\times$ to the order of vanishing of a along E . Note that $c_X(\text{ord}_E)$ is the generic point of $f(E)$.

We say $v \in \text{Val}_X$ is a *divisorial valuation* if there exists E as above and $\lambda \in \mathbf{R}_{>0}$ such that $v = \lambda \text{ord}_E$. We write $\text{DVal}_X \subset \text{Val}_X$ for the set of divisorial valuations on X .

2.3 Quasimonomial valuations

A *quasimonomial valuation* is a valuation that becomes monomial on some birational model over X . Specifically, let $f : Y \rightarrow X$ be a proper birational morphism and $p \in Y$ a closed point such that Y is regular at p . Given a regular system of parameters $y_1, \dots, y_n \in \mathcal{O}_{Y,p}$ at p and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{R}_{\geq 0}^n \setminus \{\mathbf{0}\}$, we define a valuation v_α as follows. For $r \in \mathcal{O}_{Y,p}$ we can write r in $\widehat{\mathcal{O}_{Y,p}}$ as $r = \sum_{\beta \in \mathbf{Z}_{\geq 0}^n} c_\beta y^\beta$, with $c_\beta \in \widehat{\mathcal{O}_{Y,p}}$ either zero or unit. We set

$$v_\alpha(r) = \min\{\langle \alpha, \beta \rangle \mid c_\beta \neq 0\}.$$

A quasimonomial valuation is a valuation that can be written in the above form. Note that in the above notation, if there exists $\lambda \in \mathbf{R}_{>0}$ such that $\lambda \cdot \alpha \in \mathbf{Z}_{\geq 0}^n$, then v_α is a divisorial valuation. Indeed, take a weighted blowup of Y at p to find the correct exceptional divisor.

2.4 The relative canonical divisor

Let $f : Y \rightarrow X$ be a proper birational morphism of normal varieties. If X is \mathbf{Q} -Gorenstein, that is K_X is \mathbf{Q} -Cartier, we define the *relative canonical divisor* of f to be

$$K_{Y/X} := K_Y - f^*(K_X),$$

where K_Y and K_X are chosen so that $f_*K_Y = K_X$. While K_Y and K_X are only defined up to linear equivalence, $K_{Y/X}$ is a well-defined divisor.

We say that a variety X is *klt* if X is normal, \mathbf{Q} -Gorenstein, and for any proper birational morphism of normal varieties $Y \rightarrow X$ the coefficients of $K_{Y/X}$ are > -1 . For further details on klt varieties and the relative canonical divisor, see [KM98, § 2.3].

²The *cosupport* of an ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ is defined as $\text{Cosupp}(\mathfrak{a}) := \text{Supp}(\mathcal{O}_X/\mathfrak{a})$.

2.5 The log discrepancy of a valuation

Let X be a normal \mathbf{Q} -Gorenstein variety. If $Y \rightarrow X$ is a proper birational morphism with Y normal, and $E \subset Y$ a prime divisor, then the *log discrepancy* of ord_E is defined by

$$A_X(\text{ord}_E) := 1 + (\text{coefficient of } E \text{ in } K_{Y/X}).$$

As explained in [BFFU15] (building upon [BFJ08, JM12]), the log discrepancy can be extended to a lower semicontinuous function $A_X : \text{Val}_X \rightarrow \mathbf{R} \cup \{+\infty\}$ that is homogeneous of order 1, (i.e. $A_X(\lambda v) = \lambda A_X(v)$ for $v \in \text{Val}_X$ and $\lambda \in \mathbf{R}_{>0}$). Additionally, X is klt if and only if $A_X(v) > 0$ for all $v \in \text{Val}_X$.

Note that [BFFU15] defines the log discrepancy function $A_X : \text{Val}_X \rightarrow \mathbf{R} \cup \{+\infty\}$ when X is normal and K_X is not necessarily assumed to be \mathbf{Q} -Cartier. We will not work in this level of generality.

2.6 Graded sequences of ideals

A graded sequence of ideals on a variety X is a sequence of ideals $\mathbf{a}_\bullet = \{\mathbf{a}_m\}_{m \in \mathbf{N}}$ such that $\mathbf{a}_m \cdot \mathbf{a}_n \subseteq \mathbf{a}_{m+n}$ for all $m, n \in \mathbf{N}$. By convention, we put $\mathbf{a}_0 = \mathcal{O}_X$. To simplify exposition, we always assume that \mathbf{a}_m is not equal to the zero ideal for all $m \in \mathbf{N}$.

We provide two examples of graded sequences of ideals.

(a) Let \mathfrak{b} be a nonzero ideal on X . We may define a graded sequence \mathbf{a}_\bullet by setting $\mathbf{a}_m := \mathfrak{b}^m$ for all $m \in \mathbf{N}$. This example is trivial.

(b) We fix $v \in \text{Val}_X$ and define $\mathbf{a}_\bullet(v) = \{\mathbf{a}_m(v)\}_{m \in \mathbf{N}}$ as follows. If $U \subseteq X$ is an open affine set such that $c_X(v) \in U$, then

$$\mathbf{a}_m(v)(U) := \{f \in \mathcal{O}_X(U) \mid v(f) \geq m\}.$$

If $c_x(v) \notin U$, then $\mathbf{a}_m(v)(U) := \mathcal{O}_X(U)$. If $c_X(v)$ is a closed point x , we have that each ideal $\mathbf{a}_m(v)$ is \mathfrak{m}_x -primary,³ where $\mathfrak{m}_x \subseteq \mathcal{O}_X$ denotes the ideal of functions vanishing at x .

Given $v \in \text{Val}_X$ and a graded sequence \mathbf{a}_\bullet , we may evaluate \mathbf{a}_\bullet along v by setting

$$v(\mathbf{a}_\bullet) := \inf_{m \in \mathbf{N}} \frac{v(\mathbf{a}_m)}{m} = \lim_{m \rightarrow \infty} \frac{v(\mathbf{a}_m)}{m}.$$

See [JM12, Lemma 2.3] for a proof of the previous equality.

2.7 Multiplicities

Let X be a variety of dimension n and $x \in X$ a closed point. Let $\mathfrak{m}_x \subseteq \mathcal{O}_X$ denote the ideal of functions vanishing at x . We recall that for an \mathfrak{m}_x -primary ideal \mathfrak{a} , the *Hilbert–Samuel multiplicity* of \mathfrak{a} is

$$e(\mathfrak{a}) := \lim_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X,x}/\mathfrak{a}^m)}{m^n/n!}.$$

If $\mathfrak{a} \subseteq \mathfrak{b} \subseteq \mathcal{O}_X$ are \mathfrak{m}_x -primary ideals on X , then $e(\mathfrak{a}) \geq e(\mathfrak{b})$. In addition, $e(\mathfrak{a}) = e(\bar{\mathfrak{a}})$ where $\bar{\mathfrak{a}}$ denotes the integral closure of \mathfrak{a} .

We recall the valuative definition of the *integral closure* of an ideal \mathfrak{a} on a normal variety X [Laz04, Example 9.6.8]. Let $U \subset X$ affine open subset. We have

$$\bar{\mathfrak{a}}(U) := \{f \in \mathcal{O}_X(U) \mid w(f) \geq w(\mathfrak{a}) \text{ for all } w \in \text{Val}_U \text{ divisorial}\}.$$

³ This is equivalent to saying that each ideal $\mathbf{a}_m(v)$ vanishes only at x .

2.8 Volumes

Let \mathfrak{a}_\bullet be a graded sequence of ideals with the property that each \mathfrak{a}_m is \mathfrak{m}_x -primary. The *volume* of \mathfrak{a}_\bullet is defined as

$$\text{vol}(\mathfrak{a}_\bullet) := \limsup_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X,x}/\mathfrak{a}_m)}{m^n/n!}.$$

A similar invariant is the *multiplicity* of \mathfrak{a}_\bullet , which is defined as

$$e(\mathfrak{a}_\bullet) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n}.$$

In various degrees of generality, it has been proven that

$$e(\mathfrak{a}_\bullet) = \text{vol}(\mathfrak{a}_\bullet)$$

[ELS03, Corollary C], [Mus02, Theorem 1.7], [LM09, Theorem 3.8], [Cut13, Theorem 6.5]. In our setting, the above equality will always hold. In addition, by [Cut13, Theorem 1.1], we also have that

$$\text{vol}(\mathfrak{a}_\bullet) := \lim_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X,x}/\mathfrak{a}_m)}{m^n/n!}.$$

For a valuation $v \in \text{Val}_{X,x}$, the *volume* of v is given by

$$\text{vol}(v) := e(\mathfrak{a}_\bullet(v)).$$

Note that if $\lambda \in \mathbf{R}_{>0}$, then $\text{vol}(\lambda v) = \text{vol}(v)/\lambda^n$.

2.9 Log canonical thresholds

The *log canonical threshold* is an invariant of singularities that has received considerable interest in the field of birational geometry [Kol97, § 8]. For a nonzero ideal \mathfrak{a} on a klt variety X , the log canonical threshold of \mathfrak{a} is given by

$$\text{lct}(\mathfrak{a}) := \inf_{v \in \text{DVal}_X} \frac{A_X(v)}{v(\mathfrak{a})}. \tag{2.1}$$

By [JM12, Lemma 6.7] in the case when X is smooth and a similar argument in the klt case,

$$\text{lct}(\mathfrak{a}) = \inf_{v \in \text{Val}_X} \frac{A_X(v)}{v(\mathfrak{a})}.$$

In the previous expression, we are using the convention that if $v(\mathfrak{a}) = 0$ or $A(v) < +\infty$, then $A(v)/v(\mathfrak{a}) = +\infty$. Thus, $\text{lct}(\mathcal{O}_X) = +\infty$. We say that a valuation v^* *computes* $\text{lct}(\mathfrak{a})$ if $A_X(v) < +\infty$ and $\text{lct}(\mathfrak{a}) = A(v^*)/v^*(\mathfrak{a})$.

Note the following elementary properties of this invariant. If $m \in \mathbf{Z}_{>0}$, then

$$\text{lct}(\mathfrak{a}^m) = \text{lct}(\mathfrak{a})/m.$$

If $\mathfrak{a} \subseteq \mathfrak{b}$, then

$$\text{lct}(\mathfrak{a}) \leq \text{lct}(\mathfrak{b}).$$

The log canonical threshold may be understood in terms of a log resolution of \mathfrak{a} . Recall that $\mu : Y \rightarrow X$ is a *log resolution* of \mathfrak{a} if the following hold:

- (a) μ is a projective birational morphism;
- (b) Y is smooth and $\text{Exc}(\mu)$ is pure codimension 1;
- (c) $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$ for an effective divisor D on Y ; and
- (d) $D_{\text{red}} + \text{Exc}(\mu)$ has simple normal crossing.

In this case, we have

$$\text{lct}(\mathfrak{a}) = \min_{i=1, \dots, r} \frac{A_X(\text{ord}_{E_i})}{\text{ord}_{E_i}(\mathfrak{a})},$$

where $D = \sum_{i=1}^r a_i E_i$. (Note that $\text{ord}_{E_i}(\mathfrak{a}) = a_i$.) Thus, there always exists a divisorial valuation that computes $\text{lct}(\mathfrak{a})$.

For a graded sequence of ideals \mathfrak{a}_\bullet on X , the log canonical threshold of \mathfrak{a}_\bullet is given by

$$\text{lct}(\mathfrak{a}_\bullet) := \lim_{m \rightarrow \infty} m \cdot \text{lct}(\mathfrak{a}_m) = \sup_m m \cdot \text{lct}(\mathfrak{a}_m).$$

By [JM12, Corollary 6.9] when X is smooth or as a consequence of [BFFU15, Theorem 1.2] when X is klt, we have

$$\text{lct}(\mathfrak{a}_\bullet) = \inf_{v \in \text{Val}_X} \frac{A_X(v)}{v(\mathfrak{a}_\bullet)}.$$

As before, we are using the convention that if either $v(\mathfrak{a}_\bullet) = 0$ or $A_X(v) = +\infty$, then $A(v)/v(\mathfrak{a}_\bullet) = +\infty$. We say $v^* \in \text{Val}_X$ computes $\text{lct}(\mathfrak{a}_\bullet)$ if $A(v) < +\infty$ and $\text{lct}(\mathfrak{a}_\bullet) = A_X(v^*)/v^*(\mathfrak{a}_\bullet)$. Such valuations v^* always exist (see Appendix B). When X is smooth, this is precisely [JM12, Theorem A].

3. Approximation of valuation ideals

In this section we extend the arguments of [ELS03] to approximate valuation ideals on singular varieties. We will use this approximation to determine the speed of convergence of $(e(\mathfrak{a}_m(v))/m^n)_{m \in \mathbb{N}}$ for a fixed valuation v . The main technical tool is the asymptotic multiplier ideal of a graded family of ideals. For an excellent reference on multiplier ideals, see [Laz04, ch. 9].

3.1 Multiplier ideals

Let X be a normal \mathbb{Q} -Gorenstein variety. Fix a nonzero ideal $\mathfrak{a} \subseteq \mathcal{O}_X$ and $f : Y \rightarrow X$ a log resolution of \mathfrak{a} such that $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-D)$. For a rational number $c > 0$, the *multiplier ideal* $\mathcal{J}(X, c \cdot \mathfrak{a})$ is defined by

$$\mathcal{J}(X, c \cdot \mathfrak{a}) := f_* \mathcal{O}_Y([\mathcal{K}_{Y/X} - cD]) \subseteq \mathcal{O}_X.$$

Note that if c is an integer, then $\mathcal{J}(X, c \cdot \mathfrak{a}) = \mathcal{J}(X, \mathfrak{a}^c)$. It is a basic fact that $\mathcal{J}(X, c \cdot \mathfrak{a})$ is independent of the choice of f .

Alternatively, the multiplier ideal can be understood valuably. If X is an affine variety, then [BFFU15, Theorem 1.2] implies

$$\mathcal{J}(X, c \cdot \mathfrak{a})(X) = \{f \in \mathcal{O}_X(X) \mid v(f) > cv(\mathfrak{a}) - A_X(v) \text{ for all } v \in \text{Val}_X\}.$$

When X is not necessarily affine, the above criterion allows us to understand the multiplier ideal locally.

It is important to note the relationship between the log canonical threshold and the multiplier ideal. If X is klt, then

$$\text{lct}(\mathfrak{a}) = \sup\{c \mid \mathcal{J}(X, c \cdot \mathfrak{a}) = \mathcal{O}_X\}.$$

The following lemma provides basic properties of multiplier ideals. The proof is left to the reader. See [Laz04, Proposition 9.2.32] for the case when X is smooth.

LEMMA 3.1. *Let $\mathfrak{a}, \mathfrak{b}$ be nonzero ideals on X .*

(a) *If X is klt, then*

$$\mathfrak{a} \subseteq \mathcal{J}(X, \mathfrak{a}).$$

(b) *If $\mathfrak{a} \subseteq \mathfrak{b}$ and $c > 0$ is a rational number, then*

$$\mathcal{J}(X, c \cdot \mathfrak{a}) \subseteq \mathcal{J}(X, c \cdot \mathfrak{b}).$$

(c) *For rational numbers $c \geq d > 0$, we have*

$$\mathcal{J}(X, c \cdot \mathfrak{a}) \subseteq \mathcal{J}(X, d \cdot \mathfrak{a}).$$

Multiplier ideals satisfy the following ‘subadditivity property’. The property was first observed and proved by Demailly *et al.* in the smooth case [DEL00]. The statement was extended to the singular case in [Tak06, Theorem 2.3] and [Eis11, Theorem 7.3.4].

THEOREM 3.2 (Subadditivity). *If $\mathfrak{a}, \mathfrak{b}$ are nonzero ideals on X and $c > 0$ a rational number, then*

$$\text{Jac}_X \cdot \mathcal{J}(X, c \cdot (\mathfrak{a} \cdot \mathfrak{b})) \subseteq \mathcal{J}(X, c \cdot \mathfrak{a}) \cdot \mathcal{J}(X, c \cdot \mathfrak{b}),$$

where Jac_X denotes the Jacobian ideal of X .

We recall that the *Jacobian* ideal of a variety X is $\text{Jac}_X := \text{Fitt}_n(X)$, where n is the dimension of X and Fitt_n denotes the n th fitting ideal as in [Eis95, 20.2]. The singular locus of X is equal to $\text{Cosupp}(\text{Jac}_X)$.

3.2 Asymptotic multiplier ideals

Let \mathfrak{a}_\bullet be a graded sequence of ideals on a normal \mathbf{Q} -Gorenstein variety X and $c > 0$ a rational number. We recall the definition of the asymptotic multiplier ideal $\mathcal{J}(c \cdot \mathfrak{a}_\bullet)$. By Lemma 3.1, we have that

$$\mathcal{J}\left(X, \frac{1}{p} \cdot \mathfrak{a}_{mp}\right) \subseteq \mathcal{J}\left(X, \frac{1}{pq} \cdot \mathfrak{a}_{pqm}\right)$$

for all positive integers p, q . From the above inclusion and Noetherianity, we conclude

$$\left\{ \mathcal{J}\left(X, \frac{1}{p} \cdot \mathfrak{a}_{pm}\right) \right\}_{p \in \mathbf{N}}$$

has a unique maximal element. The m th asymptotic multiplier ideal $\mathcal{J}(X, m \cdot \mathfrak{a}_\bullet)$ is defined to be this element. Like the standard multiplier ideal, the asymptotic multiplier ideal can also be understood valuably.

PROPOSITION 3.3 [BFFU15, Theorem 1.2]. *If X is affine, \mathfrak{a}_\bullet is a graded sequence of ideals on X , and $c > 0$ a rational number, then*

$$\mathcal{J}(X, c \cdot \mathfrak{a}_\bullet) = \{f \in \mathcal{O}_X(X) \mid v(f) > cv(\mathfrak{a}_\bullet) - A_X(v) \text{ for all } v \in \text{Val}_X\}.$$

The asymptotic multiplier ideals satisfy the following property. This property will allow us to approximate valuation ideals.

PROPOSITION 3.4. *If \mathbf{a}_\bullet is a graded sequence of ideals on a klt variety X and $m, \ell \in \mathbf{N}$, then*

$$(\text{Jac}_X)^{\ell-1} \mathbf{a}_m^\ell \subseteq (\text{Jac}_X)^{\ell-1} \mathbf{a}_{m\ell} \subseteq \mathcal{J}(m \cdot \mathbf{a}_\bullet)^\ell.$$

Proof. The proof is the same as the proof of [ELS03, Theorem 1.7] and is a consequence of Lemma 3.1(a) and Theorem 3.2. □

3.3 The case of valuation ideals

For a valuation $v \in \text{Val}_X$, we examine the asymptotic multiplier ideals of $\mathbf{a}_\bullet(v)$. We first prove the following elementary lemma.

LEMMA 3.5. *If v is a valuation on a variety X , then $v(\mathbf{a}_\bullet(v)) = 1$.*

Proof. Note that $v(\mathbf{a}_m(v)) \geq m$, since $\mathbf{a}_m(v)$ is the ideal of functions vanishing to at least order m along v . Next, set $\alpha := v(\mathbf{a}_1(v))$. We have $\mathbf{a}_1(v)^{\lceil m/\alpha \rceil} \subseteq \mathbf{a}_m(v)$, since $v(\mathbf{a}_1(v)^{\lceil m/\alpha \rceil}) = \alpha \lceil m/\alpha \rceil \geq m$. Thus,

$$v(\mathbf{a}_m(v)) \leq v(\mathbf{a}_1(v)^{\lceil m/\alpha \rceil}) = \alpha \lceil m/\alpha \rceil.$$

The previous two bounds combine to show

$$1 \leq \frac{v(\mathbf{a}_m(v))}{m} \leq \frac{\alpha \cdot \lceil m/\alpha \rceil}{m},$$

and the result follows. □

The following results allows us to approximate valuation ideals. In the case when X is smooth and v is an Abhyankhar valuation, the theorem below is a slight strengthening of [ELS03, Theorem A].

THEOREM 3.6. *If X is a klt variety and $v \in \text{Val}_X$ satisfies $A_X(v) < +\infty$, then*

$$(\text{Jac}_X)^{\ell-1} \cdot \mathbf{a}_m^\ell \subseteq (\text{Jac}_X)^{\ell-1} \cdot \mathbf{a}_{m\ell} \subseteq \mathbf{a}_{m-e}^\ell$$

for every $m \geq e$ and $\ell \in \mathbf{Z}_{>0}$, where $\mathbf{a}_\bullet := \mathbf{a}_\bullet(v)$ and $e := \lceil A_X(v) \rceil$.

Proof. By Proposition 3.4, we have that

$$(\text{Jac}_X)^{\ell-1} \cdot \mathbf{a}_m^\ell \subseteq (\text{Jac}_X)^{\ell-1} \cdot \mathbf{a}_{m\ell} \subseteq \mathcal{J}(X, m \cdot \mathbf{a}_\bullet)^\ell.$$

Applying Proposition 3.3 and Lemma 3.5 to \mathbf{a}_\bullet gives that

$$\mathcal{J}(X, m \cdot \mathbf{a}_\bullet) \subseteq \mathbf{a}_{m-e},$$

and the result follows. □

3.4 Uniform approximation of volumes

Given a valuation $v \in \text{Val}_X$ centered at a closed point on a n -dimensional variety X , we recall

$$\text{vol}(v) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m(v))}{m^n},$$

where n is the dimension of X . The following proposition provides a uniform rate of convergence for the terms in the above limit.

PROPOSITION 3.7. *Let X be a klt n -dimensional variety and $x \in X$ a closed point. For $\epsilon > 0$ and constants $B, E, r \in \mathbf{Z}_{>0}$, there exists $N = N(\epsilon, B, E, r)$ such that for every valuation $v \in \text{Val}_{X,x}$ with $\text{vol}(v) \leq B$, $A_X(v) \leq E$, and $v(\mathfrak{m}_x) \geq 1/r$, we have*

$$\text{vol}(v) \leq \frac{e(\mathfrak{a}_m(v))}{m^n} < \text{vol}(v) + \epsilon \quad \text{for all } m \geq N.$$

Remark 3.8. In an earlier version of this paper, we proved the following statement with the additional assumption that $x \in X$ is an isolated singularity. We are grateful to Mircea Mustața for noticing that a modification of the original proof allows us to prove the more general statement.

Proof of Proposition 3.7. For any valuation $v \in \text{Val}_{X,x}$, the first inequality is well known. Indeed, the inclusion $\mathfrak{a}_m(v)^p \subseteq \mathfrak{a}_{mp}(v)$ for $m, p \in \mathbf{N}$ implies that

$$\frac{e(\mathfrak{a}_{mp}(v))}{(mp)^n} \leq \frac{e(\mathfrak{a}_m(v))}{(m)^n}.$$

Fixing m and sending $p \rightarrow \infty$ gives

$$\text{vol}(v) \leq \frac{e(\mathfrak{a}_m(v))}{(m)^n}.$$

Next, fix $v \in \text{Val}_{X,x}$ satisfying the hypotheses in the statement of Proposition 3.7. We have

$$(\text{Jac}_X)^{\ell-1} \mathfrak{a}_{m\ell}(v) \subseteq (\mathfrak{a}_{m-e}(v))^\ell \subseteq (\mathfrak{a}_{m-E}(v))^\ell,$$

where $e = \lceil A_X(v) \rceil$. The first inclusion is the statement in Theorem 3.6, and the second follows from the assumption that $e \leq E$. After replacing m by $m + E$, we obtain

$$(\text{Jac}_X)^\ell \cdot \mathfrak{a}_{(m+E)\ell}(v) \subseteq (\text{Jac}_X)^{\ell-1} \cdot \mathfrak{a}_{(m+E)\ell}(v) \subseteq \mathfrak{a}_m(v)^\ell. \tag{3.1}$$

On the other hand, the assumption that $v(\mathfrak{m}_x) \geq 1/r$ implies that

$$\mathfrak{m}_x^{mr} \subseteq \mathfrak{a}_m(v). \tag{3.2}$$

It follows from (3.1) and (3.2) and the valuative criterion for integral closure (§ 2.7) that

$$(\text{Jac}_X + \mathfrak{m}_x^{mr})^\ell \mathfrak{a}_{(m+E)\ell}(v) \subseteq \overline{\mathfrak{a}_m(v)^\ell}. \tag{3.3}$$

Indeed, let $w \in \text{Val}_X$ be a divisorial valuation and f and g local sections of Jac_X^i and \mathfrak{m}_x^{mrj} , respectively, with $i + j = \ell$. We have $\ell \cdot w(f) + i \cdot w(\mathfrak{a}_{(m+E)\ell}(v)) \geq i \cdot w(\mathfrak{a}_m(v)^\ell)$ and $w(g) \geq j \cdot w(\mathfrak{a}_m(v))$ by the two inclusions. Thus,

$$\begin{aligned} w(fg) &= w(f) + w(g) \geq \frac{i}{\ell}(w(\mathfrak{a}_m(v)^\ell) - w(\mathfrak{a}_{(m+E)\ell}(v))) + \frac{j}{\ell}w(\mathfrak{a}_m(v)^\ell) \\ &= w(\mathfrak{a}_m(v)^\ell) - w(\mathfrak{a}_{(m+E)\ell}(v)). \end{aligned}$$

From Inclusion (3.3) and Teissier’s Minkowski inequality [Tei77], we see

$$e(\mathfrak{a}_m(v)^\ell)^{1/n} \leq e((\text{Jac}_X + \mathfrak{m}_x^{mr})^\ell)^{1/n} + e(\mathfrak{a}_{(m+E)\ell}(v))^{1/n}.$$

Next, note that if \mathfrak{a} is an \mathfrak{m}_x -primary ideal, then $e(\mathfrak{a}^m) = m^n e(\mathfrak{a})$. Applying this property and dividing by $m \cdot \ell$, gives that

$$\frac{e(\mathfrak{a}_m(v))^{1/n}}{m} \leq \frac{e(\text{Jac}_X + \mathfrak{m}_x^{mr})^{1/n}}{m} + \frac{m + E}{m} \cdot \frac{e(\mathfrak{a}_{(m+E)\ell}(v))^{1/n}}{(m + E)\ell}.$$

After letting $\ell \rightarrow \infty$, we obtain

$$\frac{e(\mathfrak{a}_m(v))^{1/n}}{m} \leq \frac{e(\text{Jac}_X + \mathfrak{m}_x^{mr})^{1/n}}{m} + \frac{m + E}{m} \text{vol}(v)^{1/n}.$$

Since $\text{vol}(v)^{1/n} \leq B^{1/n}$, the assertion will follow if we show that

$$\lim_{m \rightarrow \infty} \frac{e(\text{Jac}_X + \mathfrak{m}_x^{mr})^{1/n}}{m} = 0.$$

Choose $h \in \text{Jac}_X \cdot \mathcal{O}_{X,x}$ that is nonzero and set $R := \mathcal{O}_{X,x}/(h)$ and $\tilde{\mathfrak{m}}_x := \mathfrak{m}_x \cdot R$. We have

$$\lim_{m \rightarrow \infty} \frac{e(\text{Jac}_X + \mathfrak{m}_x^{mr})^{1/n}}{m} = \lim_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X,x}/(\text{Jac}_X + \mathfrak{m}_x^{mr}))^{1/n}}{m} \leq \lim_{m \rightarrow \infty} \frac{\text{length}(R/\tilde{\mathfrak{m}}_x^{mr})^{1/n}}{m}.$$

The last limit is 0, since

$$\lim_{m \rightarrow \infty} \frac{\text{length}(R/\tilde{\mathfrak{m}}_x^{mr})}{m^{n-1}/(n-1)!} = e(\tilde{\mathfrak{m}}_x^r) < \infty. \quad \square$$

4. Normalized volumes

For this section, we fix X an n -dimensional klt variety and $x \in X$ a closed point. As introduced in [Li15], the *normalized volume* of a valuation $v \in \text{Val}_{X,x}$ is defined as

$$\widehat{\text{vol}}(v) := A_X(v)^n \text{vol}(v).$$

In the case when $A_X(v) = +\infty$ and $\text{vol}(v) = 0$, we set $\widehat{\text{vol}}(v) := +\infty$. The word ‘normalized’ refers to the property that $\widehat{\text{vol}}(\lambda v) = \widehat{\text{vol}}(v)$ for $\lambda \in \mathbf{R}_{>0}$.

Given a graded sequence \mathfrak{a}_\bullet of \mathfrak{m}_x -primary ideals on X , we define a similar invariant. We refer to

$$\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet)$$

as the *normalized multiplicity* of \mathfrak{a}_\bullet . Similar to the normalized volume, when $\text{lct}(\mathfrak{a}_\bullet) = +\infty$ and $e(\mathfrak{a}_\bullet) = 0$, we set $\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) := +\infty$. The above invariant was looked at in [FEM04] and [Mus02].

The following lemma provides elementary information on the normalized multiplicity. The proof is left to the reader.

LEMMA 4.1. *Let \mathfrak{a} be an \mathfrak{m}_x -primary ideal and \mathfrak{a}_\bullet a graded sequence of \mathfrak{m}_x -primary ideals on X .*

(a) *If $\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) < +\infty$, then*

$$\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) = \lim_{m \rightarrow \infty} \text{lct}(\mathfrak{a}_m)^n e(\mathfrak{a}_m).$$

(b) If \mathfrak{b}_\bullet is a graded sequence given by $\mathfrak{b}_m := \mathfrak{a}^m$, then

$$\text{lct}(\mathfrak{a})^n e(\mathfrak{a}) = \text{lct}(\mathfrak{b}_\bullet)^n e(\mathfrak{b}_\bullet).$$

(c) If \mathfrak{a}_{N_\bullet} is the graded sequence whose m th term is \mathfrak{a}_{N-m} , then

$$\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) = \text{lct}(\mathfrak{a}_{N_\bullet})^n e(\mathfrak{a}_{N_\bullet}).$$

Remark 4.2. Fix $\delta > 0$. If \mathfrak{a}_\bullet a graded sequence of \mathfrak{m}_x -primary ideals such that $\mathfrak{a}_m \subseteq \mathfrak{m}_x^{\lfloor \delta m \rfloor}$ for all m , then

$$\text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) < +\infty.$$

It is always the case that $e(\mathfrak{a}_\bullet) < +\infty$, since $e(\mathfrak{a}_\bullet) \leq e(\mathfrak{a}_1)$. The assumption that $\mathfrak{a}_m \subseteq \mathfrak{m}_x^{\lfloor \delta m \rfloor}$ gives that $\text{lct}(\mathfrak{a}_\bullet) \leq \text{lct}(\mathfrak{m}_x)/\delta$, the latter of which is $< +\infty$.

The following proposition relates the normalized volume, an invariant of valuations, to the normalized multiplicity, an invariant of graded sequences of ideals.

PROPOSITION 4.3 [Liu16, Theorem 27]. *The following equality holds:*

$$\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) = \inf_{\mathfrak{a}_\bullet \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) = \inf_{\mathfrak{a} \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a}).$$

The previous statement first appeared in [Liu16]. In the case when $x \in X$ is a smooth point, it was explained in [Li15, Example 3.7]. We provide Liu’s proof, since the argument will be useful to us. The proposition is a consequence of the following lemma.

LEMMA 4.4 [Liu16]. *The following statements hold.*

(a) If \mathfrak{a}_\bullet is a graded sequence of \mathfrak{m}_x -primary ideals and $v \in \text{Val}_{X,x}$ computes $\text{lct}(\mathfrak{a}_\bullet)$ (i.e. $A(v)/v(\mathfrak{a}_\bullet) = \text{lct}(\mathfrak{a}_\bullet)$), then

$$\widehat{\text{vol}}(v) \leq \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet).$$

(b) If $v \in \text{Val}_{X,x}$, then

$$\text{lct}(\mathfrak{a}_\bullet(v))^n e(\mathfrak{a}_\bullet(v)) \leq \widehat{\text{vol}}(v).$$

Proof. To prove statement (a), we first rescale v so that $v(\mathfrak{a}_\bullet) = 1$. Thus, $A_X(v) = A_X(v)/v(\mathfrak{a}_\bullet) = \text{lct}(\mathfrak{a}_\bullet)$. Since

$$1 = v(\mathfrak{a}_\bullet) := \inf_{m \geq 0} \frac{v(\mathfrak{a}_m)}{m},$$

we see $v(\mathfrak{a}_m) \geq m$ and, thus, $\mathfrak{a}_m \subseteq \mathfrak{a}_m(v)$ for all m . This implies $e(\mathfrak{a}_\bullet(v)) \leq e(\mathfrak{a}_\bullet)$, and the desired inequality follows.

In order to show statement (b), we note

$$\text{lct}(\mathfrak{a}_\bullet(v)) := \min_w \frac{A_X(w)}{w(\mathfrak{a}_\bullet(v))} \leq \frac{A_X(v)}{v(\mathfrak{a}_\bullet(v))} = A_X(v),$$

where the last equality follows from Lemma 3.5. Thus,

$$\text{lct}(\mathfrak{a}_\bullet(v))^n e(\mathfrak{a}_\bullet(v)) \leq A_X(v)^n e(\mathfrak{a}_\bullet(v)) = \widehat{\text{vol}}(v). \quad \square$$

Proof of Proposition 4.3. The first equality follows immediately from the previous proposition and the fact that given a graded sequence \mathfrak{a}_\bullet there exists a valuation $v^* \in \text{Val}_X$ that computes $\text{lct}(\mathfrak{a}_\bullet)$ (see Theorem B.1). The last equality follows from Lemma 4.1. \square

Remark 4.5. Above, we provided a dictionary between the normalized volume of a valuation and the normalized multiplicity of a graded sequence of ideals. The normalized multiplicity also extends to a functional on the set of (formal) plurisubharmonic functions in the sense of [BFJ08]. In a slightly different setting, similar functionals, arising from non-Archimedean analogues of functionals in Kähler geometry, were explored in [BHJ16].

4.1 Normalized volume minimizers

PROPOSITION 4.6. *If there exists a graded sequence of \mathfrak{m}_x -primary ideals $\tilde{\mathfrak{a}}_\bullet$ such that*

$$\text{lct}(\tilde{\mathfrak{a}}_\bullet)^n e(\tilde{\mathfrak{a}}_\bullet) = \inf_{\mathfrak{a}_\bullet \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet),$$

then there exists $v^ \in \text{Val}_{X,x}$ that is a minimizer of $\widehat{\text{vol}}_{X,x}$. Furthermore, if there exists an \mathfrak{m}_x -primary ideal $\tilde{\mathfrak{a}}$ such that*

$$\text{lct}(\tilde{\mathfrak{a}})^n e(\tilde{\mathfrak{a}}) = \inf_{\mathfrak{a} \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a}),$$

then we may choose v^ to be divisorial.*

Proof. Assume there exists such a graded sequence $\tilde{\mathfrak{a}}_\bullet$. By Theorem B.1, we may choose a valuation v^* that computes $\text{lct}(\tilde{\mathfrak{a}}_\bullet)$. By Lemma 4.4,

$$\widehat{\text{vol}}(v^*) \leq \text{lct}(\tilde{\mathfrak{a}}_\bullet)^n e(\tilde{\mathfrak{a}}_\bullet).$$

By Proposition 4.3, the result follows.

When there exists such an ideal $\tilde{\mathfrak{a}}$, the same argument shows that if v^* computes $\text{lct}(\tilde{\mathfrak{a}})$, then v^* is our desired valuation. Furthermore, we may choose v^* divisorial. \square

LEMMA 4.7. *If v^* is a minimizer of $\widehat{\text{vol}}_{X,x}$, then*

$$A_X(v^*) \leq \frac{A_X(w)}{w(\mathfrak{a}_\bullet(v^*))}$$

for all $w \in \text{Val}_{X,x}$. Furthermore, equality holds if and only if $w = \lambda v^$ for some $\lambda \in \mathbf{R}_{>0}$.*

Remark 4.8. The above technical statement can be restated as follows. If v^* is a normalized volume minimizer, then v^* computes $\text{lct}(\mathfrak{a}_\bullet(v^*))$ and v^* is the only valuation (up to scaling) that computes $\text{lct}(\mathfrak{a}_\bullet(v^*))$.

A conjecture of Jonsson and Mustařă states that valuations computing log canonical thresholds of graded sequences on smooth varieties are always quasimonomial [JM12, Conjecture B]. Their conjecture in the klt case implies [Li15, Conjecture 6.1.3], which says that normalized volume minimizers are quasimonomial.

Proof. We fix $w \in \text{Val}_{X,x}$ and rescale w so that $w(\mathfrak{a}_\bullet(v^*)) = 1$. Thus, we are reduced to showing that $A_X(v^*) \leq A_X(w)$ and equality holds if and only if $w = v^*$.

By definition, we have

$$1 = w(\mathfrak{a}_\bullet(v^*)) := \inf_{m \geq 0} \frac{w(\mathfrak{a}_m(v^*))}{m},$$

and, thus, $w(\mathfrak{a}_m(v^*)) \geq m$. The latter implies that $\mathfrak{a}_m(v^*) \subseteq \mathfrak{a}_m(w)$, so

$$\text{vol}(w) \leq \text{vol}(v^*).$$

If $A_X(w) < A_X(v^*)$, then

$$A_X(w)^n \text{vol}(w) < A_X(v^*)^n \text{vol}(v^*)$$

and this would contradict our assumption on v^* . Furthermore, if $A(v^*) = A(w)$, then we must have that $\text{vol}(v^*) = \text{vol}(w)$. Since $v^* \leq w$ and $\text{vol}(v^*) = \text{vol}(w)$, then $v^* = w$ by [LX16, Proposition 2.12]. \square

PROPOSITION 4.9. *Let $v^* \in \text{Val}_{X,x}$ be a minimizer of $\widehat{\text{vol}}_{X,x}$. If $v^* = \text{ord}_E$, where E is a prime divisor on a normal variety which is proper and birational over X , then we have the following.*

- (a) *The graded \mathcal{O}_X -algebra $\mathcal{O}_X \oplus \mathfrak{a}_1(v^*) \oplus \mathfrak{a}_2(v^*) \oplus \dots$ is finitely generated.*
- (b) *The valuation v^* corresponds to a Kollar component (see [LX16]).*
- (c) *The number $\widehat{\text{vol}}(v^*)$ is rational.*

The previous proposition was independently observed in [LX16, Theorem 1.5]. In fact, prior to our contribution, the original draft of [LX16] proved that if (a) holds then (b) holds. Our argument is different from that of [LX16].

Proof. By Lemma 4.7, it follows that $\text{lct}(\mathfrak{a}_\bullet(v^*)) = A(v^*)$. The finite generation of the desired \mathcal{O}_X -algebra is a consequence of [Blu16, Theorem 1.4.1]. Additionally, the second sentence of Lemma 4.7 allows us to apply [Blu16, Proposition 4.4]. Thus, v^* corresponds to a Kollar component.

To show that $\widehat{\text{vol}}(v^*)$ is rational, we note that the finite generation statement of (a) implies there exists $N > 0$ so that $\mathfrak{a}_{mN}(v^*) = (\mathfrak{a}_N(v^*))^m$ for all $m \in \mathbf{N}$ [Gro61, Lemma II.2.1.6.v]. By Lemma 4.4,

$$\text{lct}(\mathfrak{a}_\bullet(v^*))^n e(\mathfrak{a}_\bullet(v^*)) \leq \widehat{\text{vol}}(v^*).$$

Lemma 4.1 implies

$$\text{lct}(\mathfrak{a}_\bullet(v^*))^n e(\mathfrak{a}_\bullet(v^*)) = \text{lct}(\mathfrak{a}_N)^n e(\mathfrak{a}_N),$$

and the latter is a rational number. \square

5. Limit points of collections of graded sequences of ideals

In this section we construct a space that parameterizes graded sequences of ideals on a fixed variety X . We use this parameter space to find ‘limit points’ of a collection of graded sequences of ideals on X . The ideas behind this construction arise from the work of de Fernex and Mustața [FM09], Kollár [Kol08], and de Fernex *et al.* [FEM10, FEM11].

Before explaining our construction, we set our notation. We fix an affine variety $X = \text{Spec } A$, where $A = k[x_1, \dots, x_r]/\mathfrak{p}$. Let φ denote the map

$$R = k[x_1, \dots, x_r] \xrightarrow{\varphi} A = k[x_1, \dots, x_r]/\mathfrak{p}.$$

We set $\mathfrak{m}_R := (x_1, \dots, x_r)$ and assume that $\mathfrak{p} \subset \mathfrak{m}_R$. Thus, $\mathfrak{m}_A = \varphi(\mathfrak{m}_R)$ is a maximal ideal of A .

5.1 Parameterizing ideals

We fix an integer $d > 0$ and seek to parameterize ideals $\mathfrak{a} \subseteq A$ containing \mathfrak{m}_A^d and contained in \mathfrak{m}_A . Since $\mathfrak{m}_R^d \subseteq \varphi^{-1}(\mathfrak{m}_A^d)$, such an ideal $\mathfrak{a} \subseteq A$ can be generated by \mathfrak{m}_A^d and images of polynomials from R of $\text{deg} < d$. Since there are $n_d = \binom{r+d-1}{r} - 1$ monomials of positive degree less than d in R , any such ideal $\mathfrak{m}_A^d \subseteq \mathfrak{a} \subseteq \mathfrak{m}_A$ can be generated by \mathfrak{m}_A^d and the image of n_d linear combinations of monomials. After setting $N_d = n_d^2$, we get a map

$$\{k\text{-valued points of } \mathbb{A}^{N_d}\} \longrightarrow \{\text{ideals } \mathfrak{a} \subseteq A \text{ s.t. } \mathfrak{m}_A^d \subseteq \mathfrak{a} \subseteq \mathfrak{m}_A\},$$

where \mathbb{A}^{N_d} parameterizes coefficients and generators of such ideals. The above map is surjective, but not injective (generators of an ideal are not unique). Additionally, we have a universal ideal $\mathcal{A} \subseteq \mathcal{O}_{X \times \mathbb{A}^{N_d}}$ such that \mathcal{A} restricted to the fibers of $p : X \times \mathbb{A}^{N_d} \rightarrow \mathbb{A}^{N_d}$ give us our \mathfrak{m}_A -primary ideals.

The construction in the previous paragraph follows the exposition of [FM09, § 3].

5.2 Parameterizing graded sequences of ideals

We proceed to parameterize graded sequences of ideals \mathfrak{a}_\bullet of A satisfying

$$\mathfrak{m}_A^m \subseteq \mathfrak{a}_m \subseteq \mathfrak{m}_A \quad \text{for all } m \in \mathbb{N}. \tag{\dagger}$$

We set

$$H_d := \mathbb{A}^{N_1} \times \dots \times \mathbb{A}^{N_d},$$

where N_i is chosen as in the previous section. For $d > c$, let $\pi_{d,c} : H_d \rightarrow H_c$ denote the natural projection maps. Our desired object is the following projective limit

$$H = \varprojlim H_d.$$

For $d > 0$, let $\pi_d : H \rightarrow H_d$ denote the natural map.

Note that the above projective limit exists in the category of schemes, since the maps in our directed system are all affine morphisms. Indeed, H is isomorphic to an infinite-dimensional affine space.

Since a k -valued point of H is simply a sequence of k -valued points of \mathbb{A}^{N_d} for all $d \in \mathbb{N}$, we have a surjection

$$\{k\text{-valued points of } H\} \longrightarrow \{\text{sequences of ideals } \mathfrak{b}_\bullet \text{ of } A \text{ satisfying } (\dagger)\}.$$

Note that the sequences of ideals on the right-hand side are not necessarily graded.

Given a sequence of ideals \mathfrak{b}_\bullet , we can construct a graded sequence \mathfrak{a}_\bullet inductively by setting $\mathfrak{a}_1 := \mathfrak{b}_1$ and

$$\mathfrak{a}_q := \mathfrak{b}_q + \sum_{m+n=q} \mathfrak{a}_m \cdot \mathfrak{a}_n.$$

If \mathfrak{b}_\bullet was graded to begin with, then $\mathfrak{a}_\bullet = \mathfrak{b}_\bullet$. By the construction, it is clear that $\mathfrak{a}_m \cdot \mathfrak{a}_n \subseteq \mathfrak{a}_{m+n}$. Thus, we have our desired map

$$\{k\text{-valued points of } H\} \longrightarrow \{\text{graded sequences of ideals } \mathfrak{a}_\bullet \text{ of } A \text{ satisfying } (\dagger)\}.$$

Additionally, we have a universal graded sequence of ideals $\mathcal{A}_\bullet = \{\mathcal{A}_m\}_{m \in \mathbb{N}}$ on $X \times H$. We will often abuse notation and refer to similarly defined ideals $\mathcal{A}_1, \dots, \mathcal{A}_d$ on $X \times H_d$.

The following technical lemma will be useful in the next proposition. The proof of the lemma relies on the fact that every descending sequence of nonempty constructible subsets of a variety over an uncountable field has nonempty intersection.

LEMMA 5.1. *If $\{W_d\}_{d \in \mathbf{N}}$ is a collection of nonempty subsets of H_d such that*

- (a) $W_d \subset H_d$ is a constructible, and
- (b) $W_{d+1} \subseteq \pi_{d+1,d}^{-1}(W_d)$

for each $d \in \mathbf{Z}_{>0}$,

then there exists a k -valued point in

$$\bigcap_{d \in \mathbf{N}} \pi_d^{-1}(W_d).$$

Proof. Note that a k -valued point in the above intersection is equivalent to a sequence of closed points $\{x_d \in W_d\}_{d \in \mathbf{N}}$ such that $\pi_{d+1,d}(x_{d+1}) = x_d$. We proceed to construct such a sequence.

We first look to find a candidate for x_1 . Assumption (b) implies

$$W_1 \supseteq \pi_{2,1}(W_2) \supseteq \pi_{3,1}(W_3) \supseteq \dots$$

is a descending sequence of nonempty sets. Note that W_1 is constructible by assumption and so are $\pi_{d,1}(W_d)$ for all d by Chevalley’s theorem [Har77, Exercise II.3.9]. Thus,

$$W_1 \cap \pi_{2,1}(W_2) \cap \pi_{3,1}(W_3) \cap \dots$$

is nonempty and we may choose a point x_1 lying in the set.

Next, we look at

$$W_2 \cap \pi_{2,1}^{-1}(x_1) \supseteq \pi_{3,2}(W_3) \cap \pi_{2,1}^{-1}(x_1) \supseteq \pi_{4,2}(W_4) \cap \pi_{2,1}^{-1}(x_1),$$

and note that for $d \geq 2$ each $\pi_{d,2}(W_d) \cap \pi_{2,1}^{-1}(x_1)$ is nonempty by our choice of x_1 . By the same argument as before, we see

$$\pi_{2,1}^{-1}(x_1) \cap W_2 \cap \pi_{3,2}(W_3) \cap \pi_{4,2}(W_4) \cap \dots$$

is nonempty and contains a closed point x_2 . Continuing in this manner, we construct the desired sequence. □

5.3 Finding limit points

The proof of the following proposition relies on the previous construction of a space that parameterizes graded sequences of ideals. The proof is inspired by arguments in [Kol08, FEM10, FEM11].

PROPOSITION 5.2. *Let X be a klt variety and $x \in X$ a closed point. Assume there exists a collection of graded sequences of \mathfrak{m}_x -primary ideals $\{\mathfrak{a}_\bullet^{(i)}\}_{i \in \mathbf{N}}$ and $\lambda \in \mathbf{R}$ such that the following hold.*

- (a) (Convergence from above) *For every $\epsilon > 0$, there exists positive constants M, N so that*

$$\text{lct}(\mathfrak{a}_m^{(i)})^n e(\mathfrak{a}_m^{(i)}) \leq \lambda + \epsilon$$

for all $m \geq M$ and $i \geq N$.

- (b) (Boundedness from below) *For each $m, i \in \mathbf{N}$, we have*

$$\mathfrak{m}_x^m \subseteq \mathfrak{a}_m^{(i)}.$$

(c) (Boundedness from above) There exists $\delta > 0$ such that

$$\mathfrak{a}_m^{(i)} \subseteq \mathfrak{m}^{\lfloor m\delta \rfloor}$$

for all $m, i \in \mathbf{N}$.

Then, there exists a graded sequence of \mathfrak{m}_x -primary ideals $\tilde{\mathfrak{a}}_\bullet$ on X such that

$$\text{lct}(\tilde{\mathfrak{a}}_\bullet)^n e(\tilde{\mathfrak{a}}_\bullet) \leq \lambda.$$

Proof. It is sufficient to consider the case when X is affine. Thus, we may assume that $X = \text{Spec } A$ and $A = k[x_1, \dots, x_r]/\mathfrak{p}$ as in the beginning of this section. In addition, we may assume that $x \in X$ corresponds to the maximal ideal \mathfrak{m}_A . We recall that §5.2 constructs a variety H parameterizing graded sequences of ideals on X satisfying (†). In addition, we have finite-dimensional truncations H_d that parameterize the first d elements of such a sequence.

Each graded sequence $\mathfrak{a}_\bullet^{(i)}$ satisfies (†) by assumptions (b) and (c). Thus, we may choose a point $p_i \in H$ corresponding to $\mathfrak{a}_\bullet^{(i)}$. Note that $\pi_d(p_i) \in H_d$ corresponds to the first d -terms of $\mathfrak{a}_\bullet^{(i)}$.

CLAIM 1. We may choose infinite subsets $\mathbf{N} \supset I_1 \supset I_2 \supset \dots$ and set

$$Z_d := \overline{\{\pi_d(p_i) \mid i \in I_d\}}$$

such that (**) holds:

If $Y \subsetneq Z_d$ is a closed set, there are only finitely many $i \in I_d$ such that $\pi_d(p_i) \in Y$. (**)

To prove Claim 1, we construct such a sequence inductively. First, we set $I_1 = \mathbf{N}$. Since $H_1 \simeq \mathbb{A}^0$ is a point, (**) is trivially satisfied for $d = 1$. After having chosen I_d , choose $I_{d+1} \subset I_d$ so that (**) is satisfied for Z_{d+1} . By the Noetherianity of H_d , such a choice is possible.

CLAIM 2. We have the inclusion $Z_{d+1} \subseteq \pi_{d+1,d}^{-1}(Z_d)$ for all $d \geq 1$.

The proof of Claim 2 follows from the definition of Z_d . Since $\pi_d(p_i) \in Z_d$ for all $i \in I_d$ and $I_d \supseteq I_{d+1}$, it follows that $\pi_{d+1,d}^{-1}(Z_d)$ is a closed set containing $\pi_{d+1}(p_i)$ for $i \in I_{d+1}$. The closure of the latter set of points is precisely Z_{d+1} .

CLAIM 3. If $p \in Z_d$ is a closed point, we have $\mathcal{A}_d|_p \subseteq \mathfrak{m}^{\lfloor d\delta \rfloor}$.

We now prove Claim 3. The set $\{p \in H_d \mid \mathcal{A}_d|_p \subseteq \mathfrak{m}^{\lfloor d\delta \rfloor}\}$ is a closed in H_d . By assumption (c), $\pi_d(p_i)$ lies in the above closed set for all $i \in I_d$. Thus, $Z_d \subseteq \{p \in H_d \mid \mathcal{A}_d|_p \subseteq \mathfrak{m}^{\lfloor d\delta \rfloor}\}$.

We now return to the proof of the proposition. We look at the normalized multiplicity of the ideals parameterized by Z_d . By Propositions A.1 and A.2, for each d , we may choose a nonempty open set $U_d \subseteq Z_d$ such that

$$\text{lct}(\mathcal{A}_d|_p)^n e(\mathcal{A}_d|_p) = \lambda_d$$

is constant for $p \in U_d$. Set

$$I_d^\circ = \{i \in I_d \mid \pi_d(p_i) \in U_d\} \subseteq I_d,$$

and note that $I_d \setminus I_d^\circ$ is finite. If this was not the case, then (**) would not hold.

The finiteness of $I_d \setminus I_d^\circ$ has two consequences. First,

$$\limsup_{d \rightarrow \infty} \lambda_d \leq \lambda,$$

since $\pi_d(p_i) \in U_d$ for all $i \in I_d^\circ$ and assumption (a) of this proposition. Second, since $I_{d+1} \subset I_d$ for $d \in \mathbf{N}$, we have

$$I_d^\circ \cap I_{d-1}^\circ \cdots \cap I_1^\circ \neq \emptyset.$$

CLAIM 4. *There exists a k -valued point $\tilde{p} \in H$ such that $\pi_d(\tilde{p}) \in U_d$ for all $d \in \mathbf{Z}_{>0}$.*

Proving this claim will complete the proof. Indeed, a point $\tilde{p} \in H$ corresponds to a graded sequence of \mathfrak{m}_x -primary ideals $\tilde{\mathfrak{a}}_\bullet$. Since $\pi_d(\tilde{p}) \in U_d$, we will have $\text{lct}(\tilde{\mathfrak{a}}_d)^n e(\tilde{\mathfrak{a}}_d) = \lambda_d$. In addition, Claim 2 implies and $\tilde{\mathfrak{a}}_d \subset \mathfrak{m}_x^{\lfloor d\delta \rfloor}$ for all $d \in \mathbf{Z}_{>0}$. Thus,

$$\text{lct}(\tilde{\mathfrak{a}}_\bullet)^n e(\tilde{\mathfrak{a}}_\bullet) = \lim_{d \rightarrow \infty} \text{lct}(\tilde{\mathfrak{a}}_d) e(\tilde{\mathfrak{a}}_d) \leq \limsup_{d \rightarrow \infty} \lambda_d \leq \lambda,$$

and the proof will be complete.

We are left to prove Claim 4. In order to do so, we will apply Lemma 5.1 to find such a point $\tilde{p} \in H$. First, we define constructible sets $W_d \subseteq H_d$ inductively. Set $W_1 = U_1$. After having chosen W_d , set $W_{d+1} = \pi_{d+1,d}^{-1}(W_d) \cap U_d$. For each $d \in \mathbf{N}$ we have the following:

- W_d is open in Z_d and, thus, constructible in H_d ;
 - W_d is nonempty, since W_d contains $\pi_d(p_i)$ for all $i \in I_d^\circ \cap I_{d-1}^\circ \cap \dots \cap I_1^\circ$, which is nonempty.
- By Lemma 5.1, there exists a k -valued point $\tilde{p} \in H$ such that $\pi_d(\tilde{p}) \in W_d \subset U_d$ for all $d \in \mathbf{Z}_{>0}$. This completes the proof. □

Remark 5.3. In the previous proof, we construct a graded sequence of ideal $\tilde{\mathfrak{a}}_\bullet$ based on a collection of graded sequences $\{\mathfrak{a}_\bullet^{(i)}\}_{i \in \mathbf{N}}$. While the construction of $\tilde{\mathfrak{a}}_\bullet$ is inspired by past constructions of generic limits, $\tilde{\mathfrak{a}}_\bullet$ is not a generic limit in the sense of [Kol08, 28].

We construct the precise analog as follows. We set

$$Z := \bigcap_d \pi_d^{-1}(Z_d) \subseteq H,$$

with Z_d defined in the previous proof. The generic point of Z gives a map $\text{Spec}(K(Z)) \rightarrow H$, where $K(Z)$ is the function field of Z . Thus, we get a graded sequence of ideals $\widehat{\mathfrak{a}}_\bullet$ on $X_{K(Z)}$, the base change of X by $K(Z)$.

In the previous proof, we wanted to construct a graded sequence on X , not a base change of X . Thus, $\tilde{\mathfrak{a}}_\bullet$ was chosen to be a graded sequence corresponding to a very general point in Z .

6. Proof of the Main Theorem

In this section we prove the Main Theorem. To prove the theorem, we apply the construction from § 5.

Proof of the Main Theorem. We fix a klt variety X and a closed point $x \in X$. Next, we choose a sequence of valuations $(v_i)_{i \in \mathbf{N}}$ in $\text{Val}_{X,x}$ such that

$$\lim_i \widehat{\text{vol}}(v_i) = \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v).$$

Additionally, after scaling our valuations, we may assume that $v_i(\mathfrak{m}_x) = 1$ for all $i \in \mathbf{N}$. Note that this implies that $\mathfrak{m}_x^m \subset \mathfrak{a}_m(v_i)$ for all m .

We claim that $\{\mathfrak{a}_\bullet(v_i)\}_{i \in \mathbf{N}}$ satisfy the hypotheses of Proposition 5.2 with $\lambda = \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v)$. After showing that this is the case, we will have that there exists a graded sequence of \mathfrak{m}_x -primary ideals $\tilde{\mathfrak{a}}_\bullet$ such that

$$\text{lct}(\tilde{\mathfrak{a}}_\bullet)^n e(\tilde{\mathfrak{a}}_\bullet) \leq \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v).$$

By Theorem B.1, there exists a valuation $v^* \in \text{Val}_{X,x}$ that computes $\text{lct}(\tilde{\mathbf{a}}_\bullet)$. Thus,

$$\widehat{\text{vol}}(v^*) \leq \text{lct}(\tilde{\mathbf{a}}_\bullet)^n e(\tilde{\mathbf{a}}_\bullet) = \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v),$$

where the first inequality follows from Lemma 4.4. Thus, v^* will be our normalized volume minimizer.

It is left to show that $\{\mathbf{a}_\bullet(v_i)\}_{i \in \mathbf{N}}$ satisfies the hypotheses of Proposition 5.2. Hypothesis (a) follows from Proposition 6.4, (b) from the assumption that $v_i(\mathbf{m}) = 1$ for all $i \in \mathbf{N}$, and (c) from Proposition 6.2. \square

We proceed to prove the two propositions mentioned in the previous paragraph. We emphasize that estimates from [Li15] are essential in the proof of the following lemma and propositions.

LEMMA 6.1. *With the notation above, there exist positive constants E, B such that: (a) $A_X(v_i) \leq E$; and (b) $\text{vol}(v_i) \leq B$ for all $i \in \mathbf{N}$.*

Proof. By [Li15, Theorem 3.3], there exists a constant C such that

$$A_X(v) \leq C \cdot v(\mathbf{m}) \widehat{\text{vol}}(v)$$

for all $v \in \text{Val}_{X,x}$. Thus, we may set $E := C \cdot \sup_i \widehat{\text{vol}}(v_i) < +\infty$.

The bound on the volume follows from the inclusion $\mathbf{m}_x^m \subset \mathbf{a}_m(v_i)$ for all $m \in \mathbf{N}$. The inclusion gives that

$$\text{vol}(v_i) = \lim_{m \rightarrow \infty} \frac{e(\mathbf{a}_m(v_i))}{m^n} \leq \lim_{m \rightarrow \infty} \frac{e(\mathbf{m}_x^m)}{m^n} = e(\mathbf{m}_x). \quad \square$$

PROPOSITION 6.2. *With the notation above, there exists $\delta > 0$ such that*

$$\mathbf{a}_m(v_i) \subseteq \mathbf{m}_x^{\lfloor \delta m \rfloor}$$

for all $m, i \in \mathbf{N}$.

Remark 6.3. Note that for an ideal \mathbf{a} , the order of vanishing of \mathbf{a} along x is defined to be

$$\text{ord}_x(\mathbf{a}) := \max\{n \mid \mathbf{a} \subseteq \mathbf{m}_x^n\}.$$

To prove the above proposition, it is sufficient to find $\delta' > 0$ such that

$$\delta' m \leq \text{ord}_x(\mathbf{a}_m(v_i))$$

for all $m, i \in \mathbf{N}$.

Proof. By [Li15, Proposition 2.3], there exists a constant C such that for all $v \in \text{Val}_{X,x}$ and $f \in \mathcal{O}_{X,x}$,

$$v(f) \leq C \cdot A_X(v) \text{ord}_x(f).$$

Thus,

$$m \leq v_i(\mathbf{a}_m(v_i)) \leq C \cdot A_X(v_i) \text{ord}_x(\mathbf{a}_m(v_i)),$$

and

$$\frac{m}{C A_X(v_i)} \leq \text{ord}_x(\mathbf{a}_m(v_i)).$$

By Lemma 6.1, there exists a positive constant E such that $A_X(v_i) \leq E$ for all i . We conclude

$$\frac{m}{C \cdot E} \leq \text{ord}_x(\mathbf{a}_m(v_i)). \quad \square$$

PROPOSITION 6.4. *With the notation above, for $\epsilon > 0$, there exist positive constants M, N such that*

$$\text{lct}(\mathbf{a}_m(v_i))^n e(\mathbf{a}_m(v_i)) \leq \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) + \epsilon.$$

for all $m \geq M$ and $i \geq N$.

Proof. Since $\widehat{\text{vol}}(v_i)$ converges to $\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v)$ as $i \rightarrow \infty$, we may choose N so that

$$\widehat{\text{vol}}(v_i) \leq \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) + \epsilon/2$$

for all $i \geq N$. By Lemma 6.1, we have $E := \sup A_X(v_i) < \infty$. Additionally, Lemma 6.1 allows us to apply Proposition 3.7 to find a constant M so that

$$e(\mathbf{a}_m(v_i)) \leq \text{vol}(v_i) + \epsilon/(2E^n)$$

for all integers $m \geq M$. Thus,

$$\text{lct}(\mathbf{a}_m(v_i))^n e(\mathbf{a}_m(v_i)) \leq A_X(v_i)^n (\text{vol}(v_i) + \epsilon/(2E^n)) \leq \widehat{\text{vol}}(v_i) + \epsilon/2.$$

for all $m \geq M$ and $i \in \mathbf{N}$. We conclude that

$$\text{lct}(\mathbf{a}_m(v_i))^n e(\mathbf{a}_m(v_i)) \leq \widehat{\text{vol}}(v_i) + \epsilon/2 \leq \inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) + \epsilon$$

for all $m \geq M$ and $i \geq N$. □

7. The normalized volume over a log pair

The normalized volume function has been studied in the setting of log pairs [LX16, LL16]. We explain that the arguments in this paper extend to the setting where (X, Δ) is a klt pair.

Recall that (X, Δ) is a *log pair* if X is a normal variety and Δ is an effective \mathbf{Q} -divisor on X such that $K_X + \Delta$ is \mathbf{Q} -Cartier.

7.1 Log discrepancies

If (X, Δ) is a log pair, the *log discrepancy* function $A_{(X,\Delta)} : \text{Val}_X \rightarrow \mathbf{R} \cup \{+\infty\}$ is defined as follows. If $Y \rightarrow X$ is a proper birational morphism with Y normal, and $E \subset Y$ a prime divisor, then the *log discrepancy* of ord_E over (X, Δ) is

$$A_{(X,\Delta)}(\text{ord}_E) := 1 + (\text{coefficient of } E \text{ in } K_Y - f^*(K_X + \Delta)).$$

As in [BFFU15], $A_{(X,\Delta)}$ can then be extended to Val_X . If (X, Δ) is a log pair, we say (X, Δ) is *klt* if $A_X(v) > 0$ for all divisorial valuations $v \in \text{Val}_X$.

7.2 Normalized volume minimizers

If (X, Δ) is a klt log pair and $x \in X$ is a closed point, the normalized volume of a valuation $v \in \text{Val}_{X,x}$ over the pair (X, Δ) is defined to be

$$\widehat{\text{vol}}_{(X,\Delta),x}(v) := A_{(X,\Delta)}(v)^n \text{vol}(v).$$

We claim that if (X, Δ) is a klt pair and $x \in X$ is a closed point, then there exists a minimizer of $\widehat{\text{vol}}_{(X,\Delta),x}$.

The main subtlety in extending our arguments to the log setting is in extending Theorem 3.6, which is a consequence of the subadditivity theorem. Takagi proved the following subadditivity theorem for log pairs.

THEOREM 7.1 (Takagi [Tak13]). *Let (X, Δ) be a log pair, $\mathfrak{a}, \mathfrak{b}$ ideals on X , and $s, t \in \mathbf{Q}_{\geq 0}$. For $r \in \mathbf{Z}_{>0}$ so that $r(K_X + \Delta)$, we have*

$$\text{Jac}_X \cdot \mathcal{J}((X, \Delta), \mathfrak{a}^s \mathfrak{b}^s \mathcal{O}_X(-r\Delta)^{1/r}) \subseteq \mathcal{J}((X, \Delta), \mathfrak{a}^s) \mathcal{J}((X, \Delta), \mathfrak{b}^t).$$

Takagi’s result implies the following generalization of Theorem 3.6 for log pairs. The remaining arguments in the paper extend to this setting.

THEOREM 7.2. *Let (X, Δ) be a log pair and $r \in \mathbf{Z}_{>0}$ such that $r(K_X + \Delta)$ is Carter. If $v \in \text{Val}_X$ satisfies $A_X(v) < +\infty$, then*

$$(\text{Jac}_X \cdot \mathcal{O}_X(-r\Delta))^{\ell-1} \cdot \mathfrak{a}_m^\ell \subseteq (\text{Jac}_X \mathcal{O}_X(-r\Delta))^{\ell-1} \cdot \mathfrak{a}_{m\ell} \subseteq \mathfrak{a}_{m-e}^\ell$$

for every $m \geq e$ and $\ell \in \mathbf{Z}_{>0}$, where $\mathfrak{a}_\bullet := \mathfrak{a}_\bullet(v)$ and $e := \lceil A_{(X, \Delta)}(v) \rceil$.

8. The toric setting

We use the notation of [Ful93] for toric varieties. Let N be a free abelian group of rank $n \geq 1$ and $M = N^*$ its dual. We write $N_{\mathbf{R}} := N \otimes \mathbf{R}$ and $M_{\mathbf{R}} := M \otimes \mathbf{R}$. There is a canonical pairing

$$\langle \cdot, \cdot \rangle : N_{\mathbf{R}} \times M_{\mathbf{R}} \rightarrow \mathbf{R}.$$

We say that an element $u \in N$ is primitive if u cannot be written as $u = au'$ for $a \in \mathbf{Z}_{>1}$ and $u' \in N$.

Fix a maximal dimension, strongly convex, rational, polyhedral cone $\sigma \subset N_{\mathbf{R}}$. From the cone σ , we get a toric variety $X_\sigma = \text{Spec } R_\sigma$, where $R_\sigma = k[\sigma^\vee \cap M]$. Let $x \in X_\sigma$ denote the unique torus invariant point of X_σ . We write $u_1, \dots, u_r \in N$ for the primitive lattice points of N that generate the one-dimensional faces of σ . Each u_i corresponds to a toric invariant divisor D_i on X_σ . Since the canonical divisor is given by $K_{X_\sigma} = -\sum D_i$, the divisor K_X is \mathbf{Q} -Cartier if and only if there exists $w \in M \otimes \mathbf{Q} \subset M_{\mathbf{R}}$ such that $\langle u_i, w \rangle = 1$ for $i = 1, \dots, r$.

Given $u \in \sigma$, we obtain a toric valuation $v_u \in \text{Val}_X$ defined by

$$v_u \left(\sum_{m \in M \cap \sigma^\vee} \alpha_v \chi^v \right) = \min \{ \langle u, v \rangle \mid \alpha_v \neq 0 \}.$$

If $u \in \sigma^\vee \cap N$ is primitive, the valuation v_u corresponds to vanishing along a prime divisor on a toric variety proper and birational over X_σ . For $u \in \sigma$, v_u has center equal to x if and only if $u \in \text{Int}(\sigma)$.

Let $\text{Val}_{X_\sigma, x}^{\text{toric}} \subset \text{Val}_{X_\sigma, x}$ denote the valuations on X_σ of the form v_u for $u \in \text{Int}(\sigma)$. We refer to these valuations as the toric valuations at x . It is straightforward to compute the normalized volume of such a valuation. Assume K_X is \mathbf{Q} -Cartier and w is the unique vector such that $\langle u_i, w \rangle = 1$ for $i = 1, \dots, s$. For $u \in \sigma$, we have

$$A_{X_\sigma}(v_u) = \langle u, w \rangle.$$

For $u \in \sigma$ and $m \in \mathbf{N}$, we set $H_u(m) = \{v \in M_{\mathbf{R}} \mid \langle u, v \rangle \geq m\}$. Note that

$$\mathfrak{a}_m(v_u) = (\chi^v \mid v \in H_u(m) \cap \sigma^\vee \cap M).$$

In the case when $u \in \text{Int}(\sigma)$,

$$\text{vol}(v_u) = n! \cdot \text{Vol}(\sigma^\vee \setminus H_u(1)),$$

where Vol denotes the Euclidean volume.

8.1 Deformation to the initial ideal

As explained in [Eis95], when $X_\sigma \simeq \mathbb{A}^n$ and $I \subset R_\sigma$, there exists a deformation of I to a monomial ideal. A similar argument works in our setting.

Following the approach of [KK14, §6], we put a $\mathbf{Z}_{\geq 0}^n$ order on the monomials of R_σ . Fix $y_1, \dots, y_n \in N \cap \sigma$ that are linearly independent in $M_{\mathbf{R}}$. Thus, we get an injective map $\rho : M \rightarrow \mathbf{Z}^n$ by sending

$$v \mapsto (\langle y_1, v \rangle, \dots, \langle y_n, v \rangle).$$

Since each $y_i \in \sigma$, we have $\rho(M \cap \sigma^\vee) \subseteq \mathbf{Z}_{\geq 0}^n$. After putting the lexicographic order on $\mathbf{Z}_{\geq 0}^n$, we get an order $>$ on the monomials of R_σ .

An element $f \in R_\sigma$ may be written as a sum of scalar multiples of distinct monomials. The *initial term* of f , denoted $\text{in}_> f$, is the greatest term of f with respect to the order $>$. For an ideal $I \subset R_\sigma$, the *initial ideal* of I is

$$\text{in}_> I = (\text{in}_> f \mid f \in I).$$

Note that if I is \mathfrak{m}_x -primary, then so is $\text{in}_> I$. Also, if $\{I_m\}_{m \in \mathbf{N}}$ is a graded sequence of ideals of R_σ , then so is $\{\text{in}_> I_m\}_{m \in \mathbf{N}}$. This follows from the fact that $\text{in}_> f \cdot \text{in}_> g = \text{in}_> fg$.

LEMMA 8.1. *If $I \subset R_\sigma$ is an \mathfrak{m}_x -primary ideal, then*

$$\text{length}(R_\sigma/I) = \text{length}(R_\sigma/\text{in}_> I).$$

Proof. The proof is similar to the proof of [Eis95, Theorem 15.3]. □

Similar to the argument in [Eis95], we construct a deformation of I to $\text{in}_> I$. Since R_σ is Noetherian, we may choose elements $g_1, \dots, g_s \in I$ such that

$$I = (g_1, \dots, g_s) \quad \text{and} \quad \text{in}_> I = (\text{in}_> g_1, \dots, \text{in}_> g_s).$$

Fix an integral weight $\lambda : M \cap \sigma^\vee \rightarrow \mathbf{Z}_{\geq 0}$ such that

$$\text{in}_{>\lambda}(g_i) = \text{in}_>(g_i)$$

for all i . Note that $>\lambda$ denotes the order on the monomials induced by the weight function λ .

Let $R_\sigma[t]$ denote the polynomial ring in one variable over R_σ . For $g = \sum \alpha_m \chi^m$, we write $b := \max\{\lambda(m) \mid \alpha_m \neq 0\}$ and set

$$\tilde{g} := t^b \sum \alpha_m t^{-\lambda(m)} \chi^m.$$

Next, let

$$\tilde{I} = (\tilde{g}_1 \dots \tilde{g}_s) \subset R_\sigma[t].$$

For $c \in k$, we write I_c for the image of \tilde{I} under the map $R_\sigma[t] \rightarrow R_\sigma$ defined by $t \mapsto c$. It is clear that $I_1 = I$ and $I_0 = \text{in}_> I$.

PROPOSITION 8.2. *If $I \subset R_\sigma$ is an \mathfrak{m}_x primary ideal, then $\text{lct}(\text{in}_<(I)) \leq \text{lct}(I)$.*

Proof. We consider the automorphism of $\varphi : R_\sigma[t, t^{-1}] \rightarrow R_\sigma[t, t^{-1}]$ that sends χ^m to $t^{\lambda(m)} \chi^m$. Note that φ sends $\tilde{I}R_\sigma[t, t^{-1}]$ to $IR_\sigma[t, t^{-1}]$. Therefore, for each $c \in k^*$, we get an automorphism $\varphi_c : R_\sigma \rightarrow R_\sigma$ such that $\varphi_c(I_c) = I$. Thus, $\text{lct}(I_c) = \text{lct}(I)$ for all $c \in k^*$. Since I_c is \mathfrak{m}_x -primary for all $c \in k$, we may apply Proposition A.3 to see $\text{lct}(I_0) \leq \text{lct}(I)$. Since $\text{in}_>(I) = I_0$, we are done. □

8.2 Proof of Theorem 1.4

Theorem 1.4 is a direct consequence of Proposition 8.3. Our proof of Proposition 8.3 is inspired by the main argument in [Mus02].

PROPOSITION 8.3. *Let \mathfrak{a}_\bullet be a graded sequence of \mathfrak{m}_x -primary ideals on X_σ . We have that*

$$\text{lct}(\text{in}_{>}(\mathfrak{a}_\bullet))^n e(\text{in}(\mathfrak{a}_\bullet)) \leq \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet).$$

Proof. We first note that

$$e(\text{in}_{>}(\mathfrak{a}_\bullet)) := \limsup_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X_\sigma, x} / \text{in}_{>}(\mathfrak{a}_m))}{m^n / n!} = \limsup_{m \rightarrow \infty} \frac{\text{length}(\mathcal{O}_{X_\sigma, x} / \mathfrak{a}_m)}{m^n / n!} =: e(\mathfrak{a}_\bullet),$$

where the second equality follows from Proposition 8.1. By Proposition 8.2,

$$\text{lct}(\text{in}_{>}(\mathfrak{a}_\bullet)) \leq \text{lct}(\mathfrak{a}_\bullet).$$

The result follows. □

Proof of Theorem 1.4. Since $\text{Val}_{X,x}^{\text{toric}} \subset \text{Val}_{X,x}$, we have

$$\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) \leq \inf_{v \in \text{Val}_{X,x}^{\text{toric}}} \widehat{\text{vol}}(v).$$

We proceed to show the reverse inequality. Note that

$$\inf_{v \in \text{Val}_{X,x}} \widehat{\text{vol}}(v) = \inf_{\mathfrak{a}_\bullet \text{ } \mathfrak{m}_x\text{-primary}} \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet) = \inf_{\substack{\mathfrak{a}_\bullet \text{ } \mathfrak{m}_x\text{-primary} \\ \text{monomial}}} \text{lct}(\mathfrak{a}_\bullet)^n e(\mathfrak{a}_\bullet), \tag{8.1}$$

where the first equality follows from Proposition 4.3 and the second from Proposition 8.3. The last infimum in (8.1) is equal to

$$\inf_{\substack{\mathfrak{a} \text{ } \mathfrak{m}_x\text{-primary} \\ \text{monomial}}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a})$$

by Lemma 4.1. Thus, it is sufficient to show that

$$\inf_{v \in \text{Val}_{X,x}^{\text{toric}}} \widehat{\text{vol}}(v) \leq \inf_{\substack{\mathfrak{a} \text{ } \mathfrak{m}_x\text{-primary} \\ \text{monomial}}} \text{lct}(\mathfrak{a})^n e(\mathfrak{a}).$$

Let \mathfrak{a} be an \mathfrak{m}_x -primary monomial ideal. Since \mathfrak{a} is a monomial ideal, there exists a toric valuation $v^* \in \text{Val}_{X,x}^{\text{toric}}$ such that v^* computes $\text{lct}(\mathfrak{a})$. (This follows from the fact that there exists a toric log resolution of \mathfrak{a} .) By Proposition 4.4,

$$\widehat{\text{vol}}(v^*) \leq \text{lct}(\mathfrak{a})^n e(\mathfrak{a}),$$

and the proof is complete. □

8.3 An example of nondivisorial volume minimizer

Let V denote \mathbb{P}^2 blown up at a point. Note that V is a Fano variety. The affine cone $C(V, -K_V) = \text{Spec}(\bigoplus_{m \geq 0} H^0(V, -K_V))$ is isomorphic to the toric variety X_σ , where $\sigma \subseteq \mathbf{R}^3$ is the cone in Figure 1. Let x denote the torus invariant point of X_σ .

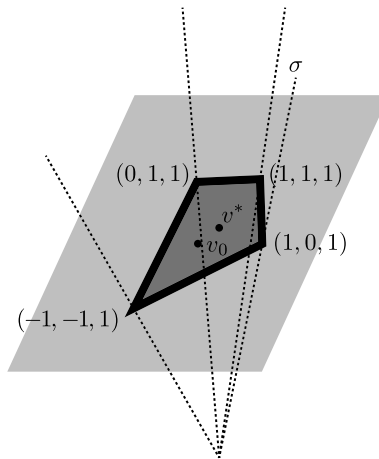


FIGURE 1. The cone σ is drawn. The toric variety X_σ is isomorphic to the cone over \mathbb{P}^2 blown up at a point.

We seek to find a minimizer of the function $\text{Val}_{X_\sigma, x}^{\text{toric}} \rightarrow \mathbf{R}_{>0}$ defined by $v_u \mapsto \widehat{\text{vol}}(v_u)$. Since the normalized volume is invariant under scaling, it is sufficient to consider elements $u \in \text{Int}(\sigma)$ of the form $u = (a, b, 1) \in \text{Int}(\sigma)$. We have

$$A_{X_\sigma}(v_{(a,b,1)}) = \langle (a, b, 1), (0, 0, 1) \rangle = 1.$$

The normalized volume of $v_{(a,b,1)}$ is

$$\widehat{\text{vol}}(v_{(a,b,1)}) := A_X(v_{(a,b,1)})^3 \text{vol}(v_{(a,b,1)}) = 3! \text{Vol}(\sigma^\vee \setminus H_{(a,b,1)}(1)).$$

After computing the previous volume, we see that the function is minimized at

$$(a^*, b^*, 1) = (4/3 - \sqrt{13}/3, 4/3 - \sqrt{13}/3, 1),$$

with $\widehat{\text{vol}}(v_{(a^*, b^*, 1)}) = \frac{1}{12}(46 + 13\sqrt{13})$. By Theorem 1.4, the toric volume minimizer $v^* = v_{(a^*, b^*, 1)}$ is also a minimizer of $\widehat{\text{vol}}_{X_\sigma, x}$. Since $\widehat{\text{vol}}(v_{(a^*, b^*, 1)})$ is irrational, Proposition 4.9 implies there cannot be a divisorial minimizer of $\widehat{\text{vol}}_{X_\sigma, x}$.

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Appendix A. Multiplicities and log canonical thresholds in families

In this section we provide information on the behavior of the Hilbert–Samuel multiplicity and log canonical threshold in a family. The content of this section is well known to experts, but does not necessarily appear in the literature in the desired form. The following statements will be useful in the proof of Proposition 5.2.

A.1 Multiplicities

We recall an interpretation of the Hilbert–Samuel multiplicity described in [Ram73]. Let X be a proper variety of dimension n and $x \in X$ a closed point. If $\mathfrak{a} \subseteq \mathcal{O}_X$ is an \mathfrak{m}_x -primary ideal and $Y \rightarrow X$ a proper birational morphism such that Y is nonsingular and $\mathfrak{a} \cdot \mathcal{O}_Y$ is an invertible sheaf, then $e(\mathfrak{a}) = (-1)^{n-1} D^n$, where D is the effective divisor on Y defined by $\mathfrak{a} \cdot \mathcal{O}_Y$.

The following proposition is well known. Related statements appear in [Ben70] and [Lip82].

PROPOSITION A.1. *Let X and T be varieties and $x \in X$ a closed point. If $\mathfrak{a} \subseteq \mathcal{O}_{X \times T}$ is an ideal such that $\mathfrak{a}_t := \mathfrak{a} \cdot \mathcal{O}_{X \times \{t\}}$ is \mathfrak{m}_x -primary for all closed points $t \in T$, then there exists a nonempty open set $U \subseteq T$ such that the function $U \ni t \mapsto e(\mathfrak{a}_t)$ is constant.*

Proof. It is sufficient to consider the case when X is proper. Indeed, let $V \subseteq X$ be an open affine subset of X containing x . Replace X with a projective closure of V .

Now, let $f : Y \rightarrow X \times T$ be a log resolution of \mathfrak{a} . Set $\mathcal{L} = \mathfrak{a} \cdot \mathcal{O}_Y$ and write $g : Y \rightarrow T$ for the map f composed with the projection $X \times T \rightarrow T$.

Choose a nonempty open set $U \subseteq T$ such that $g^{-1}(U)$ is flat over U and $Y_t \rightarrow X \times \{t\}$ is birational for all $t \in U$. By [Kol96, Proposition 2.9], $U \ni t \mapsto (\mathcal{L}_t)^n$ is constant. Thus, we are done. \square

A.2 Log canonical thresholds

We prove two statements on the behavior of the log canonical threshold along a family of ideals. For similar statements, see [Laz04, Example 9.3.17] and [Kol97, Lemma 8.6].

PROPOSITION A.2. *Let X and T be varieties and assume that X is klt. Fix an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X \times T}$, and set $\mathfrak{a}_t := \mathfrak{a} \cdot \mathcal{O}_{X \times \{t\}}$ for $t \in T$. Then, there exists a nonempty open set $U \subseteq T$ such that the function $U \ni t \mapsto \text{lct}(\mathfrak{a}_t)$ is constant.*

Proof. Let $\mu : Y \rightarrow X \times T$ be a log resolution of \mathfrak{a} , and set $p' = p \circ \mu$.

$$\begin{array}{ccc}
 Y & \xrightarrow{\mu} & X \times T \\
 & \searrow p' & \swarrow p \\
 & & T
 \end{array}$$

Let D be the divisor on Y such that $\mathfrak{a} \cdot \mathcal{O}_{X'} = \mathcal{O}_{X'}(-D)$ and E_1, \dots, E_r be the prime components of $\text{Exc}(\mu) + D_{\text{red}}$. After shrinking T , we may assume that each E_i surjects onto T and T is smooth.

We claim that on a nonempty open set $U \subset T$, $\mu_t : Y_t \rightarrow X_t$ is a log resolution of \mathfrak{a}_t for all $t \in U$, where $Y_t := Y_{p^{-1}(t)}$ and $X_t := X \times \{t\}$. Indeed, by generic smoothness [Har77, Corollary III.10.7] applied to X' , each E_i , and all the intersections of the E_i , we may find such a locus $U \subset T$.

Now, we have $(K_{Y/X \times T})|_{X_t} = K_{Y_t/X_t}$ and $\mathfrak{a}_t \cdot \mathcal{O}_{Y_t} = \mathcal{O}_{Y_t}(-D|_{Y_t})$ for $t \in U$. Therefore, $\text{lct}(\mathfrak{a}_t) = \min_{i=1, \dots, r} \text{ord}_{E_i}(K_{Y/X \times T}) / \text{ord}_{E_i}(D)$ for all $t \in U$, and we are done. \square

PROPOSITION A.3. *Let X be a klt variety, T a smooth curve, and $t_0 \in T$ a closed point. Fix an ideal $\mathfrak{a} \subseteq \mathcal{O}_{X \times T}$, and set $\mathfrak{a}_t := \mathfrak{a} \cdot \mathcal{O}_{X \times \{t\}}$. If $V(\mathfrak{a}) \subset X$ is proper over T , then there exists an open neighborhood $t_0 \in U \subseteq T$ such that*

$$\text{lct}(\mathfrak{a}_{t_0}) \leq \text{lct}(\mathfrak{a}_t)$$

for all closed points $t \in U$.

Remark A.4. The condition that $V(\mathfrak{a})$ is proper over T holds if there exists $x \in X$ such that each ideal \mathfrak{a}_t is \mathfrak{m}_x -primary for all closed points $t \in T$. Indeed, in this case $V(\mathfrak{a})_{\text{red}} = \{x\} \times T$.

Proof. By Proposition A.2, we may choose a nonempty open set $W \subseteq T$ such that $\text{lct}(\mathfrak{a}_t)$ takes the constant value λ for all $t \in W$. We will show $\text{lct}(\mathfrak{a}_{t_0}) \leq \lambda$. Then, $U := W \cup \{t_0\}$ will satisfy the conclusion of our proposition.

We first show that $p(V(\mathcal{J}(X, \times T, \lambda \cdot \mathfrak{a}))) = T$. By [Laz04, Example 9.5.34], we may shrink W so that $\mathcal{J}(X \times \{t\}, \lambda \cdot \mathfrak{a}_t) = \mathcal{J}(X, \lambda \cdot \mathfrak{a}) \cdot \mathcal{O}_{X \times \{t\}}$ for all $t \in W$. Since $\mathcal{J}(X \times \{t\}, \mathfrak{a}_t^\lambda) \neq \mathcal{O}_{X \times \{t\}}$ for all $t \in W$, we see $W \subseteq (p(V(\mathcal{J}(X, \times T, \lambda \cdot \mathfrak{a}))))$. Note that $p(V(\mathcal{J}(X \times T, \lambda \cdot \mathfrak{a})))$ is closed in T , since $\mathcal{J}(X \times T, \lambda \cdot \mathfrak{a})$ is closed in $V(\mathfrak{a})$ and $V(\mathfrak{a})$ is proper over T . Therefore, $p(V(\mathcal{J}(X \times T, \lambda \cdot \mathfrak{a}))) = T$.

Since $p(\mathcal{J}(X \times T, \lambda \cdot \mathfrak{a}) \cdot \mathcal{O}_{X \times \{t_0\}}) = T$ and $\mathcal{J}(X \times \{t_0\}, \lambda \cdot \mathfrak{a}_{t_0}) \subseteq \mathcal{J}(X \times T, \lambda \cdot \mathfrak{a}) \cdot \mathcal{O}_{X \times \{t_0\}}$ by [Laz04, Theorem 9.5.16], $\mathcal{J}(X \times \{t_0\}, \lambda \cdot \mathfrak{a}_{t_0})$ is nontrivial. Thus, $\text{lct}(\mathfrak{a}_{t_0}) \leq \lambda$, and the proof is complete. \square

Appendix B. Valuations computing log canonical thresholds of graded sequences

In [JM12], the authors prove the existence of valuations computing log canonical thresholds of graded sequences of ideals on smooth varieties. We extend the result to klt varieties. While the statement is likely known to experts, it does not appear in the literature.

THEOREM B.1. *If X is a klt variety and \mathfrak{a}_\bullet a graded sequence of ideals on X , then there exists $v^* \in \text{Val}_X$ computing $\text{lct}(\mathfrak{a}_\bullet)$.*

The proof we give is similar in spirit to the proof of [JM12, Theorem 7.3], but also relies on results in [BFFU15].

PROPOSITION B.2. *If X is a normal \mathbf{Q} -Gorenstein variety and \mathfrak{a}_\bullet a graded sequence of ideals on X , then $v \mapsto v(\mathfrak{a}_\bullet)$ is a continuous function on $\text{Val}_X \cap \{A_X(v) < +\infty\}$.*

Proof. We reduce the result to the smooth case. Take a resolution of singularities $Y \rightarrow X$ and write \mathfrak{a}_\bullet^Y for the graded sequence of ideals on Y defined by $\mathfrak{a}_m^Y = \mathfrak{a}_m \cdot \mathcal{O}_Y$. Thanks to [JM12, Corollary 6.4], the function $v \mapsto v(\mathfrak{a}_\bullet^Y)$ is continuous on $\text{Val}_Y \cap \{A_Y(v) < +\infty\}$.

Now, note that the natural map $\text{Val}_Y \rightarrow \text{Val}_X$ is a homeomorphism of topological spaces and $v(\mathfrak{a}_\bullet) = v(\mathfrak{a}_\bullet^Y)$. Since $A_X(v) = A_Y(v) + v(K_{Y/X})$, $A_X(v) < +\infty$ if and only if $A_Y(v) < +\infty$. Thus, the proof is complete. \square

Now, we recall some formalism from [BFFU15]. A *normalizing subscheme* on X is a (nontrivial) closed subscheme of X containing $\text{Sing}(X)$. If N is a normalizing subscheme of X , we set

$$\text{Val}_X^N := \{v \in \text{Val}_X \mid v(\mathcal{I}_N) = 1\}.$$

PROPOSITION B.3. *Let X be a normal \mathbf{Q} -Gorenstein variety, \mathfrak{a}_\bullet a graded sequence of ideals on X , and N a normalizing subscheme of X such that N contains the zero locus of \mathfrak{a}_1 .*

- (a) *The function $v \mapsto v(\mathfrak{a}_\bullet)$ is bounded on Val_X^N .*
- (b) *For each $M \in \mathbf{R}$, the set $\{A_X(v) \leq M\} \cap \text{Val}_X^N$ is compact.*

Proof. Statements (a) and (b) appear in [BFFU15, Proposition 2.5] and [BFFU15, Theorem 3.1], respectively. \square

PROPOSITION B.4. *If X is a klt variety and N a normalized subscheme of X , then there exists $\epsilon > 0$ such that $A_X(v) > \epsilon$ for all $v \in \text{Val}_X^N$.*

Proof. Let $\pi : Y \rightarrow X$ be a good resolution of N , and consider the continuous retraction map

$$r_\pi^N : \text{Val}_X^N \rightarrow \Delta_\pi^N$$

described in [BFFU15, § 2.2]. Since X is assumed to be klt, there exists $\epsilon > 0$ such that $A_X(v) > \epsilon$ for all $v \in \Delta_\pi^N$. Since $A_X(v) \geq A_X(r_\pi^N(v))$ for all $v \in \text{Val}_X^N$ by [BFFU15, Theorem 3.1], the proof is complete. \square

Proof of Theorem B.1. If $\text{lct}(\mathfrak{a}_\bullet) = +\infty$, then any $v \in \text{Val}_X$ with $A_X(v) < +\infty$ must compute $\text{lct}(\mathfrak{a}_\bullet)$. We now move on to the case when $\text{lct}(\mathfrak{a}_\bullet) < +\infty$.

Let N be the subscheme of X defined by the ideal $\mathcal{I}_{\text{Sing}(X)} \cdot \mathfrak{a}_1$. Since N is a normalizing subscheme of X that contains the zero locus of \mathfrak{a}_1 , we may apply the previous two propositions to choose $B \in \mathbf{R}$ and $\epsilon > 0$ so that $v(\mathfrak{a}_\bullet) < B$ and $A_X(v) > \epsilon$ for all $v \in \text{Val}_X^N$.

Note that

$$\text{lct}(\mathfrak{a}_\bullet) = \inf_{v \in \text{Val}_X^N} \frac{A_X(v)}{v(\mathfrak{a}_\bullet)}.$$

Indeed, consider $v \in \text{Val}_X$ such that $A_X(v)/v(\mathfrak{a}_\bullet) < +\infty$. Since $v(\mathfrak{a}_\bullet) > 0$, then $v(\mathfrak{a}_1) > 0$ and, thus, $v(\mathcal{I}_N) > 0$. We see $w = (1/v(\mathcal{I}_N))v \in \text{Val}_X^N$ and $A_X(w)/w(\mathfrak{a}_\bullet) = A_X(v)/v(\mathfrak{a}_\bullet)$.

Now, fix $L > \text{lct}(\mathfrak{a}_\bullet)$. If $v \in \text{Val}_X^N$ and $A_X(v)/v(\mathfrak{a}_\bullet) \leq L$, then

$$\epsilon < A_X(v) \leq Lv(\mathfrak{a}_\bullet) \leq L \cdot B.$$

Therefore,

$$\text{lct}(\mathfrak{a}_\bullet) = \inf_{v \in W} \frac{A_X(v)}{v(\mathfrak{a}_\bullet)},$$

where

$$W = \text{Val}_X^N \cap \{A_X(v) \leq L \cdot B\} \cap \{Lv(\mathfrak{a}_\bullet) \geq \epsilon\}.$$

We claim that W is compact. Indeed, $\text{Val}_X^N \cap \{A_X(v) \leq L \cdot B\}$ is compact by the previous proposition. Since $v \mapsto v(\mathfrak{a}_\bullet)$ is continuous on $\text{Val}_X^N \cap \{A_X(v) \leq L \cdot B\}$, W is closed in $\text{Val}_X^N \cap \{A_X(v) \leq L \cdot B\}$, and, thus, compact as well. Since $v \mapsto A_X(v)/v(\mathfrak{a}_\bullet)$ is lower semicontinuous on the compact set W , there exists $v^* \in W$ such that $A_X(v^*)/v^*(\mathfrak{a}_\bullet) = \text{lct}(\mathfrak{a}_\bullet)$. \square

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