Proceedings of the Edinburgh Mathematical Society (2013) **56**, 637–640
DOI:10.1017/S001309151300031X

TOPOLOGICAL CRITERIA FOR SCHLICHTNESS

ZACH TEITLER

Department of Mathematics, 1910 University Drive, Boise State University, Boise, ID 83725-1555, USA (zteitler@boisestate.edu)

(Received 11 February 2011)

Abstract We give two sufficient criteria for schlichtness of envelopes of holomorphy in terms of topology. These are weakened converses of results of Kerner and Royden. Our first criterion generalizes a result of Hammond in dimension 2. Along the way, we also prove a generalization of Royden's theorem.

Keywords: envelope of holomorphy; schlichtness; covering space

2010 *Mathematics subject classification:* Primary 32D10

Let $\Omega \subseteq \mathbb{C}^n$ be a domain. The *envelope of holomorphy* of Ω is a pair $(\tilde{\Omega}, \pi)$ consisting of a connected Stein manifold $\tilde{\Omega}$ and a locally biholomorphic map $\pi\colon \tilde{\Omega} \to \mathbb{C}^n$, together with a holomorphic inclusion $\alpha\colon \Omega \to \tilde{\Omega}$, characterized by the following properties: $\pi \circ \alpha$ is the identity, and each holomorphic function f on Ω has a unique holomorphic extension F_f on $\tilde{\Omega}$, with $f = F_f \circ \alpha$. Let $\Omega' = \pi(\tilde{\Omega})$ and let $i = \pi \circ \alpha\colon \Omega \to \Omega'$. The envelope of holomorphy $(\tilde{\Omega}, \pi)$ is *schlicht* if $\pi\colon \tilde{\Omega} \to \Omega'$ is biholomorphic. One would like to give conditions on Ω to have a schlicht envelope of holomorphy.

Two results of Kerner and Royden lead to necessary conditions. Kerner [**5**] has shown that $\alpha_*\colon \pi_1(\Omega) \to \pi_1(\tilde{\Omega})$ is surjective. Royden [**8**] has shown that $\alpha^*\colon H^1(\tilde{\Omega}; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z})$ is injective. It follows trivially that if $(\tilde{\Omega}, \pi)$ is schlicht, so $\tilde{\Omega} = \Omega'$, then $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective and $i^*\colon H^1(\Omega'; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z})$ is injective.

Neither of these conditions is sufficient, by a result of Fornæss and Zame [**1**] (see [**2**, § 3]). Following an idea of Hammond [**2**], one may seek sufficient conditions by adjoining to the surjectivity of i_* (or injectivity of i^*) the assumption that $\pi\colon \tilde{\Omega} \to \Omega'$ is a covering space. This strong assumption is still reasonable, as covering maps certainly occur among envelopes of holomorphy; indeed, Fornæss and Zame show in [**1**] that for any covering map $\pi\colon \tilde{\Omega} \to \Omega'$ there is a domain $\Omega \subseteq \Omega'$ with envelope of holomorphy $(\tilde{\Omega}, \pi)$.

Specifically, Hammond has shown that, in dimension $n = 2$, if $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective and $\pi\colon \tilde{\Omega} \to \Omega'$ is a covering map, then $(\tilde{\Omega}, \pi)$ is schlicht. We give an elementary proof of Hammond's theorem in all dimensions $n \geqslant 2$. In addition, we give a sufficient condition for schlichtness in terms of the injectivity of i^* on cohomology, again assuming π is a covering map. Along the way, we give an alternative proof of Royden's theorem, which also extends it to coefficient groups other than $\mathbb{Z}$.

Theorem 1. *If π is a covering map and $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective, then $(\tilde{\Omega}, \pi)$ is schlicht.*

This extends the theorem of Hammond for dimension $n = 2$. Hammond's proof relies on a result of Jupiter [**4**], which is special to dimension 2.

Proof. The number of sheets of the covering map π is equal to the index of $\pi_*(\pi_1(\tilde{\Omega}))$ in $\pi_1(\Omega')$ (see, for example, [**3**, Proposition 1.32]). The surjectivity of $i_* = \pi_* \circ \alpha_*$ implies that π_* is surjective. Hence, the index of the image subgroup is 1, so $\pi\colon \tilde{\Omega} \to \Omega'$ is 1-sheeted, i.e. a homeomorphism. Since π is a holomorphic homeomorphism, it is biholomorphic and so $\tilde{\Omega}$ is schlicht. $\square$

Compare with the more technical proof in [**2**].

The cohomology in Royden's result is Čech cohomology with coefficients in the sheaf of locally constant $\mathbb{Z}$-valued functions. Since our spaces are manifolds, Čech cohomology coincides with singular cohomology (with coefficients in $\mathbb{Z}$); see, for example, [**6**, Theorem 73.2]. Recall also that by the universal coefficient theorem, $H^1(X; G) = \mathrm{Hom}(\pi_1(X), G)$ for a path-connected space X and abelian coefficient group G [**3**, p. 198].

Before we go on, observe that this proves Royden's theorem as a consequence of Kerner's theorem and extends it to other coefficient groups.

Theorem 2 (Royden). *For any abelian group G, $\alpha^*\colon H^1(\Omega; G) \to H^1(\tilde{\Omega}; G)$ is injective.*

Proof. Since $\alpha_*\colon \pi_1(\Omega) \to \pi_1(\tilde{\Omega})$ is surjective,

$$\alpha^*\colon \mathrm{Hom}(\pi_1(\Omega), G) \to \mathrm{Hom}(\pi_1(\tilde{\Omega}), G)$$

is injective and these Hom groups coincide with $H^1(\Omega; G)$, $H^1(\tilde{\Omega}; G)$. $\square$

Royden proves this for $G = \mathbb{Z}$ using Čech cohomology, in particular the exponential short exact sequence (hence the restriction to $G = \mathbb{Z}$). No such result holds for higher cohomology groups [**1**, Theorem 4].

Now, we aim to give a sufficient criterion for schlichtness by assuming $i^*\colon H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective for every abelian group G, and that π is a covering map. Our strategy is to deduce that $\pi_*\colon \pi_1(\tilde{\Omega}) \to \pi_1(\Omega')$ is surjective, as in the proof of Theorem 1. This would follow if we could deduce that $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective, but, in general, injectivity of $\mathrm{Hom}(A, G) \to \mathrm{Hom}(B, G)$ does not imply surjectivity of $B \to A$. The problem is that if the image of B is a proper subgroup which is not contained in any proper normal subgroup, then there is no non-zero $f\colon A \to G$ vanishing on the image of B. For example, let $\mathfrak{S}_3$ be the symmetric group on three letters and let $B = \mathbb{Z}/2\mathbb{Z}$ be the subgroup generated by a transposition. If $f\colon \mathfrak{S}_3 \to G$ is any group homomorphism such that the restriction $f \mid B$ is zero, then f itself is zero.

We must solve this problem by adjoining a hypothesis to ensure that every proper subgroup of $\pi_1(\Omega')$ is contained in a proper normal subgroup. However, this alone is not enough. For, suppose that $B \subset \pi_1(\Omega')$ is a proper subgroup, contained in a proper normal subgroup N. We get a non-zero homomorphism $f\colon \pi_1(\Omega') \to G = \pi_1(\Omega')/N$, namely the quotient map, whose restriction to $B \subseteq N$ is zero, so $\mathrm{Hom}(\pi_1(\Omega'), G) \to \mathrm{Hom}(B, G)$ is not injective. This will prove the theorem we want, but only if G is abelian, so we can identify these Hom groups with singular cohomology.

So we need to know that every proper subgroup of $\pi_1(\Omega')$ is not only contained in a proper normal subgroup, but in one such subgroup N whose quotient $G = \pi_1(\Omega')/N$ is abelian.

Fortunately, this condition is more natural than it sounds. It holds if $\pi_1(\Omega')$ is nilpotent, as in that case every maximal proper subgroup is normal and has prime index (see [**7**, Theorem 5.40]).

We get the following.

Theorem 3. *If π is a covering map, $\pi_1(\Omega')$ is nilpotent and $i^*\colon H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective for every abelian group G, then $(\tilde{\Omega}, \pi)$ is schlicht.*

Proof. Since $i^* = \alpha^* \circ \pi^*$ is injective, π^* is injective as well. Via π_*, we regard $\pi_1(\tilde{\Omega})$ as a subgroup of $\pi_1(\Omega')$. Recall that if H is any nilpotent group, then every maximal proper subgroup N of H is normal and has prime index, and, in particular, H/N is abelian. If $\pi_1(\tilde{\Omega}) \subsetneqq \pi_1(\Omega')$, there exists a maximal subgroup $\pi_1(\tilde{\Omega}) \subseteq N \subsetneqq \pi_1(\Omega')$ and hence a surjection $\pi_1(\Omega') \to G = \pi_1(\Omega')/N$ to an abelian group with $\pi_1(\tilde{\Omega})$ mapping to zero. This surjection is non-zero and lies in the kernel of

$$\pi^*\colon H^1(\Omega'; G) = \mathrm{Hom}(\pi_1(\Omega'), G) \to \mathrm{Hom}(\pi_1(\tilde{\Omega}), G) = H^1(\tilde{\Omega}; G)$$

for the abelian group $G = \pi_1(\Omega')/N$, contradicting the injectivity of π^*.

It follows that $\pi_1(\tilde{\Omega}) = \pi_1(\Omega')$. As before, this implies that π is a degree 1 covering map, and hence a biholomorphism. □

Solvability would not be enough, as shown by the example of $\mathbb{Z}/2\mathbb{Z} \subset \mathfrak{S}_3$. This would not only obstruct the proof given above, but would actually lead to a counter-example to the version of the statement, with solvable in place of nilpotent.

Example 4. Recall that Artin's braid group on three strands, denoted B_3, is the fundamental group of the complement of the braid arrangement A_2 in $\mathbb{C}^3$, the union of the three hyperplanes defined by $(y-x)(z-x)(z-y) = 0$. Quotienting by the small diagonal, the line $x = y = z = 0$, B_3 is the fundamental group of $\Omega' \subset \mathbb{C}^2$, the complement of the union of three lines through the origin in $\mathbb{C}^2$. Let $B_2 \subset B_3$ be a subgroup corresponding to two of the strands, so $B_2 \cong \mathbb{Z}$ has index 3 in B_3 and is not normal. There exists a covering space $\tilde{\Omega} \to \Omega'$ such that $\pi_1(X) = B_2 \subset B_3$. Since U is a Stein manifold, so is $\tilde{\Omega}$ [**9**]. By [**1**, Theorem 5], there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\tilde{\Omega}$. This is not schlicht, but for every abelian group G, $\mathrm{Hom}(B_3, G) \to \mathrm{Hom}(B_2, G)$ is injective.

More generally, let H be any finitely presented group. H is the fundamental group of a 2-complex, which may be embedded in $\mathbb{R}^5$, or, for that matter, $\mathbb{C}^3$; then, a tubular neighbourhood Ω' of this complex (in $\mathbb{C}^3$) still has $\pi_1(\Omega') = H$. Any subgroup $K \subset H$

https://doi.org/10.1017/S001309151300031X Published online by Cambridge University Press

occurs as the fundamental group of a covering space $\tilde{\Omega} \to \Omega'$. Again, $\tilde{\Omega}$ is Stein since Ω' is, and there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\tilde{\Omega}$.

It is not necessary to assume that i^* is injective when coefficients are taken in any abelian group G. It would be enough to assume that i^* is injective when coefficients are taken in any finite cyclic group, in any abelian quotient G of $\pi_1(\Omega')$ or even just in a single abelian quotient $G = \pi_1(\Omega')/N$ for some proper normal subgroup N containing $\pi_1(\tilde{\Omega})$.

If, in addition, $\pi\colon \tilde{\Omega} \to \Omega'$ is a normal covering space, then $\pi_1(\tilde{\Omega}) \subseteq \pi_1(\Omega')$ is a normal subgroup and we can take G to be an abelian quotient of $\pi_1(\Omega')/\pi_1(\tilde{\Omega})$, which is the group of deck transformations.

Corollary 5. *Suppose π is a normal covering map with deck transformation group H. If there exists a non-zero abelian quotient G of H such that $i^*\colon H^1(\Omega';G) \to H^1(\Omega;G)$ is injective, then $(\tilde{\Omega}, \pi)$ is schlicht.*

Acknowledgements. The author thanks Chris Hammond for explaining his theorem, and Emil Straube, Craig Westerland and Jens Harlander for helpful and patient conversations.

References

1. J. E. Fornæss and W. R. Zame, Riemann domains and envelopes of holomorphy, *Duke Math. J.* **50** (1983), 273–283.
2. C. Hammond, Schlicht envelopes of holomorphy and topology, *Math. Z.* **266** (2010), 285–288.
3. A. Hatcher, *Algebraic topology* (Cambridge University Press, 2002).
4. D. Jupiter, A schlichtness theorem for envelopes of holomorphy, *Math. Z.* **253** (2006), 623–633.
5. H. Kerner, Überlagerungen und Holomorphiehüllen, *Math. Annalen* **144** (1961), 126–134.
6. J. R. Munkres, *Elements of algebraic topology* (Addison-Wesley, 1984).
7. J. J. Rotman, *An introduction to the theory of groups*, 4th edn, Graduate Texts in Mathematics, Volume 148 (Springer, 1995).
8. H. L. Royden, One-dimensional cohomology in domains of holomorphy, *Annals Math.* **78** (1963), 197–200.
9. K. Stein, Überlagerungen holomorph-vollständiger komplexer Räume, *Arch. Math.* **7** (1956), 354–361.