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A GROWTH-FRAGMENTATION MODEL CONNECTED TO THE RICOCHETED STABLE PROCESS

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Abstract

Growth-fragmentation processes describe the evolution of systems in which cells grow slowly and fragment suddenly. Despite originating as a way to describe biological phenomena, they have recently been found to describe the lengths of certain curves in statistical physics models. In this note, we describe a new growth-fragmentation process connected to random planar maps with faces of large degree, having as a key ingredient the ricocheted stable process recently discovered by Budd. The process has applications to the excursions of planar Brownian motion and Liouville quantum gravity.

Keywords: random planar maps; statistical physics; Lévy process; stable process; self-similar Markov process

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1. Introduction

A random planar map is a random graph embedded in the plane. One approach to understanding features of random planar maps is to explore them, starting from a distinguished 'root face' and sequentially discovering adjoining faces at random. Part-way through such an exploration, the boundary of the explored region forms a set of cycles of vertices, and subsequent exploration may either enlarge certain cycles by adding newly discovered vertices, or cause them to split apart as a face is discovered which cuts across one. Bertoin et al. [6] considered exploring a Boltzmann random planar map with faces of large degree and, in a scaling limit as the degree of the root face grows, they found and characterised a growth-fragmentation process which describes the growth and splitting of perimeters of these cycles.

Denote Bertoin et al.'s growth-fragmentation process by $\mathbf{Z}^* = (\mathbf{Z}^*(t)\colon t \geq 0)$, where $\mathbf{Z}^*(t) = (Z_1^*(t), Z_2^*(t), \ldots)$ describes the collection of cell 'masses' (rescaled cycle perimeters) at time t, listed in decreasing order. The cell masses $\mathbf{Z}^*$ have both upwards and downwards jumps. Whenever a cell of $\mathbf{Z}^*$ jumps down in mass from u to ux (with $0 < x < 1$), an additional cell of mass $u(1-x)$ is added to the system (thus conserving mass at downward jumps, i.e. fragmentation events). However, when a cell mass jumps up, no additional cells are added.

In this work, we consider the effect of additionally introducing cells at the *upward* jump times of cells in $\mathbf{Z}^*$. If a cell jumps in mass from u to ux, where $x > 1$, then with probability r we introduce a new cell of mass $u(x-1)$. This new growth-fragmentation will be denoted $\mathbf{Z}$.

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In terms of the exploration of a Boltzmann map, the downward jumps of cell masses in $\mathbf{Z}^*$ come from splitting of cycles, whereas the upward jumps represent the discovery of faces of macroscopically large degree. Heuristically, we may think of each new cell arising from the upward jumps in $\mathbf{Z}$ as corresponding to the discovery, with probability r, of a further conditionally independent Boltzmann map embedded within a face of the original map.

In [9], Budd studied the metric properties of random planar maps coupled with an $O(n)$ loop model. Such maps can be decomposed along the loops to form a *gasket*, which is another random planar map of the type studied by [6], together with conditionally independent submaps to be inserted into certain faces of the gasket. It was found that the scaling limit of perimeters of certain distinguished cycles in such maps is related to the so-called *ricocheted stable process*. This is a perturbation of the stable process in which, at each attempted jump below zero, the process is 'ricocheted' with some probability and restarts from a positive value.

The main result of this note, Theorem 1 in Section 4, is that the ricocheted stable process and its conditionings are intimately connected to the process $\mathbf{Z}$, and that the ricochet probability is proportional to the probability r of discovering a new cell at each upward jump. This is natural, given the similarity in the descriptions of the growth-fragmentation and random planar map model above, but the connection does not appear to be known. A rigorous description of the appearance of this process in the scaling limit of loop-decorated random maps may be beyond reach at present, so we confine ourselves here to discussion of the growth-fragmentation processes themselves.

As it turns out, the same process $\mathbf{Z}$ can be used to describe the excursions of a planar Brownian motion, as explored by [1]. We discuss this in more detail in Section 4, and give an application to functionals of small excursions and, in the general setting, large cells.

Similarly, the process $\mathbf{Z}^*$ arises in the context of Liouville quantum gravity [19], where the idea of introducing new cells at the upward jumps is also discussed, giving rise to the tree structure denoted there by $\tilde{\mathcal{T}}$. This tree can be viewed as representing the genealogy of the growth-fragmentation $\mathbf{Z}$.

2. Ricocheted stable processes

We begin with a precise description of the ricocheted stable process, which is a self-similar Markov process derived from a Lévy process. Lévy processes are real-valued stochastic processes with stationary, independent increments and initial value 0. A Lévy process X can be characterised by its Laplace exponent ψ, which is defined by the relation $\mathbb{E}[\mathrm{e}^{qX_t}] = \mathrm{e}^{t\psi(q)}$ and satisfies the *Lévy–Khintchine formula*

$$\psi(q) = \frac{1}{2}\sigma^2 q^2 + aq + \int_{\mathbb{R}} (\mathrm{e}^{qy} - 1 - qy\mathbf{1}_{\{|y|\leq 1\}})\,\Pi(\mathrm{d}y),$$

where $\sigma \geq 0$, $a \in \mathbb{R}$, and Π is a measure (the Lévy measure) on $\mathbb{R}\setminus\{0\}$ satisfying $\int_{\mathbb{R}} \min\{y^2, 1\}\,\Pi(\mathrm{d}y) < \infty$. In general, ψ is finite, at least for $q \in \mathrm{i}\mathbb{R}$, but in many cases it is finite even for some neighbourhood of 0 in $\mathbb{C}$. A Lévy process with initial value 0 can be started from any $x \in \mathbb{R}$ by considering $t \mapsto X_t + x$.

Consider a stochastic process X with associated probability measures $(\mathbb{P}_x)_{x\in E}$, for some state space E which is closed under multiplication by positive scalars, where $\mathbb{P}_x(X_0 = x) = 1$. We say that X satisfies the *scaling property* with index θ if, for any $x \in E$ and any $c > 0$,

$$(cX_{tc^{-\theta}})_{t\geq 0} \text{ under } \mathbb{P}_x \text{ has the same distribution as } X \text{ under } \mathbb{P}_{cx}. \tag{1}$$

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Let parameters θ, ρ be chosen from the set

$$\mathcal{A}_{\theta,\rho} = \{(\theta, \rho)\colon \theta \in (0, 1), \rho \in (0, 1)\} \cup \{(\theta, \rho)\colon \theta \in (1, 2); \theta\rho, \theta(1-\rho) \in (0, 1)\} \cup \{(\theta, \rho) = (1, 1/2)\}.$$

Then, the function

$$\psi_{\theta,\rho}(q) = -c|q|^{\theta}\big(1 - \mathrm{i}\beta \tan(\pi\theta/2)\mathrm{sgn}(q/\mathrm{i})\big), \qquad q \in \mathrm{i}\mathbb{R}, \tag{2}$$

where $\beta = \tan(\pi\theta(\rho - 1/2))/\tan(\pi\theta/2)$, is the Laplace exponent of a Lévy process, known as a *stable process* with index θ (or a θ-*stable process*) and positivity parameter ρ.

Stable processes fulfil the scaling property (1) with $E = \mathbb{R}$. Indeed, every Lévy process satisfying the scaling property has Laplace exponent of form (2), up to a multiplicative factor and with a few exceptions: our parameter restrictions mean that in this work we neglect symmetric Cauchy processes with drift ($\theta = 1,\ \rho \neq 1/2$), Brownian motion ($\theta = 2$), processes with monotone paths ($\theta \in (0, 1)$, $\rho = 0$ or $\rho = 1$), and other processes which jump only in one direction ($\theta \in (1, 2)$, $\theta\rho = 1$ or $\theta(1-\rho) = 1$).

We are now in a position to define the ricocheted stable process, as detailed in [9, 17]. Let $x > 0$ and take a stable process X under $\mathbb{P}_x$ with index θ and positivity parameter ρ. Take an additional parameter $\mathfrak{p} \in [0, 1]$, the *ricochet probability*. Define the first passage time below zero: $\tau = \inf\{t \geq 0\colon X_t < 0\}$. We define a new process Y, which follows X up until time τ. At time τ, with probability $1 - \mathfrak{p}$, Y is killed (i.e. sent to the absorbing state 0), and with probability $\mathfrak{p}$ it moves not to X_τ but to $Y_\tau = -X_\tau$; the jump of the stable process below zero is 'ricocheted' back above the origin. Subsequently, Y follows a stable process under $\mathbb{P}_{-X_\tau}$, until the first passage time of this process below zero, at which point the probabilistic ricochet occurs again. This iteratively constructs an entire path of the process Y, up until possibly reaching the point 0, at which point Y is killed.

The process Y is called the *ricocheted stable process*. It is an example of a *positive, self-similar Markov process* (pssMp). In general, a process is called a pssMp with index θ if it is a standard Markov process with state space $[0, \infty)$, has 0 as an absorbing state, and satisfies the scaling property (1).

An important property of pssMps, which explains their usefulness and will allow us to explore the ricocheted stable process in more detail, is that they are in bijection with Lévy processes via the *Lamperti transformation* [15, Chapter 13]. For a pssMp Y with index θ, let $S(t) = \int_0^t (Y_u)^{-\theta}\,\mathrm{d}u$, and define T as the inverse of S. Then, the process $\xi_s = \log Y_{T(s)}$, $s \geq 0$, is a Lévy process (possibly killed and sent to $-\infty$), the Lamperti transformation of Y.

The Lamperti transformations of the pssMp given by the stable process killed at first passage below zero, and conditionings thereof, are well known [13], and belong to the class of 'hypergeometric' Lévy processes. This identifiability extends to ricocheted stable processes as well [9, 17]: the Lamperti transformation of the ricocheted stable process Y is the Lévy process ξ with Laplace exponent

$$\psi(q) = -2^{\theta}\frac{\Gamma\left(\frac{\theta-q}{2}\right)\Gamma\left(\frac{1+\theta-q}{2}\right)}{\Gamma\left(\frac{b-\sigma-q}{2}\right)\Gamma\left(\frac{2-\sigma-b-q}{2}\right)}\frac{\Gamma\left(\frac{1+q}{2}\right)\Gamma\left(\frac{2+q}{2}\right)}{\Gamma\left(\frac{\sigma+b+q}{2}\right)\Gamma\left(\frac{2+\sigma-b+q}{2}\right)}, \qquad q \in (-1, \theta),$$

where $\sigma = \frac{1}{2} - \theta(1-\rho)$ and $b = (1/\pi)\arccos(\mathfrak{p}\cos(\pi\sigma))$.

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We will say that Y is a *ricocheted stable process with parameters* (θ, σ, b). If we specify the set of admissible parameters

$$\begin{aligned}\mathcal{A}_{\theta,\sigma,b} = &\{(\theta,\sigma,b)\colon \theta\in(0,1),\ \sigma\in(1/2-\theta,1/2),\ b\in[|\sigma|,1/2]\}\\ &\cup\{(\theta,\sigma,b)\colon \theta\in(1,2),\ \sigma\in(-1/2,3/2-\theta),\ b\in[|\sigma|,1/2]\}\\ &\cup\{(\theta,\sigma,b)\colon \theta=1,\sigma=0,b\in[0,1/2]\},\end{aligned}$$

then these are in one-to-one correspondence with the admissible parameters θ, ρ, and $\mathfrak{p}$.

The Lamperti transform of Y has other nice properties. Its Wiener–Hopf factorisation is known, and it is an example of a *double hypergeometric Lévy process* [17].

Note that when $\rho = \frac{1}{2}$, meaning that X is symmetric, the ricocheted stable process is equal to $|X|$, the absolute value of X, with additional probability of killing at the times when X changes sign. When $\mathfrak{p} = 0$, Y is the stable process killed upon first passage below zero.

3. A novel growth-fragmentation

$\mathbf{Z}^*$ is an example of a *Markovian growth-fragmentation*, in the sense of [4]. These processes can be described more formally as follows. For each $x > 0$, we construct a process, under the probability measure $\mathbb{P}_x$, starting from a single cell of mass x. For $u \in \mathbb{U} = \bigcup_{n\geq 0}\mathbb{N}^n$, the set of Ulam–Harris labels, we define processes $\mathcal{X}_u = (\mathcal{X}_u(t))_{t\geq 0}$ as follows. The process $\mathcal{X}_\varnothing$ is a Markov process with state space $[0,\infty)$ and with prescribed distribution, known as the *cell process*, which we interpret as the mass of the initial cell, such that $\mathcal{X}_\varnothing(0) = x$. We assign $b_\varnothing = 0$, its birth time. If we have constructed $\mathcal{X}_u$, then we list the jump times and jump ratios $\{(t_i, x_i)\colon i \geq 1\}$ in some appropriate order, such as in decreasing order of jump ratio; recall here that if $\mathcal{X}_u$ jumps at time t_i, the jump ratio is $x_i = \mathcal{X}_u(t_i)/\mathcal{X}_u(t_i-)$. A *selection probability* $c\colon [0,\infty)\to[0,1]$ determines which jumps lead to additional cells. For each $i \geq 1$, with probability $c(x_i)$ we define the process $\mathcal{X}_{ui}$ to be equal in distribution to the cell process, started from $\mathcal{X}_{ui}(0) = |\mathcal{X}_u(t_i) - \mathcal{X}_u(t_i-)| = |\mathcal{X}_u(t_i-)(x_i-1)|$, and let $b_{ui} = b_u + t_i$. (On the complementary event with probability $1 - c(x_i)$, no new process is defined.) The Markovian growth-fragmentation process at time $t \geq 0$, $\mathbf{X}(t)$, is then given by the decreasing rearrangement $(X_1(t), X_2(t), \dots)$ of $(\mathcal{X}_u(t-b_u)\colon u\in\mathbb{U})$. For clarity, we will call $\mathcal{X} = (\mathcal{X}_u\colon u\in\mathbb{U})$ the generational growth-fragmentation associated with $\mathbf{X}$.

When the cell process in a Markovian growth-fragmentation is a pssMp of index θ, we say that the growth-fragmentation itself is *self-similar*, with *index* $\alpha = -\theta$. This is the case for the process $\mathbf{Z}^*$ found by [6], parametrised by $\theta\in(1/2,3/2]$, in which the cell process is a pssMp of index θ and can be characterised by its Lamperti transformation, which is a Lévy process whose Laplace exponent is given [6, p. 702] by

$$\psi^*(q) = aq + \int_{(-\log 2,\infty)} (\mathrm{e}^{qy} - 1 + q(1-\mathrm{e}^y))\,\Lambda(\mathrm{d}y),$$

where

$$a = \frac{\Gamma(2-\theta)}{2\Gamma(2-2\theta)\sin(\pi\theta)} + \frac{\Gamma(\theta+1)B_{1/2}(-\theta,2-\theta)}{\pi},$$

with B the incomplete beta function, and $\Lambda = \nu\circ\log^{-1}$, with ν defined by

$$\begin{aligned}\nu(\mathrm{d}x) = \frac{\Gamma(\theta+1)}{\pi}\big((x(1-x))^{-(\theta+1)}\mathbf{1}_{\{1/2<x\leq 1\}}\\ + \sin\left(\pi\left(\theta-\tfrac{1}{2}\right)\right)(x(x-1))^{-(\theta+1)}\mathbf{1}_{\{x>1\}}\big)\,\mathrm{d}x,\end{aligned}$$

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known as the *dislocation measure* of $\mathbf{Z}^*$. In the case of $\mathbf{Z}^*$, the selection probability is given by $c(x) = \mathbf{1}_{\{x<1\}}$, meaning that only negative jumps of the cell masses lead to the introduction of new cells.

A self-similar growth-fragmentation $\mathbf{X}$ can be characterised [21, Theorem 1.2] by its index α and its *cumulant*, which is defined (see [6, (10)]) by the formula

$$\mathbb{E}_1\left[\sum_{u\in\mathbb{U},|u|=1} \mathcal{X}_u(0)^q\right] = 1 - \frac{\kappa(q)}{\psi(q)} \tag{3}$$

for those q where the left-hand side is finite; where ψ is the Laplace exponent belonging to the Lamperti transform of $\mathbf{X}$'s cell process. Recall that $|u|$ is the generation of u; in other words, the length of the word u. It is known that κ can be expressed (see [6, (5)]) in the form $\kappa(q) = \psi(q) + \int_{(0,\infty)} c(x)|1-x|^q\, \nu(\mathrm{d}x)$.

In general, if $\omega \in (\text{dom } \kappa)^\circ$ is such that $\kappa(\omega) \le 0$, then the stochastic process $\mathcal{M}_\omega(n) = \sum_{u\in\mathbb{U},\,|u|=n+1} \mathcal{X}_u(0)^\omega$, $n \ge 0$, is a supermartingale in the filtration $\mathcal{G}_n = \sigma(\mathcal{X}_u,\, |u| \le n)$; and if $\kappa(\omega) = 0$, then it is a martingale. This is shown in [6, Section 2.3], and can be seen from (3), recalling that $\psi < \kappa$. We define a change of measure

$$\left.\frac{\mathrm{d}\mathbb{P}_x^\omega}{\mathrm{d}\mathbb{P}_x}\right|_{\mathcal{G}_n} = x^{-\omega}\mathcal{M}_\omega(n),$$

and additionally single out a distinguished cell $U = (U(n)\colon n \ge 0)$ under $\mathbb{P}_\cdot^\omega$ by

$$\mathbb{P}_x^\omega(U(n+1) = u \mid \mathcal{G}_n) = \frac{\mathcal{X}_u(0)^\omega}{\mathcal{M}_\omega(n)}, \qquad u \in \mathbb{U},\ |u| = n+1.$$

If we now define, by a slight abuse of notation, $U(t)$ to be the unique value of $U(n)$ such that $b_{U(n)} \le t < b_{U(n+1)}$, then the (sub-Markov) process $Y_\omega(t) = \mathcal{X}_{U(t)}(t - b_{U(t)})$, $t \ge 0$, plays a special role, and is called the *spine*. It is a pssMp with index $-\alpha$ whose Lamperti transformation has Laplace exponent $q \mapsto \kappa(\omega + q)$.

In the case of $\mathbf{Z}^*$, let us denote its cumulant by κ^*, an explicit formula for which was found in [6, (19)]:

$$\kappa^*(q) = -\Gamma(1+2\theta-q)\Gamma(q-\theta)\frac{\cos(\pi(\theta+1-q))}{\pi} = -\frac{\Gamma(1+2\theta-q)\Gamma(q-\theta)}{\Gamma\left(\frac{3}{2}+\theta-q\right)\Gamma\left(-\frac{1}{2}-\theta+q\right)},$$

for $q \in (\theta,\, 1+2\theta)$. If we define

$$\omega_- = \theta + 1/2, \qquad \omega_0 = \theta + 1, \qquad \omega_+ = \theta + 3/2,$$

then $\kappa^*(\omega_-) = \kappa^*(\omega_+) = 0$, and $\kappa^*(\omega_0) < 0$, and the corresponding spines are identifiable. $Y^*_{\omega_0}$ is a stable process with parameters θ, ρ such that $\theta(1-\rho) = \frac{1}{2}$, killed upon going below zero; $Y^*_{\omega_-}$ is the same process conditioned to hit zero continuously in the sense of [11]; and $Y^*_{\omega_+}$ is the process conditioned to stay positive in the sense of [18]. The spines corresponding to $\omega_\pm$ were exploited in [6] to describe Boltzmann maps.

Our novel growth-fragmentation process is obtained by additionally introducing cells at the upward jump times of cells in $\mathbf{Z}^*$. When a cell jumps in mass from u to ux, where $x > 1$, we introduce a new cell of mass $u(x-1)$ with probability r. Write $\mathbf{Z}$ for this process. Mathematically, the process $\mathbf{Z}$ is a Markovian growth-fragmentation with the same cell process

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as $\mathbf{Z}^*$, but with selection probability $c(x) = \mathbf{1}_{\{x<1\}} + r\mathbf{1}_{\{x>1\}}$, meaning that all downward jumps lead to daughter cells, and upward jumps lead to daughter cells with probability r.

In principle, the process $\mathbf{Z}$ may not be well defined, in that it may not be possible to rank its elements in descending order. However, this can be done provided that it is *locally finite*, meaning that, at all times and for all compact sets $K \subset (0, \infty)$, the number of cells with size in K is finite.

Our work in the next section will be to prove that $\mathbf{Z}$ is locally finite, establish its cumulant, and characterise its spines.

Remark 1.

(i) Our description of Markovian growth-fragmentations is a minor elaboration on the setting of [4, 21], in which only cell processes with negative jumps were considered, and the selection probability was absent. These changes do not affect the arguments there.

(ii) Unlike in $\mathbf{Z}^*$, there is no preservation of total mass (i.e. ℓ^1-norm) of $\mathbf{Z}$ at fragmentation events. The cells described above could be thought of as cells of negative initial mass $u(1-x)$ (which would conserve mass) 'ricocheted' back above the origin. We will still refer to $\mathbf{Z}$ as a growth-fragmentation, though perhaps it would be equally appropriate to call it a branching self-similar Markov process, à la Bertoin and Mallein [5].

(iii) The process $\mathbf{Z}^*$ is denoted $\mathbf{X}_\theta^{(-\theta)}$ in [6]. That work also identifies the growth-fragmentation arising from the cycle perimeters of the metric balls in a Boltzmann map. The two processes are very similar, and differ only by a stopping-line time change which shifts the self-similarity index from $\alpha = -\theta$ to $\alpha = 1-\theta$.

4. Results and proofs

Theorem 1. *The process* $\mathbf{Z}$ *is locally finite, is self-similar of index* $-\theta$*, and has cumulant given by*

$$\kappa(q) = -2^\theta \frac{\Gamma\left(\frac{1+2\theta-q}{2}\right)\Gamma\left(\frac{2+2\theta-q}{2}\right)}{\Gamma\left(\frac{1+b+\theta-q}{2}\right)\Gamma\left(\frac{3-b+\theta-q}{2}\right)} \frac{\Gamma\left(\frac{-\theta+q}{2}\right)\Gamma\left(\frac{1-\theta+q}{2}\right)}{\Gamma\left(\frac{1-b-\theta+q}{2}\right)\Gamma\left(\frac{b-1-\theta+q}{2}\right)}, \qquad q \in (\theta, 1+2\theta),$$

where $b \in \left(0, \frac{1}{2}\right]$ *is chosen so that* $r \sin\left(\pi\left(\theta - \frac{1}{2}\right)\right) = \cos(\pi b)$*. Define*

$$\omega_- = \theta + 1 - b, \qquad \omega_0 = \theta + 1, \qquad \omega_+ = \theta + 1 + b.$$

Then, ω_- *and* ω_+ *are the unique zeroes of* κ*, and* $\kappa(q) < 0$ *for* $q \in (\omega_-, \omega_+)$*.*

The spine Y_{ω_0} *corresponding to* ω_0 *is the ricocheted stable process with parameters* $(\theta, 0, b)$*; that is,* $\theta(1-\rho) = \frac{1}{2}$ *and* $\mathfrak{p} = r \sin\left(\pi\left(\theta - \frac{1}{2}\right)\right)$*. The spine* Y_{ω_-} *is the same process conditioned to reach zero continuously in the sense of [11], and the spine* Y_{ω_+} *is the process conditioned to avoid zero in the sense of [18].*

Proof. The process $\mathbf{Z}$, if it is indeed locally finite, has cumulant

$$\kappa(q) = \kappa^*(q) + r\int_1^\infty (x-1)^q \, \nu(\mathrm{d}x),$$

where ν is the dislocation measure of $\mathbf{Z}^*$.

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Using a beta integral we find that

$$\int_1^\infty (x-1)^{q-(\theta+1)} x^{-(\theta+1)}\,\mathrm{d}x = \frac{\Gamma(q-\theta)\Gamma(1-q+2\theta)}{\Gamma(1+\theta)}, \qquad q \in (\theta, 1+2\theta),$$

whence

$$\begin{aligned}\kappa(q) &= \Gamma(1+2\theta-q)\Gamma(q-\theta)\frac{\cos(\pi b) - \cos(\pi(\theta+1-q))}{\pi} \\ &= -\frac{2}{\pi}\Gamma(1+2\theta-q)\Gamma(q-\theta)\sin\left(\pi\left(\frac{1+b+\theta-q}{2}\right)\right)\sin\left(\pi\left(\frac{b-1-\theta+q}{2}\right)\right) \\ &= -2\pi\frac{\Gamma(1+2\theta-q)\Gamma(q-\theta)}{\Gamma\left(\frac{1+b+\theta-q}{2}\right)\Gamma\left(\frac{3-b+\theta-q}{2}\right)\Gamma\left(\frac{1-b-\theta+q}{2}\right)\Gamma\left(\frac{b-1-\theta+q}{2}\right)}, \end{aligned} \tag{4}$$

using product-sum identities and the reflection formula. The expression for κ given in the statement then follows by applying the Legendre duplication formula to the gamma functions in the numerator.

Treated as a meromorphic function, κ has poles at

$$\begin{aligned}\rho_k &= 2\theta+1+k, && k=0,1,2,\ldots,\\ \hat{\rho}_k &= \theta-k, && k=0,1,2,\ldots,\end{aligned}$$

and zeroes at

$$\begin{aligned}\zeta_k^{(1)} &= b+\theta+2k+1, && k=0,1,2,\ldots,\\ \zeta_k^{(2)} &= \theta-b+2k+3, && k=0,1,2,\ldots,\\ \hat{\zeta}_k^{(1)} &= b+\theta-(2k+1), && k=0,1,2,\ldots,\\ \hat{\zeta}_k^{(2)} &= \theta-b-(2k-1), && k=0,1,2,\ldots.\end{aligned}$$

Comparing these, we see that $\omega_- = \hat{\zeta}_0^{(2)}$ and $\omega_+ = \zeta_0^{(1)}$ are the only zeroes of κ within its domain $(\theta, 1+2\theta)$.

We address the local finiteness of $\mathbf{Z}$. For this, it is enough to assume $r=1$, since other values of r can be obtained by randomly removing cells from this one. Since there are values of q such that $\kappa(q) \le 0$, [4, Lemma 2] implies that $x \mapsto x^q$ is excessive for the cell process X of $\mathbf{Z}$, in the sense that $\mathbb{E}_x\big[X_t^q + \sum_{0<s\le t}|\Delta X_s|^q\big] \le x^q$. In turn, [4, Theorem 1] implies that $\mathbf{Z}$ is locally finite. (These results are proven under the assumption that the cell process only jumps downward, but their statements remain valid in our more general setting, if we use the above definition of excessiveness.)

The claim about the spine processes follows by comparing $q \mapsto \kappa(q+\omega_0)$ with the description of the ricocheted stable process in Section 2. As shown in [11, 18] and summarised in [15, Section 13.4.4], the conditionings of the ricocheted stable process to reach zero continuously and avoid zero have Lamperti transformation with Laplace exponent $q \mapsto \kappa(q+\omega_-)$ and $q \mapsto \kappa(q+\omega_+)$, respectively. □

The relation $\mathfrak{p} = r\sin\pi\left(\theta - \frac{1}{2}\right)$ reflects the differing rates of ricochets, which occur as a result of upward jumps in the growth-fragmentation and downward jumps in the ricocheted stable process.

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The expression of κ as a ratio of eight gamma functions, instead of as formula (4), may seem complex, but it makes much clearer the Wiener–Hopf factorisations of the spine processes. For instance, Y_{ω_0} is a pssMp whose Lamperti transformation has Laplace exponent

$$\psi(q) = -2^{\theta} \frac{\Gamma\left(\frac{\theta-q}{2}\right)\Gamma\left(\frac{1+\theta-q}{2}\right)}{\Gamma\left(\frac{2-b-q}{2}\right)\Gamma\left(\frac{b-q}{2}\right)} \times \frac{\Gamma\left(\frac{1+q}{2}\right)\Gamma\left(\frac{2+q}{2}\right)}{\Gamma\left(\frac{2-b+q}{2}\right)\Gamma\left(\frac{b+q}{2}\right)},$$

and the two functions on either side of the $\times$ represent its Wiener–Hopf factors [17].

When $\theta \to \frac{1}{2}$ or $\theta = \frac{3}{2}$, b takes the value $\frac{1}{2}$ and κ approaches the cumulant of the Brownian fragmentation [2, p. 338] or the exploration of a Boltzmann triangulation [7], respectively.

A particularly relevant situation appears in the following result, part of which was observed, using properties of conformal loop ensembles rather than the explicit description of $\mathbf{Z}$, in [19, p. 33].

Corollary 1. *When $r = 1$, $b = |\theta - 1|$. In the case $\theta > 1$, we have $\omega_- = 2$ and $\omega_+ = 2\theta$; in the case $\theta \leq 1$, we have $\omega_- = 2\theta$ and $\omega_+ = 2$.*

Specialising still further, we have a relation with planar Brownian motion.

Corollary 2. *When $\theta = 1$ and $r = 1$, the cell process of $\mathbf{Z}$ is the same as that found in [1], where it describes the size of excursions in a planar Brownian motion. The process $\mathbf{Z}$ itself can be obtained from the process $\overline{\mathbf{X}}$ in that work by taking absolute values of the cell 'masses'. Its cumulant is given by*

$$\kappa(q) = -2\frac{\Gamma\left(\frac{3-q}{2}\right)\Gamma\left(\frac{q-1}{2}\right)}{\Gamma\left(\frac{2-q}{2}\right)\Gamma\left(\frac{q-2}{2}\right)}, \qquad q \in (1, 3).$$

Denote by $\mathcal{Z}$ the generational growth-fragmentation associated with $\mathbf{Z}$. In this case, $\omega_- = \omega_0 = \omega_+ = 2$, the process $\mathcal{M}_2(n) = \sum_{u\in\mathbb{U},|u|=n+1} \mathcal{Z}_u(0)^2$, $n \geq 0$, is a martingale, and the corresponding spine is the absolute value of a symmetric Cauchy process.

The second part of this corollary provides a proof of [1, Proposition 3.3], once the identification of the growth-fragmentations is known. However, the process $\overline{\mathbf{X}}$ constructed in that work contains more information than the approach using $\mathbf{Z}$; the ricochet step loses some information about the 'sign' of excursions. This information is preserved (and the setting extended to cover excursions involving stable processes) by Da Silva's framework of signed growth-fragmentations [12]. It must also be said that ricocheted processes are not strictly required for this application: when $\theta = 1$ we are in a symmetric setting, so $\mathbf{Z}$ can be described in terms of the absolute value of a Cauchy process [10, 14].

Using this description, it is possible to learn something about the appearance of small excursions. In the generational growth-fragmentation $\mathcal{Z}$, the cells u describe excursions in the upper half-plane above some barrier, represented by time. For each $u \in \mathbb{U}$ and $t \geq 0$, let $u(t)$ be the most recent ancestor of u which was alive at time t (which may be u itself). If we define $T_u^-(\epsilon) = \inf\{t \geq 0 \colon \mathcal{Z}_{u(t)}(t - b_{u(t)}) < \epsilon\}$, $u \in \mathbb{U}$, then

$$\mathcal{Z}(T^-(\epsilon)) := \left(\mathcal{Z}_{u(T_u^-(\epsilon))}\left(T_u^-(\epsilon) - b_{u(T_u^-(\epsilon))}\right) : u \in \mathbb{U},\, u(T_u^-(\epsilon)) = u\right)$$

represents the collection of excursions frozen at the first time that their size drops below ϵ. (The condition '$u(T_u^-(\epsilon)) = u$' avoids multiple counting.) Additive functionals of the set can be described using the growth-fragmentation.

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Corollary 3. *Let $\theta = 1$ and $r = 1$. For an integrable function f and $\epsilon > 0$,*

$$\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^-(\epsilon))} f(z)z^2\right] = \int_0^\epsilon \frac{2}{\pi}\frac{1}{1-y^2}\sqrt{\frac{1-\epsilon^2}{\epsilon^2-y^2}}f(y)\,\mathrm{d}y.$$

Proof. Using a version of [6, Proposition 4.1], and writing $g(z) = f(z)z^2$, we obtain

$$\begin{aligned}\mathbb{E}_1&\left[\sum_{u\in\mathbb{U},u(T_u^-(\epsilon))=u} g(\mathcal{Z}_u(T_u^-(\epsilon)) - b_{u(T_u^-(\epsilon))})\right]\\ &= \mathbb{E}_1\left[\sum_{u\in\mathbb{U}} g(\mathcal{Z}_u(t))\mathbf{1}_{\{\forall s\le t:\ \mathcal{Z}_{u(s)}(s)\ge\epsilon\}}\mathbf{1}_{\{\mathcal{Z}_u(t)<\epsilon\}}\right]\\ &= \mathbb{E}_1^{\omega_0}\left[g(Y_{T^-(\epsilon)})Y_{T^-(\epsilon)}^{-2}\right] = \mathbb{E}_1^{\omega_0}\left[f(Y_{T^-(\epsilon)})\right],\end{aligned}$$

where Y is the spine process corresponding to exponent $\omega_0 = 2$, and $T^-(\epsilon)$ is its first passage time below ϵ. Since Y is the absolute value of a symmetric Cauchy process, the result follows using the hitting distribution in [8]. □

We can also observe a variant of [3, Proposition 1.12]:

$$\lim_{\epsilon\downarrow 0}\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^-(\epsilon))} f(z/\epsilon)z^2\right] = \int_0^1 \frac{2f(y)\,\mathrm{d}y}{\pi\sqrt{1-y^2}}.$$

In principle, it is possible to prove similar results for the process $\mathbf{Z}^*$, using [20], or $\mathbf{Z}$ with $\theta \neq 1$, using [17, Theorem 2.1]. The latter is probably the most interesting for this note, but the analogue of the right-hand side in Corollary 3 involves an integral over a product of sums of hypergeometric functions, and no simplification appears to be possible. However, by a remarkable coincidence of distributions, it is possible to obtain the first passage upward.

Define the first passage time $T_u^+(x) = \inf\{t \ge 0\colon \mathcal{Z}_{u(t)}(t - b_{u(t)}) > x\}$, $u \in \mathbb{U}$, of the size of cell u above level x, and $\mathcal{Z}(T^+(x))$ analogously with the downward passage case. We allow that if the size of a cell never exceeds level x before its death, it is omitted from $\mathcal{Z}(T^+(x))$. Additive functionals of the growth-fragmentation frozen as the cells exceed size x are given as follows.

Corollary 4. *Let $r = 1$, $x > 1$, and let f be an integrable function. Then*

$$\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^+(x))} f(z)z^2\right] = \int_x^\infty f(y)h_\theta(x,y)\,\mathrm{d}y$$

where, if $\theta \le 1$,

$$h_\theta(x,y) = 2\frac{\sin\pi(\theta-1/2)}{\pi}(x^2-1)^{\theta-1/2}(y^2-x^2)^{1/2-\theta}\frac{y}{y^2-1}, \qquad y > x,$$

and, if $\theta > 1$,

$$\begin{aligned}h_\theta(x,y) &= 2\frac{\sin\pi(\theta-1/2)}{\pi}(x^2-1)^{\theta-1/2}(y^2-x^2)^{1/2-\theta}\frac{y}{y^2-1}\\ &\quad - 2(\theta-1)\frac{\sin\pi(\theta-1/2)}{\pi}\frac{x}{y}(y^2-x^2)^{1/2-\theta}\int_1^{x^2}(t-1)^{\theta-3/2}t^{-1/2}\,\mathrm{d}t, \quad y > x.\end{aligned}$$

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Proof. We begin with the $\theta \le 1$ case, and recall first that when $r = 1$, we have $\omega_+ = 2$, and the corresponding spine is the ricocheted stable process conditioned to stay positive, which we will denote Y under measure $\mathbb{P}^{\omega_+}_{\cdot}$. Analogously with the proof of the preceding corollary, we can express

$$\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^+(x))} f(z)z^2\right] = \mathbb{E}_1^{\omega_+}[f(Y_{T^+(x)})],$$

where $T^+(x)$ is the first passage time of Y above x. The process Y can be studied via its Lamperti transform, say ξ, which has Laplace exponent

$$\psi_+(q) = \kappa(q+2) = -2^\theta \frac{\Gamma\left(\frac{2\theta-1-q}{2}\right)}{\Gamma\left(\frac{-q}{2}\right)} \frac{\Gamma\left(\frac{2-\theta+q}{2}\right)\Gamma\left(\frac{3-\theta+q}{2}\right)}{\Gamma\left(\frac{2+q}{2}\right)\Gamma\left(\frac{2-2\theta+q}{2}\right)}.$$

It is convenient to rescale time and space by defining $\xi'_t = 2\xi_{2^{-\theta}t}$, which has Laplace exponent $\psi'(q) = 2^{-\theta}\psi_+(2q)$. We see directly that $\mathbb{E}_1^{\omega_+}[f(Y_{T^+(x)})] = \mathbb{E}_0\left[f\left(\mathrm{e}^{H_{S^+(2\log x)/2}}\right)\right]$, where H is the ascending ladder height process of ξ' and $S^+(\cdot)$ its first passage time upward.

Examining ψ', we see that the Laplace exponent of the ascending ladder height process of ξ' is given by

$$q \mapsto \frac{\Gamma(\theta - 1/2 + q)}{\Gamma(q)}, \qquad q \ge 0.$$

This is identical to the ascending ladder height of the Lamperti transform of the path-censored stable process [16, Section 5.4], though we stress that the descending ladder height processes differ. This allows us to express

$$\mathbb{E}_0\left[f\left(\mathrm{e}^{H_{S^+(2\log x)/2}}\right)\right] = \mathbb{E}_1\left[f\left(X^{1/2}_{T^+(x^2)}\right)\right] = \mathbb{E}_{x^{-2}}\left[f\left(xX^{1/2}_{T^+(1)}\right)\right],$$

where X is the stable process with index θ and positivity parameter $1 - (1/2\theta)$, and $T^+(\cdot)$ is its first passage time upward. Putting together the pieces, we have

$$\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^+(x))} f(z)z^2\right] = \int_1^\infty f(xz^{1/2})g(z)\,\mathrm{d}z = \int_1^\infty f(y)2yx^{-2}g(y^2x^{-2})\,\mathrm{d}y,$$

where $g(y) = \mathbb{P}_{x^{-2}}(X_{T^+(1)} \in \mathrm{d}y)/\mathrm{d}y$. This quantity was calculated in [20] and appears as equation (3) in [16]. Substituting g completes the proof.

The proof for the case $\theta > 1$ is similar, with the key difference being that now $\omega_- = 2$, so we make use of the spine given by the ricocheted stable process conditioned to hit zero, with measures $\mathbb{P}^{\omega_-}_{\cdot}$, and obtain

$$\begin{aligned}\mathbb{E}_1\left[\sum_{z\in\mathcal{Z}(T^+(x))} f(z)z^2\right] &= \mathbb{E}_1^{\omega_-}\left[f(Y_{T^+(x)})\mathbf{1}_{\{T^+(x)<T^0\}}\right] \\ &= \mathbb{E}_{x^{-2}}\left[f(xX^{1/2}_{T^+(1)})\mathbf{1}_{\{T^+(1)<T^0\}}\right], \end{aligned} \tag{5}$$

where T^0 is the time at which Y (in the middle expression) or X (in the latter) hits zero. In deriving this equality, we follow the same proof structure, the difference being that we use [16, Section 5.5], corresponding to our value of θ. In order to evaluate the right-hand side of (5), we use [16, Theorem 1.5], and this gives the expression in the statement. □

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The preceding proof relies on the curious fact that, up to a space transformation, the ricocheted stable process conditioned to stay positive (when $\theta \leq 1$) or to hit zero (when $\theta > 1$) attains new maxima in the same way as the stable process (killed upon reaching zero). We are not aware of any pathwise explanation of this property.

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