Bull. Aust. Math. Soc. **98** (2018), 113–121
doi:10.1017/S0004972718000321

BOX DIMENSION OF BILINEAR FRACTAL INTERPOLATION SURFACES

QING-GE KONG, HUO-JUN RUAN✉ and SHENG ZHANG

(Received 10 March 2018; accepted 20 March 2018; first published online 30 May 2018)

Abstract

Bilinear fractal interpolation surfaces were introduced by Ruan and Xu in 2015. In this paper, we present the formula for their box dimension under certain constraint conditions.

2010 *Mathematics subject classification*: primary 28A80; secondary 41A30.

Keywords and phrases: bilinear fractal interpolation surfaces, box dimension, iterated function systems, vertical scaling factors.

1. Introduction

Fractal interpolation functions (FIFs for short) were introduced by Barnsley in 1986 [1]. Their graphs are invariant sets of certain iterated function systems which are effective in modelling natural curves [11]. In the classical case, the graphs of FIFs are self-affine sets, and their box dimension was obtained by Hardin and Massopust [8] and Barnsley *et al.* [2]. In 2015, Barnsley and Massopust constructed one-dimensional bilinear FIFs and obtained the box dimension of their graphs under certain conditions [3].

There are also results on the construction and the box dimension of FIFs on rectangular grids (see, for example, [4, 6, 7, 9, 10, 12]). Ruan and Xu [13] presented a general framework to generate FIFs on rectangular grids and also introduced bilinear fractal interpolation surfaces (bilinear FISs). However, it seems too complicated to obtain the box dimension of bilinear FISs if we use the method in [3]. In this paper, we will calculate the box dimension of bilinear FISs by estimating the oscillation of functions (see Theorem 3.8). Our method is more straightforward than that in [3], although the basic ideas are similar. We remark that our method can also be used to obtain the box dimension of the one-dimensional bilinear FIFs in [3].

The paper is organised as follows. In Section 2, we recall the definition of bilinear FISs. In Section 3, we present the formula for the box dimension of bilinear FISs.

The research was supported in part by the NSFC grants 11271327, 11771391 and by ZJNSFC grant LR14A010001.

2. Definition of bilinear FISs

We are given a data set $\{(x_i, y_j, z_{ij}) \in \mathbb{R}^3 : i \in \{0, 1, \dots, N\}, j \in \{0, 1, \dots, M\}\}$ with

$$x_0 < x_1 < \cdots < x_N, \quad y_0 < y_1 < \cdots < y_M,$$

where $N, M \geq 2$ are positive integers. Ruan and Xu [13] constructed bilinear FIFs on $[x_0, x_N] \times [y_0, y_M]$ as follows.

Denote $\Sigma_N = \{1, 2, \dots, N\}$, $\Sigma_{N,0} = \{0, 1, \dots, N\}$. Similarly, we can define Σ_M and $\Sigma_{M,0}$. Denote $I = [x_0, x_N]$ and $J = [y_0, y_M]$. For any $i \in \Sigma_N$ and $j \in \Sigma_M$, we denote $I_i = [x_{i-1}, x_i]$, $J_j = [y_{j-1}, y_j]$ and $D_{ij} = I_i \times J_j$.

For any $i \in \Sigma_N$, let $u_i : I \to I_i$ be the linear function satisfying

$$u_i(x_0) = x_{i-1}, \quad u_i(x_N) = x_i \quad \text{if } i \text{ is odd,}$$
$$u_i(x_0) = x_i, \quad u_i(x_N) = x_{i-1} \quad \text{if } i \text{ is even.}$$

Clearly, this implies that $u_1(x_0) = x_0$, $u_1(x_N) = u_2(x_N) = x_1$, $u_2(x_0) = u_3(x_0) = x_2$ and so on. Similarly, for any $j \in \Sigma_M$, let $v_j : J \to J_j$ be the linear function satisfying

$$v_j(y_0) = y_{j-1}, \quad v_j(y_M) = y_j \quad \text{if } j \text{ is odd,}$$
$$v_j(y_0) = y_j, \quad v_j(y_M) = y_{j-1} \quad \text{if } j \text{ is even.}$$

Let g be the bilinear function on $I \times J$ satisfying

$$g(x_i, y_j) = z_{ij} \quad \text{for all } (i, j) \in \{0, N\} \times \{0, M\}.$$

Equivalently,

$$g(x, y) = \frac{1}{(x_N - x_0)(y_M - y_0)}((x_N - x)(y_M - y)z_{0,0} + (x - x_0)(y_M - y)z_{N,0} + (x_N - x)(y - y_0)z_{0,M} + (x - x_0)(y - y_0)z_{N,M}).$$

Then we define $h : I \times J \to \mathbb{R}$ to be the function satisfying

$$h(x_i, y_j) = z_{ij} \quad \text{for all } (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0},$$

and such that $h|_{D_{ij}}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_M$.

Let $\{s_{ij} : (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ be a given data set with $|s_{ij}| < 1$ for all i, j. We define $s : I \times J \to \mathbb{R}$ to be the function satisfying

$$s(x_i, y_j) = s_{ij} \quad \text{for all } (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0},$$

and such that $s|_{D_{ij}}$ is bilinear for all $(i, j) \in \Sigma_N \times \Sigma_M$.

For all $(i, j) \in \Sigma_N \times \Sigma_M$, we define $F_{ij} : I \times J \times \mathbb{R} \to \mathbb{R}$ by

$$F_{ij}(x, y, z) = s(u_i(x), v_j(y))(z - g(x, y)) + h(u_i(x), v_j(y)).$$

Now, for each $(i, j) \in \Sigma_N \times \Sigma_M$, we define $W_{ij} : I \times J \times \mathbb{R} \to I_i \times J_j \times \mathbb{R}$ by

$$W_{ij}(x, y, z) = (u_i(x), v_j(y), F_{ij}(x, y, z)). \tag{2.1}$$

Then $\{I \times J \times \mathbb{R}, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ is an iterated function system.

https://doi.org/10.1017/S0004972718000321 Published online by Cambridge University Press

Theorem 2.1 [13]. *Let* $\{I \times J \times \mathbb{R}, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$ *be the iterated function system defined in* (2.1). *Then there exists a unique continuous function* $f : I \times J \to \mathbb{R}$ *such that* $f(x_i, y_j) = z_{ij}$ *for all* $(i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}$ *and* $\Gamma f = \bigcup_{(i,j)\in\Sigma_N\times\Sigma_M} W_{ij}(\Gamma f)$, *where* $\Gamma f = \{(x, y, f(x, y)) : (x, y) \in I \times J\}$ *denotes the graph of* f.

With the notation in Theorem 2.1, we call Γf the *bilinear FIS* and f the *bilinear FIF* with respect to the iterated function system $\{I \times J \times \mathbb{R}, W_{ij} : (i, j) \in \Sigma_N \times \Sigma_M\}$.

From $W_{ij}(x, y, f(x, y)) \in \Gamma f$, we have the following useful property:

$$f(u_i(x), v_j(y)) = F_{ij}(x, y, f(x, y)) \quad \text{for all } (x, y) \in I \times J. \tag{2.2}$$

From the above construction, a bilinear FIS is determined by the interpolation points $\{(x_i, y_j, z_{ij}) : (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$ in conjunction with the *vertical scaling factors* $\{s_{ij} : (i, j) \in \Sigma_{N,0} \times \Sigma_{M,0}\}$. A natural question is:

Question 2.2 [13, Question 5.1]. How can we obtain the box dimension of a bilinear FIS?

By estimating the number of the squares in small columns covering the graph of the interpolation function, Barnsley and Massopust [3] obtained the box dimension of a one-dimensional bilinear FIF under the condition that $x_i = i/N$, $s_i \geq 0$ for $0 \leq i \leq N$ and $s_0 = s_N$. However, dealing with FISs is more involved. In this paper, we will obtain their box dimension by estimating their oscillation. This is a more straightforward approach than that in [3].

3. Box dimension of bilinear FISs

For any $k_1, k_2, \ldots, k_d \in \mathbb{Z}$ and $\varepsilon > 0$, we call $\Pi_{i=1}^d [k_i\varepsilon, (k_i + 1)\varepsilon]$ an ε-coordinate cube in $\mathbb{R}^d$. Let E be a bounded set in $\mathbb{R}^d$ and $\mathcal{N}_E(\varepsilon)$ the number of ε-coordinate cubes intersecting E. We define

$$\overline{\dim}_B E = \varlimsup_{\varepsilon\to 0} \frac{\log \mathcal{N}_E(\varepsilon)}{\log 1/\varepsilon} \quad \text{and} \quad \underline{\dim}_B E = \varliminf_{\varepsilon\to 0} \frac{\log \mathcal{N}_E(\varepsilon)}{\log 1/\varepsilon}, \tag{3.1}$$

and call them the *upper box dimension* and the *lower box dimension* of E, respectively. If $\overline{\dim}_B E = \underline{\dim}_B E$, we use $\dim_B E$ to denote the common value and call it the *box dimension* of E. It is easy to see that in the definition of the upper and lower box dimensions we need only consider $\varepsilon_n = N^{-n}$, where $n \in \mathbb{Z}^+$. That is,

$$\overline{\dim}_B E = \varlimsup_{n\to\infty} \frac{\log \mathcal{N}_E(\varepsilon_n)}{n \log N} \quad \text{and} \quad \underline{\dim}_B E = \varliminf_{n\to\infty} \frac{\log \mathcal{N}_E(\varepsilon_n)}{n \log N}. \tag{3.2}$$

It is also well known that $\underline{\dim}_B E \geq 2$ when E is the graph of a continuous function on a domain of $\mathbb{R}^2$. See [5] for details.

In this section, we will calculate the box dimension of Γf, where f is the bilinear FIF defined in Section 2. It is difficult to obtain the box dimension of general bilinear FISs. In this paper, we always assume that $M = N$.

For each $n \in \mathbb{Z}^+$, we define $\Sigma_N^n = \{i_1 i_2 \cdots i_n : i_k \in \Sigma_N \text{ for all } k\}$. Further, for each $\mathbf{i} = i_1 i_2 \cdots i_n \in \Sigma_N^n$, we define $u_{\mathbf{i}} = u_{i_1} \circ u_{i_2} \circ \cdots \circ u_{i_n}$. Similarly, we define $v_{\mathbf{j}}$ for $\mathbf{j} \in \Sigma_N^n$. Denote $D_{\mathbf{ij}} = u_{\mathbf{i}}(I) \times v_{\mathbf{j}}(J)$ for $\mathbf{i}, \mathbf{j} \in \Sigma_N^n$. Define

$$O(f, n) = \sum_{\mathbf{i},\mathbf{j} \in \Sigma_N^n} O(f, D_{\mathbf{ij}}),$$

where we use $O(f, E)$ to denote the oscillation of f on $E \subset I \times J$, that is,

$$O(f, E) = \sup\{f(\mathbf{x}') - f(\mathbf{x}'') : \ \mathbf{x}', \mathbf{x}'' \in E\}.$$

We will use the following simple lemma, which presents a method to calculate the box dimension of the graph of a function from its oscillation. Similar results can be found in [5, 14].

Lemma 3.1.

(1) $\dim_B(\Gamma f) = 2$ *if* $\overline{\lim}_{n\to\infty}(\log O(f, n)/n \log N) \le 1$, *and*

(2) $\dim_B(\Gamma f) = 1 + \lim_{n\to\infty} (\log O(f, n)/n \log N)$ *if the limit exists and is larger than* 1,

where we define $\log 0 = -\infty$ *according to the usual convention.*

Proof. It is clear that

$$\mathcal{N}_{\Gamma f}(\varepsilon_n) \ge \varepsilon_n^{-1} \sum_{\mathbf{i},\mathbf{j} \in \Sigma_N^n} O(f, D_{\mathbf{ij}}) = N^n O(f, n).$$

Since $\underline{\dim}_B(\Gamma f)$ is always larger than or equal to 2,

$$\underline{\dim}_B(\Gamma f) \ge \max\Big\{2, 1 + \underline{\lim}_{n\to\infty} \frac{\log O(f, n)}{n \log N}\Big\}. \tag{3.3}$$

On the other hand, we note that $\mathcal{N}_E(\varepsilon)$ and $\mathcal{N}_E(\varepsilon_n)$ can be replaced by $\widetilde{\mathcal{N}}_E(\varepsilon)$ and $\widetilde{\mathcal{N}}_E(\varepsilon_n)$ in (3.1) and (3.2) respectively, where $\widetilde{\mathcal{N}}_E(\varepsilon)$ is the smallest number of cubes of side ε that cover E (see [5] for details). In our case,

$$\widetilde{\mathcal{N}}_{\Gamma f}(\varepsilon_n) \le \sum_{\mathbf{i},\mathbf{j} \in \Sigma_N^n} \Big(\frac{O(f, D_{\mathbf{ij}})}{\varepsilon_n} + 2\Big) = N^n O(f, n) + 2N^{2n}.$$

Hence

$$\overline{\dim}_B(\Gamma f) \le 1 + \overline{\lim}_{n\to\infty} \frac{\log(O(f, n) + 2N^n)}{n \log N}. \tag{3.4}$$

If $\overline{\lim}_{n\to\infty} \log O(f, n)/n \log N \le 1$, then from (3.4) $\overline{\dim}_B(\Gamma f) \le 2$. Combining this with (3.3) shows that $\dim_B(\Gamma f) = 2$.

If $\lim_{n\to\infty} \log O(f, n)/n \log N$ exists and is larger than 1, we denote it by t. From (3.4) and $t > 1$, we have $\overline{\dim}_B(\Gamma f) \le 1 + t$. Combining this with (3.3) shows that $\dim_B(\Gamma f) = 1 + t$. □

https://doi.org/10.1017/S0004972718000321 Published online by Cambridge University Press

We say that the vertical scaling factors $\{s_{ij} : i, j \in \Sigma_{N,0}\}$ are *steady* if for each $(i, j) \in \Sigma_N \times \Sigma_N$ all of $s_{i-1,j-1}$, $s_{i-1,j}$, $s_{i,j-1}$ and $s_{i,j}$ are nonnegative or all of them are nonpositive.

Lemma 3.2. *Assume that the vertical scaling factors are steady, and*

$$\sum_{i,j\in\Sigma_N} |s(u_i(x_0), v_j(y_0))| = \sum_{i,j\in\Sigma_N} |s(u_i(x_0), v_j(y_N))| = \sum_{i,j\in\Sigma_N} |s(u_i(x_N), v_j(y_0))| = \sum_{i,j\in\Sigma_N} |s(u_i(x_N), v_j(y_N))|. \tag{3.5}$$

Denote the common value by γ. Then for all $(x, y) \in I \times J$,

$$\sum_{i,j\in\Sigma_N} |s(u_i(x), v_j(y))| = \gamma.$$

Proof. Define

$$\widetilde{s}(x, y) = \sum_{i,j\in\Sigma_N} |s(u_i(x), v_j(y))| - \gamma, \quad (x, y) \in I \times J.$$

Since the vertical scaling factors are steady, $s(x, y)$ is nonnegative or nonpositive on D_{ij}, for each $(i, j) \in \Sigma_N \times \Sigma_N$. It follows that $s(u_i(x), v_j(y))$ is nonnegative or nonpositive on $I \times J$. As a result, $\widetilde{s}$ is a bilinear function on $I \times J$. But $\widetilde{s} = 0$ on $(x_0, y_0), (x_0, y_N), (x_N, y_0)$ and (x_N, y_N), so $\widetilde{s} = 0$ on $I \times J$ which proves the lemma. □

Remark 3.3. The vertical scaling factors are steady if $s_{ij} \geq 0$ for all $i, j \in \Sigma_{N,0}$, or $s_{ij} \leq 0$ for all $i, j \in \Sigma_{N,0}$. Furthermore, if there exists $d \in (-1, 1)$, such that $s_{ij} = d$ for all $i, j \in \Sigma_{N,0}$, then the vertical scaling factors are steady, and (3.5) is satisfied with $\gamma = N^2|d|$.

Remark 3.4. In the one-dimensional case studied by Barnsley and Massopust [3], it is assumed that $s_i \geq 0$ for all i, and $u_i(x_0) = x_{i-1}$ and $u_i(x_N) = x_i$ for all $i \in \Sigma_N$. Thus their assumption $s_0 = s_N$ is equivalent to $\sum_{i\in\Sigma_N} s(u_i(x_0)) = \sum_{i\in\Sigma_N} s(u_i(x_N))$.

Example 3.5. Let $N = 4$ and $s_{2j} = s_{i2} = 0$ for all $i, j \in \Sigma_{N,0}$. In this case, (3.5) is equivalent to

$$|s_{00}| + |s_{04}| + |s_{40}| + |s_{44}| = 2(|s_{01}| + |s_{03}| + |s_{41}| + |s_{43}|) = 2(|s_{10}| + |s_{14}| + |s_{30}| + |s_{34}|) = 4(|s_{11}| + |s_{13}| + |s_{31}| + |s_{33}|).$$

Let $d \in [0, 1/4)$ and define $s_{00} = s_{44} = 4d$, $s_{04} = s_{40} = -4d$, $s_{01} = s_{10} = s_{34} = s_{43} = 2d$, $s_{03} = s_{30} = s_{14} = s_{41} = -2d$, $s_{11} = s_{33} = d$ and $s_{13} = s_{31} = -d$. Then the vertical scaling factors are steady, while (3.5) is also satisfied with $\gamma = 16d$.

Lemma 3.6. *Let f be the bilinear FIF determined in Section 2 with $M = N$. Assume that the conditions of Lemma 3.2 are satisfied. Then there exists a positive constant C, such that for all $n \in \mathbb{Z}^+$,*

$$\gamma O(f, n) - CN^n \leq O(f, n + 1) \leq \gamma O(f, n) + CN^n.$$

Proof. Notice that

$$O(f, n+1) = \sum_{i,j\in\Sigma_N} \sum_{\mathbf{k},\mathbf{l}\in\Sigma_N^n} O(f, u_i(I_{\mathbf{k}}) \times v_j(J_{\mathbf{l}})). \tag{3.6}$$

Denote $M^* = \max\{|f(x,y)| : (x,y) \in I \times J\}$. For $(i,j) \in \Sigma_N \times \Sigma_N$ and $z^* \in [-M^*, M^*]$, we define

$$\hat{F}_{i,j,z^*}(x,y) = F_{ij}(x,y,z^*), \quad (x,y) \in I \times J.$$

Since $s(u_i(x), v_j(y)), g(x,y), h(u_i(x), v_j(y))$ are all bilinear functions on $I \times J$,

$$C_{ij} = \sup_{\substack{(x,y)\in \mathrm{int}(I\times J)\\ z^*\in[-M^*,M^*]}} \|\nabla \hat{F}_{i,j,z^*}(x,y)\| < \infty,$$

where $\mathrm{int}(I \times J) = (x_0, x_N) \times (y_0, y_N)$ and $\|\cdot\|$ is the standard Euclidean norm.

Given $x \in I_{\mathbf{k}}$ and $y \in J_{\mathbf{l}}$, where $\mathbf{k}, \mathbf{l} \in \Sigma_N^n$, it is clear that

$$|F_{ij}(u_{\mathbf{k}}(x_0), v_{\mathbf{l}}(y_0), f(x,y)) - F_{ij}(x,y,f(x,y))| \le \sqrt{2}C_{ij}\varepsilon_n. \tag{3.7}$$

On the other hand, for all $(x', y'), (x'', y'') \in I_{\mathbf{k}} \times J_{\mathbf{l}}$, we know from the definition that

$$\begin{aligned}
&F_{ij}(u_{\mathbf{k}}(x_0), v_{\mathbf{l}}(y_0), f(x',y')) - F_{ij}(u_{\mathbf{k}}(x_0), v_{\mathbf{l}}(y_0), f(x'',y''))\\
&\quad = s(u_i \circ u_{\mathbf{k}}(x_0), v_j \circ v_{\mathbf{l}}(y_0))(f(x',y') - f(x'',y'')).
\end{aligned}$$

Combining this with (2.2) and (3.7),

$$\begin{aligned}
&|f(u_i(x'), v_j(y')) - f(u_i(x''), v_j(y''))|\\
&\quad \le |s(u_i \circ u_{\mathbf{k}}(x_0), v_j \circ v_{\mathbf{l}}(y_0))| \cdot |f(x',y') - f(x'',y'')| + 2\sqrt{2}C_{ij}\varepsilon_n
\end{aligned}$$

so that

$$O(f, u_i(I_{\mathbf{k}}) \times v_j(J_{\mathbf{l}})) \le |s(u_i \circ u_{\mathbf{k}}(x_0), v_j \circ v_{\mathbf{l}}(y_0))| \cdot O(f, D_{\mathbf{kl}}) + 2\sqrt{2}C_{ij}\varepsilon_n.$$

Denote $C^* = \max\{2\sqrt{2}C_{ij} : i,j \in \Sigma_N\}$. From (3.6) and Lemma 3.2,

$$\begin{aligned}
O(f, n+1) &\le \sum_{\mathbf{k},\mathbf{l}\in\Sigma_N^n} \sum_{i,j\in\Sigma_N} |s(u_i \circ u_{\mathbf{k}}(x_0), v_j \circ v_{\mathbf{l}}(y_0))| \cdot O(f, D_{\mathbf{kl}}) + C^* N^{n+2}\\
&= \gamma O(f,n) + C^* N^{n+2}.
\end{aligned}$$

Similarly, we have $O(f, n+1) \ge \gamma O(f,n) - C^* N^{n+2}$ so that the lemma holds. □

The interpolation points $\{(x_i, y_j, z_{ij})\}_{(i,j)\in\Sigma_{N,0}\times\Sigma_{N,0}}$ are *co-bilinear* if $z_{ij} = g(x_i, y_j)$ for all $(i,j) \in \Sigma_{N,0} \times \Sigma_{N,0}$. Since g is bilinear, we can easily see that the property holds if and only if the following two conditions hold:

(1) for each $i \in \Sigma_{N,0}$, the points $\{(x_i, y_j, z_{ij}) : j \in \Sigma_{N,0}\}$ are collinear; and
(2) for each $j \in \Sigma_{N,0}$, the points $\{(x_i, y_j, z_{ij}) : i \in \Sigma_{N,0}\}$ are collinear.

https://doi.org/10.1017/S0004972718000321 Published online by Cambridge University Press

LEMMA 3.7. *Assume that the interpolation points* $\{(x_i, y_j, z_{ij})\}$ *are not co-bilinear, and satisfy the conditions of Lemma 3.2. Then there exists a positive constant* δ *such that, for all* $n \in \mathbb{Z}^+$,

$$O(f, n) \geq \gamma^n \delta.$$

PROOF. Since the interpolation points $\{(x_i, y_j, z_{ij})\}$ are not co-bilinear, we can assume without loss of generality that the points $\{(x_i, y_\ell, z_{i\ell}) : i \in \Sigma_{N,0}\}$ are not collinear for some $\ell \in \Sigma_{N,0}$. Thus, there exists k with $0 < k < N$, such that $(x_k, y_\ell, z_{k\ell})$ does not lie on the line passing through $(x_0, y_\ell, z_{0\ell})$ and $(x_N, y_\ell, z_{N\ell})$. Set $\lambda = (x_N - x_k)/(x_N - x_0)$. Then

$$\begin{aligned}\delta &:= |f(x_k, y_\ell) - (\lambda f(x_0, y_\ell) + (1 - \lambda) f(x_N, y_\ell))| \\ &= |z_{k\ell} - (\lambda z_{0\ell} + (1 - \lambda) z_{N\ell})| > 0.\end{aligned}$$

Since $\lambda \in (0, 1)$,

$$\max\{|z_{0\ell} - z_{k\ell}|, |z_{N\ell} - z_{k\ell}|\} \geq |z_{k\ell} - (\lambda z_{0\ell} + (1 - \lambda) z_{N\ell})|$$

so that

$$O(f, I \times J) \geq \delta.$$

For all $i, j \in \Sigma_N$, we define $\theta_{ij} = 1$ if $s(x, y)$ is nonnegative on D_{ij}, and define $\theta_{ij} = -1$ otherwise. Then $|s(x, y)| = \theta_{ij} s(x, y)$ for all $x, y \in D_{ij}$. From (2.2),

$$\theta_{ij} f(u_i(x_k), v_j(y_\ell)) = |s(u_i(x_k), v_j(y_\ell))|(f(x_k, y_\ell) - g(x_k, y_\ell)) + \theta_{ij} h(u_i(x_k), v_j(y_\ell))$$

so that

$$\sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_k), v_j(y_\ell)) = \gamma(f(x_k, y_\ell) - g(x_k, y_\ell)) + \sum_{i,j\in\Sigma_N} \theta_{ij} h(u_i(x_k), v_j(y_\ell)).$$

Similarly, $\sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_0), v_j(y_\ell))$ and $\sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_N), v_j(y_\ell))$ can be expressed in the same way. From

$$\begin{aligned}g(x_k, y_\ell) &= \lambda g(x_0, y_\ell) + (1 - \lambda) g(x_N, y_\ell), \\ h(u_i(x_k), v_j(y_\ell)) &= \lambda h(u_i(x_0), v_j(y_\ell)) + (1 - \lambda) h(u_i(x_N), v_j(y_\ell)),\end{aligned}$$

it follows that

$$\begin{aligned}&\sum_{i,j\in\Sigma_N} |f(u_i(x_k), v_j(y_\ell)) - (\lambda f(u_i(x_0), v_j(y_\ell)) + (1 - \lambda) f(u_i(x_N), v_j(y_\ell)))| \\ &\quad\geq \bigg| \sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_k), v_j(y_\ell)) - \lambda \sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_0), v_j(y_\ell)) \\ &\qquad - (1 - \lambda) \sum_{i,j\in\Sigma_N} \theta_{ij} f(u_i(x_N), v_j(y_\ell)) \bigg| \\ &\quad= \gamma |f(x_k, y_\ell) - (\lambda f(x_0, y_\ell) + (1 - \lambda) f(x_N, y_\ell))|.\end{aligned}$$

Thus

$$O(f,1)=\sum_{i,j\in\Sigma_N} O(f,D_{ij})\geq \gamma\delta.$$

Continuing the above process, for all $n \geq 2$,

$$\sum_{\mathbf{i},\mathbf{j}\in\Sigma_N^n} |f(u_{\mathbf{i}}(x_k),v_{\mathbf{j}}(y_\ell))-(\lambda f(u_{\mathbf{i}}(x_0),v_{\mathbf{j}}(y_\ell))+(1-\lambda)f(u_{\mathbf{i}}(x_N),v_{\mathbf{j}}(y_\ell)))|\geq \gamma^n\delta$$

so that

$$O(f,n)=\sum_{\mathbf{i},\mathbf{j}\in\Sigma_N^n} O(f,D_{\mathbf{ij}})\geq \gamma^n\delta.$$

This completes the proof of the lemma. □

THEOREM 3.8. *Let f be the bilinear FIF determined in Section 2 with $M = N$, and satisfying the conditions of Lemma 3.2. If $\gamma > N$ and the interpolation points $\{(x_i, y_j, z_{ij})\}$ are not co-bilinear, then*

$$\dim_B \Gamma f = 1+\frac{\log\gamma}{\log N}.$$

Otherwise, $\dim_B \Gamma f = 2$.

PROOF. If the interpolation points $\{(x_i, y_j, z_{ij})\}$ are co-bilinear, then $f = g$ so that $\dim_B \Gamma f = 2$.

Otherwise, from Lemma 3.6, there exists a constant $C > 0$, such that for all $n \geq 1$,

$$O(f,n+1)\leq \gamma O(f,n)+CN^n.$$

In the case where $N \neq \gamma$,

$$O(f,n)\leq \gamma^{n-1}O(f,1)+\frac{CN(N^{n-1}-\gamma^{n-1})}{N-\gamma},$$

and in the case where $N = \gamma$,

$$O(f,n)\leq N^{n-1}(O(f,1)+(n-1)C).$$

Thus

$$\varlimsup_{n\to\infty}\frac{\log O(f,n)}{n\log N}\leq \max\left\{1,\frac{\log\gamma}{\log N}\right\}. \tag{3.8}$$

If $\gamma \leq N$, from Lemma 3.1, $\dim_B \Gamma f = 2$. If $\gamma > N$, from Lemma 3.7,

$$\varliminf_{n\to\infty}\frac{\log O(f,n)}{n\log N}\geq \frac{\log\gamma}{\log N}.$$

Combining this with (3.8) and Lemma 3.1 gives $\dim_B \Gamma f = 1 + \log\gamma/\log N$. □

https://doi.org/10.1017/S0004972718000321 Published online by Cambridge University Press

References

[1] M. F. Barnsley, 'Fractal functions and interpolation', *Constr. Approx.* **2** (1986), 303–329.

[2] M. F. Barnsley, J. Elton, D. Hardin and P. R. Massopust, 'Hidden variable fractal interpolation functions', *SIAM J. Math. Anal.* **20** (1989), 1218–1242.

[3] M. F. Barnsley and P. R. Massopust, 'Bilinear fractal interpolation and box dimension', *J. Approx. Theory* **192** (2015), 362–378.

[4] P. Bouboulis and L. Dalla, 'A general construction of fractal interpolation functions on grids of $\mathbb{R}^n$', *Eur. J. Appl. Math.* **18** (2007), 449–476.

[5] K. J. Falconer, *Fractal Geometry: Mathematical Foundation and Applications* (Wiley, New York, NY, 1990).

[6] Z. Feng, 'Variation and Minkowski dimension of fractal interpolation surfaces', *J. Math. Anal. Appl.* **345** (2008), 322–334.

[7] J. S. Geronimo and D. Hardin, 'Fractal interpolation surfaces and a related 2-D multiresolution analysis', *J. Math. Anal. Appl.* **176** (1993), 561–586.

[8] D. P. Hardin and P. R. Massopust, 'The capacity of a class of fractal functions', *Commun. Math. Phys.* **105** (1986), 455–460.

[9] R. Małysz, 'The Minkowski dimension of the bivariate fractal interpolation surfaces', *Chaos Solitons Fractals* **27** (2006), 1147–1156.

[10] P. R. Massopust, 'Fractal surfaces', *J. Math. Anal. Appl.* **151** (1990), 275–290.

[11] D. S. Mazel and M. H. Hayes, 'Using iterated function systems to model discrete sequences', *IEEE Trans. Signal Process.* **40** (1992), 1724–1734.

[12] W. Metzler and C. H. Yun, 'Construction of fractal interpolation surfaces on rectangular grids', *Internat. J. Bifur. Chaos* **20** (2010), 4079–4086.

[13] H.-J. Ruan and Q. Xu, 'Fractal interpolation surfaces on rectangular grids', *Bull. Aust. Math. Soc.* **91** (2015), 435–446.

[14] H.-J. Ruan, W.-Y. Su and K. Yao, 'Box dimension and fractional integral of linear fractal interpolation functions', *J. Approx. Theory* **161** (2009), 187–197.

QING-GE KONG, School of Mathematical Science,
Zhejiang University, Hangzhou 310027, China
e-mail: kongqingge@163.com

HUO-JUN RUAN, School of Mathematical Science,
Zhejiang University, Hangzhou 310027, China
e-mail: ruanhj@zju.edu.cn

SHENG ZHANG, School of Mathematical Science,
Zhejiang University, Hangzhou 310027, China
and
Department of Mathematics,
Purdue University, West Lafayette, IN 47907, USA
e-mail: zhan2694@purdue.edu