ON SEGREGATED RINGS AND ALGEBRAS

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Segregated algebras have been nicely characterized by M. Ikeda [4]. In this paper §1, we consider segregated rings and study the structure of such rings in Theorems 1.1 and 1.2. In §2, we specialize to the case of segregated algebras of finite dimension over a field. Theorem 2.1 gives a new characterization of such algebras. Theorem 2.2 shows an interesting property of segregated algebras; two segregated algebras S and T, with radicals N and P respectively, are isomorphic if and only if S/N^2 and T/P^2 are isomorphic.

§1. Segregated Rings

Following [3], if $\theta: A \to B$ is a ring homomorphism of A onto B, we shall say that B is segregated in A if there exists a subring A' of A such that θ restricted to A' is an isomorphism. Clearly $A = A' + \text{kernel } \theta$ is a direct sum. If B is a ring and M an abelian group we say that M is a (B, B) module if M is both a left and right B module and the associativity condition $\beta(m\gamma) = (\beta m)\gamma$ holds for m in M and β, γ in B.

Consider a class $\mathfrak{C}$ of rings with the property that if B belongs to $\mathfrak{C}$ and A is a subring of B then A belongs to $\mathfrak{C}$.

Definition 1.1: A ring B is *segregated in* $\mathfrak{C}$ will mean B belongs to $\mathfrak{C}$ and, when B is the homomorphic image of D belonging to $\mathfrak{C}$, B is segregated in D.

Definition 1.2: A ring B is *separated in* $\mathfrak{C}$ if, 1. B is segregated in $\mathfrak{C}$, and 2. if B is a subring of D belonging to $\mathfrak{C}$ and M is a subset of D which is a (B, B) module then M is completely reducible as a (B, B) module.

One sees that a separated ring with identity is the direct sum of a finite number of simple ideals with identity and is therefore semisimple. (We would like to say that B is separable if every (B, B) module is completely reducible, and then prove a separable ring is segregated. Since we do not know that this

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can be done, we must assume this property in the definition. Perhaps this will explain the use of the term "separated" in the above definition.)

The class of rings which motivated this investigation is the class $\mathfrak{F}_\Phi$ of algebras of finite dimension over a field Φ. In this class the concepts of segregated and separated defined above are equivalent to segregated and separable in the usual sense [3]. We shall return to this class in 2 for applications of the general results.

Let B be a ring and let M be a (B, B) module.

DEFINITION 1.3: Let $M_1 = M$, $M_{r+1} = M_r \times {}_B M$ (Kronecker product over B) $r = 1, 2, \ldots$. See [2] for definition and properties.

DEFINITION 1.4: Denote by $F(B, M)$ the (weak) direct sum $B + \sum_1^\infty M_r$.

Each M_r is a (B, B) module under the compositions $\beta(m_{i_1} \times \ldots \times m_{i_r}) = (\beta m_{i_1} \times \ldots \times m_{i_r})$ and $(m_{i_1} \times \ldots \times m_{i_r})\beta = (m_{i_1} \times \ldots \times m_{i_r}\beta)$. Also $M_r \times {}_B M_n$ is isomorphic to M_{r+n} under the identification $(m_{i_1} \times \ldots \times m_{i_r}) \times (m_{j_1} \times \ldots \times m_{j_n}) \times (m_{i_1} \times \ldots \times m_{i_r} \times m_{j_1} \times \ldots \times m_{j_n})$, see [2]. For simplicity we denote the elements of M_r by m_r and the above compositions by βm_r, $m_r\beta$, and $m_r m_n$ respectively. This defines a multiplication for generators of $F(B, M)$ and we have

LEMMA 1.1: *Under the above compositions $F(B, M)$ is a ring, the free ring on B and M.*

Proof: By the definition of the product we have $\beta(m_r m_n) = (\beta m_r)m_n$, $m_r(m_n\beta) = (m_r m_n)\beta$, and $(m_r\beta)m_n = m_r(\beta m_n)$, the latter because the product was taken over B. In addition, M_{r+n+p}, $(M_r \times {}_B M_n) \times {}_B M_p$, and $M_r \times {}_B (M_n \times {}_B M_p)$ are all isomorphic under the identification of $m_r(m_n m_p)$ and $(m_r m_n)m_p$ in M_{r+n+p}; see [2]. This assures associativity of the product. Distributivity of the multiplication follows from the rules for Kronecker products, the fact that $F(B, M)$ is a (B, B) module, and the extension of the multiplication linearly from generators of $F(B, M)$.

The ring $F(B, M)$ also has a natural topology. Let $K(B, M) = K$ be the subset $\sum_1^\infty M_r$; clearly K is an ideal of $F(B, M)$. We see that $K^2 = \sum_2^\infty M_r$ and in general $K^n = \sum_n^\infty M_r$. Since K is the direct sum of the modules M_r, we have that $\bigcap_1^\infty K^n = (0)$.

DEFINITION 1.5: Let K^n, $n=1, 2, \ldots$ be a basis for neighborhoods of (0) in $F(B, M)$. With respect to the topology generated by these neighborhoods, we see that $F(B, M)$ is a topological ring.

The following theorem shows that under certain conditions on B and M, the free ring $F(B, M)$ on B and M is segregated.

THEOREM 1.1: *If B is separated in $\mathfrak{C}$, M a (B, B) module and $F(B, M)$ belongs to $\mathfrak{C}$ then $F(B, M)$ is segregated in $\mathfrak{C}$.*

Proof: Suppose that A belongs to $\mathfrak{C}$ and there exists $\theta: A \to F(B, M)$, a ring homomorphism onto $F(B, M)$. Since B is a subring of $F(B, M)$, the set $\theta^{-1}(B)$ is a subring of A. Then θ maps $\theta^{-1}(B)$ onto B and there exists a subring A' of $\theta^{-1}(B)$ such that θ restricted to A' is an isomorphism of A' onto B. This is because B is separated in $\mathfrak{C}$ and therefore segreated in $\mathfrak{C}$. Let ψ be the inverse of θ restricted to A', $\psi: B \to A'$.

Let $N_1 = \theta^{-1}(M)$ in A. N_1 can be considered as a (B, B) module under the compositions $n\beta = n\psi(\beta)$ and $\beta n = \psi(\beta)n$ for β in B and n in N_1. With this definition, θ restricted to N_1 becomes a (B, B) homomorphism of N_1 onto M. Let N_0 be its kernel. Since B is separated, N_0 has a complementary (B, B) submodule N in N_1 and θ restricted to N is an isomorphism of N onto M. Let ψ_1 be its inverse, $\psi_1: M \to N$.

Let C be the subring of A generated by A' and N. Since $A' = \psi(B)$ and N is a (B, B) module under this identification, $A'N \subseteqq N$, $NA' \subseteqq N$ and in general $A'N^r \subseteqq N^r$ and $N^rA' \subseteqq N^r$. Thus C has the form $A' + N + N^2 \ldots.$ We shall show that θ restricted to C is an isomorphism onto $F(B, M)$.

Since $F(B, M)$ is a direct sum $B + \sum_1^\infty M_r$, it is enough to show that θ restricted to N^r is an isomorphism onto M_r. Inducing on r, we already have that θ restricted to N is an isomorphism onto M. Assume that θ restricted to N^r is an isomorphism onto M_r and let ψ_r be its inverse.

Since $\theta[\psi_r(\beta m_r) - \psi(\beta)\psi_r(m_r)] = \beta m_r - \beta m_r = 0$, and $\theta[\psi_r(m_r\beta) - \psi_r(m_r)\psi(\beta)] = m_r\beta - m_r\beta = 0$ and θ restricted to N^r is an isomorphism, we have $\psi_r(\beta m_r) = \psi(\beta)\psi_r(m_r)$ and $\psi_r(m_r\beta) = \psi_r(m_r)\psi(\beta)$, for β in B and m_r in M_r. By defining $\beta n_r = \psi(\beta)n_r$ and $n_r\beta = n_r\psi(\beta)$, ψ_r becomes a (B, B) isomorphism of M_r onto N_r. Now define $\psi_{r+1}(m_r m)$ equal to $\psi_r(m_r)\psi_1(m)$ in N^{r+1} for m_r in M_r and m in M. This induces a (B, B) homomorphism of M_{r+1} onto N^{r+1} since both ψ_r

and ψ_1 are onto N^r and N respectively. Let n_r be in N^r and n in N, then the elements of the form $n_r n$ generate N^{r+1}. But $\psi_{r+1}[\theta(n_r n)] = \psi_{r+1}[\theta(n_r)\theta(n)] = \psi_r(\theta(n_r))\psi_1(\theta(n)) = n_r n$, so $\psi_{r+1}\theta$ is identity on generators of N^{r+1} and θ restricted to N^{r+1} is an isomorphism. Also we have $\theta\psi_{r+1}(m_r m) = \theta[\psi_r(m_r)\psi_1(m)] = m_r m$ for generators $m_r m$ of M_{r+1}. Thus $\theta\psi_{r+1}$ is identity on M_{r+1} and θ restricted to N^{r+1} maps it onto M_{r+1}. This completes the induction and the proof of the theorem.

We would like to prove a converse to Theorem 1.1, *i.e.* that every ring segregated in $\mathfrak{C}$ has the form $F(B, M)$ where B is separated and M is a (B, B) module. Since we have been unable to do this, we give next the closest thing to a converse to Theorem 1.1 that we have been able to prove.

THEOREM 1.2: *If S is segregated in $\mathfrak{C}$ and $S = B + L$ direct sum, where B is a separated subring, L an ideal of S such that $L = M + L^2$, (B, B) direct sum and M generates L as a subring, then if $F(B, M)$ belongs to $\mathfrak{C}$ S is a dense subring of $F(B, M)$.*

Proof: We have $F(B, M) = B + M + M_2 + \ldots$ and we can define a ring homomorphism θ of $F(B, M)$ into S by $\theta(\beta) = \beta$ for β in B and $\theta(m_1 \times \ldots \times m_r) = m_1 \ldots m_r$ for $m_1 \times \ldots \times m_r$ in M_r and extending linearly to $F(B, M)$. Since L is generated by M and $S = B + L$, θ is onto S.

Since $F(B, M)$ is assumed to be in $\mathfrak{C}$ and S is segregated in $\mathfrak{C}$, there exists S_0, a subring of $F(B, M)$ such that θ restricted to S_0 is an isomorphism. We show that S_0 is dense in $F(B, M)$. Let ψ be the inverse of θ restricted to S_0, $\psi : S \to S_0 \subseteq F(B, M)$.

The kernel of θ is contained in $K^2 = \sum_2^\infty M_r$ because θ restricted to $B + M$ is an isomorphism onto $B + M$ in S and S is $B + M + L^2$ (direct). Also for x in $F(B, M)$, $x - \psi(\theta(x))$ is in the kernel of θ. Hence, for any set T in $F(B, M)$,

(1) $$T + \text{kernel}\,\theta = \psi(\theta(T)) + \text{kernel}\,\theta \text{ or}$$

(1′) $$T + K^2 = \psi(\theta(T)) + K^2.$$

Applying this equation when $T = M$, we have

(2) $$K = M + K^2 = \psi(\theta(M)) + K^2,$$

and when $T = B + M$ we have

(3) $$F(B, M) = B + M + K^2 = \psi(\theta(B)) \dot{+} \psi(\theta(M)) + K^2.$$

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Equation (2) can be used recursively to show

$$K^r = \psi(\theta(M_r)) + K^{r+1}, \tag{4}$$

for assuming (4) and multiplying (2) and (4), we have

$$K^{r+1} = \psi(\theta(M))\psi(\theta(M_r)) + K^{r+2} \text{ or} \tag{5}$$

$$K^{r+1} = \psi(\theta(M_{r+1})) + K^{r+2}, \tag{5'}$$

thus proving (4) by induction on r.

Using (2), (3), and (4) for $r = 1, 2, \ldots, n$, we see

$$F(B, M) = \psi(\theta(B)) + \sum_1^n \psi(\theta(M_r)) + K^{n+1} \text{ or} \tag{6}$$

$$F(B, M) = \psi(\theta(F(B, M))) + K^{n+1}. \tag{6'}$$

Since this latter equation holds for all n, $\psi(\theta(F(B, M))) = S_0$ is dense in $F(B, M)$.

The following lemma is useful in deciding when a ring is not segregated.

LEMMA 1.2: *Let R be a ring and L an ideal of R with the property that if a set S satisfies $S + L^2 = L$ then S generates L as a subring. If $L^2 \supseteqq P$, P an ideal of $R \neq (0)$, then R/P is not segregated in R under the natural homomorphism.*

Proof: Suppose $R_0 + P = R$ a direct sum and R_0 is a subring of R. Then $L \cap R_0 + P = L$ and $L^2 \cap R_0 + P = L^2$. Let S be the set $\{s \mid s \in L \cap R_0,\ s \notin L^2 \cap R_0\}$ $\cup (0)$, then $S + L^2 \cap R_0 + P$ equals $S + L^2 = L$. By the assumptions on L, S generates L. But $S \subseteqq R_0$ so $L \subseteqq R_0$ and hence $(0) \neq P \subseteqq R_0$. This contradicts the assumption that the sum $R_0 + P$ was direct.

In particular we note the following corollary to Lemma 1.2.

LEMMA 1.3: *If a ring R contains a nilpotent ideal L and $L^2 \supseteqq P \neq (0)$, P an ideal in R, then R/P is not segregated in R under the natural homomorphism.*

Proof: Let $S + L^2 = L$. We show that S generates L as a subring. $L^{r+1} = (0)$ so $S^{r+1} = (0)$, let $S_0 = S \cup (0)$. By induction on n $S_0^n + L^{n+1} = L^n$, for assuming this equation and multiplying by $S_0 + L^2 = L$, we have $S_0^{n+1} + L^{n+2} = L^{n+1}$. The distributive law holds for set multiplication because all the sets contain 0, see [1]. But then $\sum_1^r S_0^n = L$, so the subring generated by S is L. Lemma 1.2 then gives the result.

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§ 2. Finite Dimensional Algebras over a Field

In the following we consider the class $\mathfrak{F}_{\Phi}$ of algebras finite dimensional over a field Φ. Here the concept of segregated is equivalent to segregated in the usual sense [3]. Also we have

LEMMA 2.1: *B is separated in $\mathfrak{F}_{\Phi}$ if and only if B is separable.*

Proof: We see that B is separable over Φ if and only if every (B, B) module is completely reducible; this is equivalent to the vanishing of the first cohomology groups of B, see [3]. Also B separable implies B segregated [3]. Thus B separable implies B is separated in $\mathfrak{F}_{\Phi}$. The converse is clear.

DEFINITION 2.1: The (B, B) module M is nilpotent if $M_r = (0)$ for some r. Clearly, in that case $M_{r+k} = (0)$ for $k \geqq 0$.

From the definition of $F(B, M)$ if $(B : \Phi) < \infty$ and $(M : \Phi) < \infty$, we have $(F(B, M) : \Phi) < \infty$ if and only if M is nilpotent. We now prove a structure theorem for segregated algebras over Φ.

THEOREM 2.1: *The algebra S is segregated over Φ if and only if S/N is separable and S is isomorphic to $F(S/N, N/N^2)$, N the radical of S.*

Proof: If S/N is separable and S is isomorphic to $F(S/N, N/N^2)$, then the latter belongs to $\mathfrak{F}_{\Phi}$ and by Theorem 1.1 it is segregated in $\mathfrak{F}_{\Phi}$.

Conversely, assume S is segregated. By Ikeda's Theorem [4], we have S/N is separable. Then S can be written $S = S' + M + N^2$ where S' is a subalgebra isomorphic to S/N, and $N = M + N^2$ where M is the (S', S') complement of N^2 in N. The natural homomorphism of S onto S/N when restricted to S' is an isomorphism and the natural homomorphism of N onto N/N^2 is an isomorphism when restricted to M. Under these identifications it is clear that $F(S', M)$ is isomorphic to $F(S/N, N/N^2)$. It is more convenient to work with the former.

By the proof of Theorem 1.2, there exists a homomorphism θ of $F(S', M)$ onto S, where the kernel of θ is contained in $K^2 = \sum_2^{\infty} M_r$. Suppose that $N^{r+1} = (0)$. Then the kernel of θ contains K^{r+n} for $n \geqq 1$. But then S is the homomorphic image of $F(S', M)/K^{r+n}$ which is finite dimensional and has the nilpotent radical K/K^{r+n}. Since S is segregated, Lemma 1.3 shows that kernel $\theta = K^{r+n}$ for $n \geqq 1$. But $K^{r+1} = K^{r+2}$ implies that M is a nilpotent module and the algebra $F(S', M)$ is finite dimensional. Theorem 1.2 then implies that S is isomorphic to a dense subring S_0 of $F(S', M)$. But since $K^{r+1} = (0)$ the topology is discrete and $S_0 = F(S', M)$.

The structure theorem just proved has an interesting corollary:

THEOREM 2.2: *If S and T are segregated algebras over Φ, where N is the radical of S and P the radical of T, then S and T are isomorphic if and only if S/N^2 and T/P^2 are isomorphic.*

Proof: Clearly S and T isomorphic implies that S/N^2 and T/P^2 are isomorphic under the induced isomorphism.

Conversely, let S and T be segregated and let $\theta : S/N^2 \to T/P^2$ be an isomorphism. This induces an isomorphism of S/N onto T/P and each is separable. Identifying isomorphic images, θ restricted to N/N^2 becomes an $(S/N, S/N)$ isomorphism onto P/P^2, where N/N^2 and P/P^2 are considered as $(S/N, S/N)$ modules. This induces an isomorphism of $F(S/N, N/N^2)$ onto $F(T/P, P/P^2)$. Theorem 2.1 then gives the result.

Addendum: For further investigations concerning algebras and rings with these properties, see "On the Dimensions of Modules and Algebras, VII" by the author and T. Nakayama, which appears later in this volume. This joint paper is a combination of the author's rather meager results with some truely fine investigations by Professor Nakayama. The author wishes to express his gratitude to Professor Nakayama for his generosity in making it a joint paper.

BIBLIOGRAPHY

[1] E. Artin, C. J. Nesbitt, and R. M. Thrall, *Rings with Minimum Condition*, University of Michigan Press 1948.

[2] M. Auslander, S. Eilenberg, *Notes on Cohomology Theory of Groups*, available from Dept. of Math., University of Chicago.

[3] G. Hochschild, *On the Cohomology of an Associative Algebra*, Ann. of Math. vol. **47** (1946).

[4] M. Ikeda, *On Absolutely Segregated Algebras*, Nagoya Math. Journal vol. **6** (1953).

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