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Meromorphic Functions with Prescribed Asymptotic Behaviour, Zeros and Poles and Applications in Complex Approximation

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Abstract. We construct meromorphic functions with asymptotic power series expansion in z^{-1} at ∞ on an Arakelyan set A having prescribed zeros and poles outside A. We use our results to prove approximation theorems where the approximating function fulfills interpolation restrictions outside the set of approximation.

1 Introduction

The notion of asymptotic expansions or more precisely asymptotic power series is classical and one usually refers to Poincaré [Po] for its definition (see also [Fo], [O], [Pi], and [R1, pp. 293–301]).

A function $f\colon A \to \mathbb{C}$ where $A \subset \mathbb{C}$ is unbounded, possesses an asymptotic expansion (in A) at ∞ if there exists a (formal) power series $\sum a_n z^{-n}$ such that $f(z) - \sum_{n=0}^{N} a_n z^{-n} = O(|z|^{-(N+1)})$ as $z \to \infty$ in A.

This imitates the properties of functions with convergent Taylor expansions. In fact, if f is holomorphic at ∞ its Taylor expansion and asymptotic expansion coincide. We will be mainly concerned with entire functions possessing an asymptotic expansion. Well known examples are the exponential function (in the left half plane) and Sterling's formula for the behaviour of the Γ-function at ∞.

In Sections 2 and 3 we introduce a suitable algebraical and topological structure on the set of all entire functions with an asymptotic expansion. Using this in the following sections, we will prove existence theorems in the spirit of the Weierstrass product theorem and Mittag-Leffler's partial fraction theorem. It will turn out that zeros and poles can be prescribed arbitrarily (not accumulating at a finite point of course) outside an Arakelyan set A, whereas in A the asymptotic behaviour of the function is as described above. As a corollary we get for a given Jordan path γ to ∞, $a \in \mathbb{C}$ the existence of a function f meromorphic in $\mathbb{C}$ with $f \to a$ on γ having prescribed zeros and poles outside γ.

The main tool in our constructions is an approximation theorem of Arakelyan. To a certain amount our results overcome the problem that functions constructed from approximation theorems are usually difficult to control outside the set of approximation (which we always denote by A). In this direction we will prove in Section 5 approximation theorems where the approximating function has prescribed zeros outside A or solves an interpolation problem outside A.

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2 Definitions and Basic Results

First let us fix some notation. We will denote by $\hat{\mathbb{C}}$ the one point compactification of $\mathbb{C}$, by $\mathbb{N}$ the positive integers and we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $r \geq 0$ let $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$ be the open disc of radius r and $\overline{\mathbb{D}}_r$ be its topological closure. We set $\mathbb{D} := \mathbb{D}_1$. The interior and complement of a set $A \subset \mathbb{C}$ will be denoted by A° and A^c, respectively.

Definition 2.1 Let A be an unbounded set in $\mathbb{C}$. We say that a function $f\colon A \to \mathbb{C}$ possesses an *asymptotic expansion* in A if there exists a complex sequence a_ν such that for all $n \in \mathbb{N}_0$

$$z^{n+1}\Big(f(z) - \sum_{\nu=0}^{n} a_\nu z^{-\nu}\Big) \to a_{n+1}$$

as $z \to \infty$ in A. We write $f(z) \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$. Further if we set for $n \in \mathbb{N}_0$

$$R_n(f,z) := f(z) - \sum_{\nu=0}^{n-1} a_\nu z^{-\nu}$$

then it is easy to show that $f(z) \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ is equivalent to $R_n(f,z) = O(|z|^{-n})$ for all $n \in \mathbb{N}_0$. Here for $n = 0$ we have $R_0(f,z) = f(z)$.

Remark (a) It is easy to see that the asymptotic expansion of a function is uniquely determined. But even for holomorphic functions with asymptotic expansion at ∞ the identity theorem does not carry over: Let A be a closed subsector of the left half plane. Then $e^z \sim 0$ and of course for $f \equiv 0$ we have $f \sim 0$. (If $a_\nu = 0$ for $\nu \geq 1$ we write a_0 for $\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$.) Note that the asymptotic expansion of f need not converge and is therefore a "formal" power series. Further, in case the expansion converges, it need not represent f as can be seen from the above example $f(z) = e^z$.

(b) Suppose $f(z) \sim a_n z^{-n} + a_{n+1} z^{-(n+1)} + \cdots$, *i.e.*, that the first n coefficients in the asymptotic expansion are zero. One easily shows $z^n f(z) \sim \sum_{\nu=0}^{\infty} a_{\nu+n} z^{-\nu}$. More generally, if p is a polynomial with degree $\deg p \leq n$, then $p(z) \cdot f(z)$ possesses an asymptotic expansion. This property gives the oppurtunity to bring polynomials into play, although polynomials have no finite limits at ∞. This fact will play a key role in this paper and can be considered as the reason to use asymptotic power series in the context we will be dealing with.

We now fix an unbounded set $A \subset \mathbb{C}$ and consider the set F of all complex valued functions on A that posses an asymptotic expansion in A. With pointwise addition and multiplication F is a unital algebra. This follows from the following well known proposition. We omit its simple proof.

Proposition 2.2 *Let $f, g \in F$ with $f \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ and $g \sim \sum_{\nu=0}^{\infty} b_\nu z^{-\nu}$ in A and and $\alpha, \beta \in \mathbb{C}$. Then the following holds:*

(i) $\alpha f(z) + \beta g(z) \sim \sum_{\nu=0}^{\infty} (\alpha a_\nu + \beta b_\nu) z^{-\nu}$
(ii) $f(z) \cdot g(z) \sim \sum_{\nu=0}^{\infty} c_\nu z^{-\nu}$ *where* $c_\nu = \sum_{n=0}^{\nu} a_n b_{\nu-n}$.

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Now let us consider the sub-algebra B of all bounded functions in F. We introduce on B for $k \in \mathrm{N}_0$ the semi-norms

$$M_k(f) := \sup_{z\in A} |z|^k \left| f(z) - \sum_{\nu=0}^{k-1} a_\nu z^{-\nu} \right| = \sup_{z\in A} |z|^k \, |R_k(f,z)|.$$

Note that the number $M_k(f)$ is the infimum of all admissible constants in the O-terms in Definition 2.1.

The following convergence theorem can be found in [Pi, p. 38].

Theorem 2.3 *Let f_n be a sequence in B with $f_n(z) \sim \sum_{\nu=0}^{\infty} a_\nu^{(n)} z^{-\nu}$ such that f_n converges pointwise to a function f. Assume that f_n is bounded in the norms M_k, i.e., that for all $k \in \mathrm{N}$ there exists $C_k \geq 0$ such that $M_k(f_n) \leq C_k$ uniformly in n. Then the limits $a_k := \lim\limits_{n\to\infty} a_k^{(n)}$ exist and $f(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}$ with $M_k(f) \leq C_k$. In particular $f \in B$.*

We will need the following estimation of $M_k(f \cdot g)$.

Lemma 2.4 *For $f, g \in B$ we have $M_k(fg) \leq \sum_{\kappa=0}^{k} M_\kappa(f) M_{k-\kappa}(g)$.*

Proof By induction. Let $f \sim \sum a_\nu z^{-\nu}$ and $g \sim \sum b_\nu z^{-\nu}$. For $k = 0$ we have $M_0(fg) = \sup_{z\in A} |f(z)g(z)| \leq M_0(f)M_0(g)$. It is easy to show that $|a_0| \leq M_0(f)$ and $M_k\big(z(f-a_0)\big) = M_{k+1}(f)$. Using this the step $k \to k+1$ is done by

$$\begin{aligned} M_{k+1}(fg) &= M_k\big(z(fg - a_0 b_0)\big) = M_k\big(z(f - a_0)g + z(g - b_0)a_0\big) \\ &\leq M_k\big(z(f - a_0)g\big) + M_k\big(z(g - b_0)a_0\big) \\ &\leq \sum_{\kappa=0}^{k} M_\kappa\big(z(f - a_0)\big) M_{k-\kappa}(g) + |a_0| M_k\big(z(g - b_0)\big) \\ &\leq \sum_{\kappa=0}^{k} M_{\kappa+1}(f) M_{k-\kappa}(g) + M_0(f) M_{k+1}(g) = \sum_{\kappa=0}^{k+1} M_\kappa(f) M_{k-\kappa}(g). \end{aligned}$$

■

3 Entire Functions with Asymptotic Expansions and Prescribed Zeros

Let now E be the sub-algebra of all entire functions in B. First we define the usual norms that induce the topology of compact convergence: $|f|_n := \max_{|z|=n} |f(z)|$. As is well-known the algebra of all entire functions is complete in the topology induced by these norms. This is in general not the case for our algebra E. Let A be a subsector of the left half-plane. Then $e^z \in E$ and $e^z \sim 0$. Let f be an arbitrary entire function. Further let e_n and f_n be polynomial approximations of e^{-z} and f respectively in the topology of compact convergence (*e.g.*, the Taylor expansions). Then $F_n := f_n e_n e^z \sim 0$ and therefore $F_n \in E$, but $F_n \to f$ compactly.

It is natural to introduce a topology on E that is finer than the topology of compact convergence such that E becomes a complete space. One way to do this, is to enlarge the

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system of norms by the norms M_k defined above. Theorem 2.3 tells us that E is complete in the topology τ_A induced by the norms $\{|\cdot|_n \mid n \in \mathrm{N}\} \cup \{M_k \mid k \in \mathrm{N}_0\}$. With Lemma 2.4 one can prove that E with τ_A is a Fréchet algebra. Further, by a result of Carpenter (see [Go, p. 163]) this topology is unique. So the above constructed topology on E is very natural. From now on we endow E with τ_A.

We prove a useful characterisation of convergence in (E, τ_A):

Theorem 3.1 *A sequence f_n in E converges to $f \in E$ iff $f_n \to f$ compactly and for each $k \in \mathrm{N}_0$ there exists $C_k \geq 0$ such that $M_k(f_n) \leq C_k$ for all $n \in \mathrm{N}$.*

Proof "$\Rightarrow$" is trivial. Suppose $f_n \to f$ in the topology of compact convergence. Clearly f is entire. Let $f_n(z) \sim \sum_{\nu=0}^{\infty} a_\nu^{(n)} z^{-\nu}$. Since compact convergence implies pointwise convergence Theorem 2.3 shows that the limits $\lim_{n\to\infty} a_\nu^{(n)} = a_\nu$ exist and that $f(z) \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ (in particular $f \in E$). It is left to show $\lim_{n\to\infty} M_k(f_n - f) = 0$ for all $k \in \mathrm{N}$. For $|z| \geq C(\varepsilon)$ it holds

$$|z|^k \left| f_n(z) - f(z) - \sum_{\nu=0}^{k-1} (a_\nu^{(n)} - a_\nu) z^{-\nu} \right| \leq |a_k^n - a_k| + \frac{\varepsilon}{2} \leq \varepsilon$$

for $n \geq n_0$. Now $\overline{A} \cap \overline{\mathrm{D}}_{C(\varepsilon)}$ is compact. Since convergence of the coefficients of polynomials with fixed degree implies compact convergence and by the assumed compact convergence of f_n it follows

$$\sup_{z \in \overline{A} \cap \overline{\mathrm{D}}_{C(\varepsilon)}} |z|^k \left| f_n(z) - f(z) - \sum_{\nu=0}^{k-1} (a_\nu^{(n)} - a_\nu) z^{-\nu} \right| \leq \varepsilon$$

for $n \geq n_1$. It follows $M_k(f_n - f) \leq \varepsilon$ for $n \geq \max\{n_0, n_1\}$ and the proof is complete. ■

With Montel's convergence theorem for normal families it follows from Theorem 3.1:

Corollary 3.2 *E is a Montel space.*

For the construction of non-trivial elements in E it is unavoidable to impose restrictions on the set A. For example, if A is a neighbourhood of ∞ then E contains only the constant functions by the Liouville theorem. Since we will use an approximation theorem of Arakelyan we need the following definition.

Definition 3.3 A closed unbounded proper subset A of C is called an *Arakelyan set* if $\hat{\mathrm{C}} \setminus A$ is connected and locally connected at ∞. We will call a sequence α_n (possibly finite or empty) in C *admissible* (with respect to A) if α_n has no finite accumulation point and all α_n are contained in $\mathrm{C} \setminus A$.

Simple examples of Arakelyan sets are sectors or Jordan paths going to ∞. We will need a charaterisation of local connectedness at ∞ (*cf.* [Ga, p. 126]).

Lemma 3.4 *The set $\hat{\mathrm{C}} \setminus A$ is locally connected at ∞ if and only if for each neighbourhood U of ∞ there exists a neighbourhood $V \subset U$ of ∞ such that every $z \in V \setminus A$ can be connected with ∞ by a Jordan path in $U \setminus A$.*

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It is easy to check that it is sufficient that U and V in the above statement belong to a basis of neighbourhoods of ∞, *e.g.*, all sets of the form $\overline{\mathrm{D}}_r^c$ with $r > 0$. With the notion of an Arakelyan set the mentioned approximation theorem of Arakelyan can be stated (see [Fu, p. 39] or [Ga, p. 145]).

***Theorem 3.5* (Arakelyan)** *Let A be an Arakelyan set and $\varepsilon\colon [0,\infty) \to (0,\infty)$ be continuous such that*

$$\int_1^\infty t^{-3/2} \log \varepsilon(t)\, dt > -\infty. \tag{1}$$

Then for every function f that is continuous on A and holomorphic in A° there exists an entire function g such that

$$|f(z) - g(z)| \leq \varepsilon(|z|)$$

for all $z \in A$.

It is easy to show that $\varepsilon(t) := \eta \exp(-ct^{1/3})$ with $\eta, c > 0$ fulfills (1). Now we can prove the main result of the paper.

Theorem 3.6 *Let A be an Arakelyan set and let α_n be an admissible sequence. Further let o_n be a sequence in N. Then there exists an entire function f with exactly the zeros α_n of order o_n and $f \sim 1$ in A.*

Proof We assume that α_n occurs o_n times in the sequence. This sequence will be denoted by α_j. Further since $A^c \neq \varnothing$ simple transformations show that we can assume $A \cap \overline{\mathrm{D}} = \varnothing$. Since A is an Arakelyan set, for every $n \in \mathrm{N}_0$ there exists $r_n > n$ such that every point in $\overline{\mathrm{D}}_{r_n}^c \setminus A$ can be connected with ∞ by a Jordan path in $\overline{\mathrm{D}}_n^c \setminus A$ (Lemma 3.4). We can assume that $r_n \to \infty$ monotonically.

Let B_n be the union of all bounded components of $(A \cup \overline{\mathrm{D}}_n)^c$ and set $A_n := A \cup \overline{\mathrm{D}}_n \cup B_n$. For consistency of notation set $A_0 := A$. We claim that A_n is an Arakelyan set. Clearly A_n is closed and unbounded. Further A_n^c is connected in $\hat{\mathrm{C}}$ since all components of A_n^c are unbounded. Using Lemma 3.4 we show that A_n is locally connected at ∞. Let V be a neighbourhood of ∞ in $\hat{\mathrm{C}}$. It exists $l > n$ such that $\overline{\mathrm{D}}_l^c \subset V$ and $B_n \subset \mathrm{D}_l$. Hence $U := \overline{\mathrm{D}}_{r_l}^c \subset V$ and every point in $\overline{\mathrm{D}}_{r_l}^c \setminus A$ can be connected with ∞ by a Jordan-path γ in $\overline{\mathrm{D}}_l^c \setminus A$. Since $\overline{\mathrm{D}}_l^c \cap \overline{\mathrm{D}}_n = \varnothing$ and $\overline{\mathrm{D}}_l^c \cap B_n = \varnothing$ we deduce that γ lies in $\overline{\mathrm{D}}_l^c \setminus A_n \subset V \setminus A_n$ and the claim follows.

We define a partition of the sequence α_j as follows. Set $\Theta_0 := \{\alpha_j \mid \alpha_j \in \overline{\mathrm{D}}_{r_1}\}$ and $\Theta_n := \{\alpha_j \mid \alpha_j \in \mathrm{D}_{r_{n+1}} \setminus \overline{\mathrm{D}}_{r_n}\}$ for $n \in \mathrm{N}$. By the definition of r_n every point in $\Theta_n \subset A_n^c$ can be connected with ∞ by a path in A_n^c. Hence it is possible to define $\log(z - \alpha_j)$ as a holomorphic function in a neighbourhood of A_n for every $\alpha_j \in \Theta_n$. By Arakelyan's approximation Theorem 3.5 it exists for all $k \in \mathrm{N}$ an entire function $g_{n,k,j}$ such that $|g_{n,k,j}(z) + \log(z - \alpha_j)| \leq \varepsilon_{n,k}(|z|)$ on A_n where

$$\varepsilon_{n,k}(t) := \min\left\{ \left(\frac{1}{2}\right)^{k+1}, 1/2 \exp(-t^{1/3} r_{n+1}^k) \right\}.$$

We define

$$E_{n,k,j}(z) := (z-\alpha_j)\exp\bigl(g_{n,k,j}(z)\bigr).$$

Hence $E_{n,k,j}$ possesses exactly one zero α_j of order one. We show $E_{n,k,j} \sim 1$. For all $l \in \mathrm{N}$ we have

$$\begin{aligned}|z^l\bigl(E_{n,k,j}(z)-1\bigr)| &= |z|^l\,|\exp\bigl(g_{n,k,j}(z)+\log(z-\alpha_j)\bigr)-1|\\ &= |z|^l\,|\exp\bigl(O(|z|^{-(l+1)})\bigr)-1| = O(|z|^{-1}) \to 0\end{aligned}$$

as $z \to \infty$ in A. The claim follows from Definition 2.1. It is easy to check that $\varepsilon_{n,k}(t) \leq 1/2$. Using the elementary inequality $|e^w - 1| \leq 2|w|$ for $|w| \leq 1/2$ we obtain:

$$|E_{n,k,j}(z)-1| = |\exp\bigl(g_{n,k,j}(z)+\log(z-\alpha_j)\bigr)-1| \leq 2\left(\frac{1}{2}\right)^{k+1} = \left(\frac{1}{2}\right)^{k}$$

for $z \in \overline{\mathrm{D}}_n$. Let c_n be the (finite) cardinality of Θ_n. Consider the compact exhaustion of C by the sets $\overline{\mathrm{D}}_l$, $l \in \mathrm{N}$. Then from the estimation above we deduce for $z \in \overline{\mathrm{D}}_l$:

$$\sum_{n\geq l}\sum_{\alpha_j\in\Theta_n}|E_{n,k_n,j}(z)-1| \leq \sum_{n=l}^{\infty} c_n\left(\frac{1}{2}\right)^{k_n} < \infty$$

for suitable k_n. Thus the product

$$f(z) := \prod_{n=0}^{\infty}\prod_{\alpha_j\in\Theta_n} E_{n,k_n,j}(z)$$

converges normally in C and hence represents an entire function with exactly the zeros α_n of order o_n (*cf.* [R1, p. 8]). To show that the product also converges in E it is by Theorem 3.1 sufficient to show that $f_N(z) := \prod_{n=0}^{N}\prod_{\alpha_j\in\Theta_n} E_{n,k_n,j}(z)$ is bounded in the norms M_l for all $l \in \mathrm{N}_0$. We deal the case $l = 0$. Since M_0 is submultiplicative we get

$$\begin{aligned}M_0(f_N) &\leq \prod_{n=0}^{N}\prod_{\alpha_j\in\Theta_n} M_0(E_{n,k_n,j}) \leq \prod_{n=0}^{N}\prod_{\alpha_j\in\Theta_n}\exp\bigl(\max_{z\in A}\varepsilon_{n,k_n}(|z|)\bigr)\\ &\leq \exp\left(\frac{1}{2}\sum_{n=0}^{N} c_n\exp(-r_{n+1}^{k_n})\right).\end{aligned} \tag{2}$$

(Recall $A \cap \overline{\mathrm{D}} = \varnothing$.) For large n clearly $\exp(-r_{n+1}^{k_n}) \leq (1/2)^{k_n}$ and the series on the right converges. For $l > 0$ first note that since $E_{n,k_n,j} \sim 1$ we have $f_N \sim 1$. Thus the norms can be expressed by

$$\begin{aligned}M_l(f_N) &= \sup_{z\in A}|z|^l\,|f_N(z)-1|\\ &= \sup_{z\in A}|z|^l\left|\exp\Bigl(\sum_{n=0}^{N}\sum_{\alpha_j\in\Theta_n} g_{n,k_n,j}(z)+\log(z-\alpha_j)\Bigr)-1\right|.\end{aligned} \tag{3}$$

By enlarging k_n if needed we can assume $\sum_{n=0}^{\infty} c_n \exp(-r_{n+1}^{k_n}) \le 1$. Then

$$\Big|\sum_{n=0}^{N} \sum_{\alpha_j \in \Theta_n} g_{n,k_n,j}(z) + \log(z-\alpha_j)\Big| \le \frac{1}{2}\sum_{n=0}^{N} c_n \exp(-r_{n+1}^{k_n}) \le \frac{1}{2}.$$

Using again $|e^w - 1| \le 2|w|$ for $|w| \le 1/2$ we obtain from (3) for $z \in A$

$$\begin{aligned} M_l(f_N) &\le 2 \sup_{z\in A} |z|^l \Big|\sum_{n=0}^{N} \sum_{\alpha_j \in \Theta_n} g_{n,k_n,j}(z) + \log(z-\alpha_j)\Big| \\ &\le \sup_{z\in A} \sum_{n=0}^{N} c_n |z|^l \exp(-|z|^{1/3} r_{n+1}^{k_n}). \end{aligned} \tag{4}$$

It is elementary to show that for a constant $c > 0$ the function $t^l \exp(-ct^{1/3})$ takes its maximum on the positive reals in $t = (3l/c)^3$. Hence

$$M_l(f_N) \le \sum_{n=0}^{N} c_n \left(\frac{3l}{r_{n+1}^{k_n}}\right)^{3l} \exp(-3l) \le (3l)^{3l} \sum_{n=0}^{N} c_n r_{n+1}^{-3lk_n}. \tag{5}$$

Enlarging k_n if needed the last series converges hence f_N converges in E. Since $f_N \sim 1$ it follows $f \sim 1$. The proof is complete. ∎

Because of the importance of asymptotic values in the theory of entire functions we state the following immediate corollary.

Corollary 3.7 *Let γ be a Jordan path in* C *going to ∞ and let α_n be a complex sequence going to ∞ non of the α_n lying on γ. Then there exists an entire function f having exactly the zeros α_n (with prescribed order) and $f \to a$ on γ for any given $a \in$* C.

Proof Set $A := \gamma$. If $a \neq 0$ then multiply the function constructed in Theorem 3.6 by a. For the case $a = 0$ we show the existence of $f_1 \in E$ with $f \sim z^{-1}$ having no finite zero. W.l.o.g. we assume $0 \notin A$. It is possible to define a branch of $\log(z)$ on A which is holomorphic in a neighbourhood of A. By Arakelyan's approximation theorem it is possible to construct an entire function g such that $|g(z) + \log(z)| \le \exp(-|z|^{1/3})$ for all $z \in A$. Set $f_1 := e^g$. Clearly f_1 has no finite zero. Further it holds $|f_1(z)| = |\exp\big(g(z) + \log(z)\big)\frac{1}{z}| \le \exp\big(\exp(-|z|^{1/3})\big)\frac{1}{|z|} \to 0$ as $z \to \infty$ in A. The second coefficient is determined by $zf_1(z) = \exp\big(g(z)+\log(z)\big) \to 1$ since $g(z)+\log(z) \to 0$ in A. For $n \ge 2$ we get $z^n\big(f_1(z)-z^{-1}\big) = z^{n-1}\Big(\exp\big(g(z)+\log(z)\big)-1\Big) = z^{n-1}\Big(\exp\big(O(z^{-n})\big)-1\Big) = O(z^{-1})$. Hence all coefficients in the asymptotic expansion of f_1 with index greater than 1 are zero. Now choose f_2 with the zeros α_n and $f_2 \sim 1$. Set $f := f_1 f_2$. ∎

Another consequence is:

Corollary 3.8 *Let A be an Arakelyan set and α_n, β_n be admissible sequences. Then there exists a function f meromorphic in the plane having exactly the zeros α_n and poles β_n (both with prescribed order) and $f \sim 1$ in A.*

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Proof Take the quotient of two suitable functions constructed in Theorem 3.6. ■

We are now in the position to construct meromorphic functions with prescribed asymptotic expansion, zeros and poles. We will use an existence theorem in [V, Proposition 10] which is a generalization of a theorem of Ritt (see [R1, p. 299]). We formulate it in a way which fits to our situation.

Theorem 3.9 *Let K be a continuum in $\hat{\mathbb{C}}$ containing ∞ such that $\hat{\mathbb{C}} \setminus K$ is unbounded in $\mathbb{C}$. Then for every formal power series $\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ there exists a function f holomorphic on $\hat{\mathbb{C}} \setminus K$ such that $f \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ in $\hat{\mathbb{C}} \setminus K$.*

Theorem 3.10 *Let A be an Arakelyan set. For every formal series $\sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ there exists an entire function f such that $f \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ in A.*

Proof Let z_0 be a finite point in A^c. Since A is an Arakelyan set there exists a Jordan path γ connecting z_0 and ∞ in A^c. According to Theorem 3.9 there exists f_1 holomorphic on $\hat{\mathbb{C}} \setminus \gamma$ with $f_1 \sim \sum_{\nu=0}^{\infty} a_\nu z^{-\nu}$ in A. Now Arakelyan's approximation theorem gives the existence of an entire function f with $|f_1(z) - f(z)| \leq \exp(-|z|^{1/3})$ on A. This implies $f_1 - f \sim 0$ in A which shows the assertion. ■

Multiplying the function from Theorem 3.10 with a suitable function from Corollary 3.8 we obtain:

Theorem 3.11 *Let A be an Arakelyan set. For admissible sequences α_n, β_n and every formal power series $\sum a_n z^{-n}$ there exists f meromorphic in the plane with exactly the zeros α_n and poles β_n such that $f \sim \sum a_n z^{-n}$ in A.*

4 A Mittag-Leffler Type Construction

We now construct meromorphic functions with prescribed principal parts outside A and asymptotic expansion in A. This can be done without any explicit construction: Let f_1 be a Mittag-Leffler series with prescribed principal parts and f_2 an entire function such that $f := f_1 - f_2 \sim 0$ in A. f_2 can easily be constructed with Arakelyan's Theorem 3.5. The reason why we construct such a function f explicitely are inequalities (6) and (7). They give estimations of $M_k(f)$ which we will use in section 5.

Theorem 4.1 *Let A be an Arakelyan set and α_n be an admissible sequence. For every n let h_n be a principal part with pole at α_n. Then there exists a function f that is meromorphic in $\mathbb{C}$ with exactly the principal parts h_n and $f \sim 0$ in A.*

Proof We use the same notation as in the proof of Theorem 3.6 and again we can assume $A \cap \overline{\mathbb{D}} = \varnothing$. For every $j \in \mathbb{N}$ choose an entire function $g_{n,k,j}$ such that

$$|h_j(z) - g_{n,k,j}(z)| \leq \varepsilon_{n,k}(|z|)$$

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for $z \in A$ where

$$\varepsilon_{n,k}(t) := \min\left\{\left(\frac{1}{2}\right)^k, \exp(-t^{1/3} r_{n+1}^k)\right\}.$$

It follows that $h_j - g_{n,k,j} \sim 0$ in A. Now consider $z \in \overline{\mathrm{D}}_l$:

$$\sum_{n \geq l} \sum_{\alpha_j \in \Theta_n} |h_j(z) - g_{n,k_n,j}(z)| \leq \sum_{n=l}^{\infty} c_n \left(\frac{1}{2}\right)^{k_n} < \infty$$

for suitable k_n. Thus the series

$$f(z) := \sum_{n=1}^{\infty} \sum_{\alpha_j \in \Theta_n} \big(h_j(z) - g_{n,k_n,j}(z)\big)$$

converges compactly in $\mathrm{C} \setminus \{\alpha_j \mid j \in \mathrm{N}\}$ to a meromorphic function with exactly the principal parts h_n.

As in the proof of Theorem 3.6 it is left to show that the sequence

$$f_N(z) := \sum_{n=1}^{N} \sum_{\alpha_j \in \Theta_n} \big(h_j(z) - g_{n,k_n,j}(z)\big)$$

is bounded in the norms M_l. For $l = 0$ we get

$$M_0(f_N) \leq \sum_{n=1}^{N} \sum_{\alpha_j \in \Theta_n} \exp(-r_{n+1}^{k_n}) < \infty. \tag{6}$$

Since $h_j - g_{n,k,j} \sim 0$ in A it follows $f_N \sim 0$. Hence for $l > 0$:

$$M_l(f_N) \leq \sup_{z \in A} |z|^l \sum_{n=1}^{N} \sum_{\alpha_j \in \Theta_n} \exp(-|z|^{1/3} r_{n+1}^{k_n}). \tag{7}$$

Now the rest follows from the argumentation after (4). ■

With this we can prove analogues of the usual conclusions from the Weierstraß product theorem and Mittag-Leffler's partial fraction theorem:

Corollary 4.2 *Let A be an Arakelyan set and α_n be an admissible sequence. Set*

$$R_n(z) := \sum_{\nu=p_n}^{q_n} a_\nu^{(n)} (z - \alpha_n)^\nu$$

with integers $p_n \leq q_n$. Then there exists a function f meromorphic in the plane with poles only at the points α_n such that the first terms of the Laurent expansions of f at α_n coincide with R_n and such that $f \sim 0$ in A.

Proof The proof is virtually the same as in the classical case: First Theorem 3.6 gives the existence of an entire function f_1 with zeros exactly at the points α_n of order $k_n > q_n$ and $f_1 \sim 1$ in A. Now let g_n be the principal part of the function $\frac{R_n}{f_1}$ at α_n, *i.e.*, $\frac{R_n}{f_1} = g_n + P_n$ locally at α_n with P_n holomorphic at α_n. According to Theorem 4.1 there exists a function g meromorphic in C with poles only in α_n and principal part g_n at α_n such that $g \sim 0$ in A. Hence locally $g = g_n + Q_n$ with Q_n holomorphic at α_n. Set $f := f_1 g$. In a neighbourhood of α_n it follows

$$f = f_1(g_n + Q_n) = f_1\left(\frac{R_n}{f_1} + Q_n - P_n\right) = R_n + f_1(Q_n - P_n).$$

Clearly $f_1(Q_n - P_n)$ is holomorphic at α_n with a zero of order at least k_n. Hence the Laurent expansion begins with R_n. Further from $f_1 \sim 1$ and $g \sim 0$ it follows $f \sim 0$. ■

Corollary 4.3 *Let A be an Arakelyan set and α_n be an admissible sequence. Further let a_n be a complex sequence. Then there exists an entire function f with $f(\alpha_n) = a_n$ for all n and $f \sim 0$ in A.*

5 Some Approximation Theorems

So far we were only concerned with qualitative statements like $f \sim 1$, *i.e.*, we did not determine the constants occuring in the O-terms of Definition 2.1. In order to prove approximation theorems we need to control the behaviour of the functions constructed in Sections 3 and 4 also at finite points.

Lemma 5.1 *Let A be an Arakelyan set and α_n be an admissible sequence. Then the function constructed in Theorem 3.6 can be chosen such that for given $\varepsilon > 0$ and $n \in$ N we have $M_0(f) \leq 1 + \varepsilon$ and $M_k(f) \leq \varepsilon$ for $k = 1, \ldots, n$.*

Proof Under the assumption $A \cap \overline{\mathrm{D}} = \varnothing$ this follows directly from inequalities (2) and (5) by enlarging k_n. Now simple transformations show the general case. ■

We can now prove approximation theorems where the approximating function has prescribed zeros and poles outside the set of approximation. Unfortunately we have to impose growth restrictions on the functions to be approximated. It would be interesting to know whether condition (8) can be dropped.

Theorem 5.2 *Let A be an Arakelyan set and g be continuous on A and holomorphic in A° such that for some $k \in$ N it holds*

$$|g(z)| \leq C|z|^k \tag{8}$$

for $z \in A$. Then for all $\varepsilon > 0$ and $n \in$ N there exists an entire function f having no zeros outside A such that

$$|f(z) - g(z)| \leq \varepsilon |z|^{-n}$$

for all $z \in A$.

Proof W.l.o.g. we assume $A \cap \overline{\mathrm{D}} = \varnothing$. By Arakelyan's Theorem 3.5 there exists an entire function h with $|h(z) - g(z)| \leq \eta \exp(-|z|^{1/3})$ for $z \in A$ and we choose $\eta > 0$ such that $\eta \exp(-|z|^{1/3}) \leq \frac{\varepsilon}{2}|z|^{-n}$. Now according to Theorem 3.6 we can construct an entire function f_1 having exactly the same zeros as h outside A (and no other zeros) with the same multiplicity and $f_1 \sim 1$. Then $f := h/f_1$ is entire and zero-free outside A. Since g satisfies (8) we deduce that h satisfies (8) with some constant $C_1 > C$. Set $K := \max\{C_1, 1\}$. It follows for $z \in A$

$$
\begin{aligned}
|f(z) - g(z)| &= \left| h(z)\left(\frac{1}{f_1(z)} - 1\right) + h(z) - g(z) \right| \\
&\leq \left| h(z)\frac{f_1(z) - 1}{f_1(z)} \right| + \eta \exp(-|z|^{1/3}) \\
&\leq K|z|^k \frac{1}{\min_{z\in A}|f_1(z)|} M_{n+k}(f_1)|z|^{-(n+k)} + \frac{\varepsilon}{2}|z|^{-n} \\
&= K\frac{1}{\min_{z\in A}|f_1(z)|} M_{n+k}(f_1)|z|^{-n} + \frac{\varepsilon}{2}|z|^{-n}.
\end{aligned}
$$

Set $\delta := \varepsilon/(2K + \varepsilon)$. According to Lemma 5.1 we can choose f_1 such that $M_{n+k}(f_1) = \sup_{z\in A}|z|^{n+k}|f_1(z) - 1| \leq \delta$. This shows in particular $\min_{z\in A}|f_1(z)| \geq 1 - \delta$. It follows

$$
K\frac{1}{\min_{z\in A}|f_1(z)|} M_{n+k}(f_1) \leq K\frac{1}{1-\delta}\delta = \frac{\varepsilon}{2}
$$

which shows the assertion. ■

Theorem 5.3 *Let A be an Arakelyan set and α_ν be an admissible sequence. Let o_ν be a sequence in* N *and g be a continuous function on A that is holomorphic in A° and such that for some $k \in$* N

$$
|g(z)| \leq C|z|^k
$$

for $z \in A$. Then for all $\varepsilon > 0$ and $n \in$ N *there exists an entire function f with exactly the zeros α_ν of order o_ν such that*

$$
|f(z) - g(z)| \leq \varepsilon|z|^{-n}
$$

for $z \in A$.

Proof Theorem 5.2 shows the existence of an entire function f_1 that has no zeros outside A with

$$
|f_1(z) - g(z)| \leq \frac{\varepsilon}{2}|z|^{-n}
$$

for $z \in A$. Further from Theorem 3.6 we get an entire function f_2 having exactly the zeros α_ν of order o_ν with $f_2 \sim 1$ in A. Set $f := f_1 f_2$. With the notation of the foregoing proof it follows for $z \in A$:

$$
\begin{aligned}
|f(z) - g(z)| &\leq |f_1(z)\big(f_2(z) - 1\big)| + |f_1(z) - g(z)| \\
&\leq C_1|z|^k M_{n+k}(f_2)|z|^{-(n+k)} + \frac{\varepsilon}{2}|z|^{-n}.
\end{aligned}
$$

Choosing f_2 such that $M_{n+k}(f_2) \leq \frac{\varepsilon}{2C_1}$ (Lemma 5.1) shows the assertion. ■

For our last theorem we need the following lemma which is analoguous to Lemma 5.1.

Lemma 5.4 *The functions f constructed in Theorem 4.1 and Corollary 4.2 can be chosen such that for all $\varepsilon > 0$ and $n \in \mathrm{N}$ we have $M_k(f) \le \varepsilon$ for $k = 0, \dots, n$.*

Proof For the function in Theorem 4.1 this follows from (6) and (7). The function f from the proof of Corollary 4.2 was defined as a product $f = f_1 g$. Now f_1 was constructed by Theorem 3.6. Therefore f_1 can be chosen such that $M_k(f_1) \le 2$ for $k = 0, \dots, n$ by Lemma 5.1. The function g comes from Theorem 4.1 and can be chosen such that $M_k(g) \le \frac{\varepsilon}{2n}$ for $k = 0, \dots, n$. This follows again from (6) and (7). The estimation $M_k(f \cdot g) \le \sum_{\kappa=0}^{k} M_\kappa(f) M_{k-\kappa}(g)$ of Lemma 2.4 shows immediately $M_k(f) \le \varepsilon$ for $k = 0, \dots, n$. ■

Using Lemma 5.4 we can prove an approximation theorem where the approximating function solves an interpolation problem outside A. (The problem of approximation and simultaneous interpolation inside A was treated, *e.g.*, in [GH].)

Here the growth restriction (8) can be omitted.

Theorem 5.5 *Let A be an Arakelyan set and α_n be an admissible sequence. Further let g be a continuous function on A that is holomorphic in A°. Set*

$$R_n(z) := \sum_{\nu=p_n}^{q_n} a_\nu^{(n)} (z - \alpha_n)^\nu$$

with integers $p_n \le q_n$. Then for all $\varepsilon > 0$, $k \in \mathrm{N}$ there exists a function f meromorphic in the plane with poles only at the points α_n such that the first terms of the Laurent expansions of f at α_n coincide with R_n and such that

$$|f(z) - g(z)| \le \varepsilon |z|^{-k}$$

in A.

Proof By Arakelyan's approximation theorem we have an entire function f_1 such that $|f_1(z) - g(z)| \le \frac{\varepsilon}{2}|z|^{-k}$ on A. Let $P_n(z) := \sum_{\nu=0}^{q_n} c_\nu^{(n)} (z - \alpha_n)^\nu$ be the first terms in the Taylor expansion of f_1 at α_n and set $S_n := R_n - P_n$. According to Theorem 4.1 and Lemma 5.4 there exists a meromorphic function f_2 with prescribed first terms S_n in the Laurent expansions around the points α_n and $|f_2(z)| \le \frac{\varepsilon}{2}|z|^{-k}$ on A. Set $f := f_1 + f_2$ and the rest follows easily. ■

Corollary 5.6 *Let A be an Arakelyan set and α_n be an admissible sequence. Further let a_n be a complex sequence and g be a continuous function on A that is holomorphic in A°. Then for all $\varepsilon > 0$, $k \in \mathrm{N}$ there exists an entire function f such that $f(\alpha_n) = a_n$ and*

$$|f(z) - g(z)| \le \varepsilon |z|^{-k}$$

in A.

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