DIRECTED COVERS AND PARACOMPACT SPACES

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A cover $\mathfrak{U}$ is *directed* (*is well-ordered*, *totally ordered*) if it is directed (is well-ordered, totally ordered, respectively) by set inclusion. Specifically $\mathfrak{U}$ is directed if, given U, V in $\mathfrak{U}$, there exists W in $\mathfrak{U}$ such that $U \cup V \subset$ W.

Directed covers have been used implicitly in numerous theorems dealing with paracompactness and related properties. In particular, characterizations of paracompactness in terms of product spaces seem to require the use of directed covers (**3**; **11**, § 2; or **12**). However, the utility of directed covers has not been fully exploited.

In this note we shall use directed covers to obtain characterizations of $\mathfrak{M}$-paracompact spaces. The principal result of this paper is given in Theorem 5. A corollary to that theorem is the following: *A topological space is paracompact if and only if every directed open cover has a locally finite closed refinement* (**7**).

Here the term *cover* will mean a collection of sets (not necessarily open) whose union is the entire space. A topological space X is $\mathfrak{M}$-*paracompact* ($\mathfrak{M}$—an infinite cardinal) if every open cover having power $\leqslant \mathfrak{M}$ has a locally finite open refinement. The space is *paracompact* if it is $\mathfrak{M}$-paracompact for every infinite cardinal $\mathfrak{M}$.

LEMMA 1. (i) *Each countable directed cover contains a well-ordered subcover.*
(ii) *Each totally ordered cover contains a well-ordered subcover.*

Proof. This lemma is a restatement of the well-known fact that each countable directed set and each totally ordered set contains a well-ordered cofinal subset.

The following is essentially due to Alexandroff and Urysohn (**1**).

THEOREM 2. *The following are equivalent for any topological space S and any infinite cardinal $\mathfrak{M}$.*

(a) *X is $\mathfrak{M}$-compact.*
(b) *X is an element of each directed open cover of X having power $\leqslant \mathfrak{M}$.*
(c) *X is an element of each well-ordered open cover of X having power $\leqslant \mathfrak{M}$.*

Proof. A finite directed cover contains X as an element. Hence (a) implies (b). Obviously, (b) implies (c). The implication (c) implies (a) is proved as in (**1**, Theorem I, on p. 8).

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A collection $\mathfrak{A}$ of subsets of a topological space is *closure-preserving* if, for every subcollection $\mathfrak{B}$ of $\mathfrak{A}$, the closure of the union is the union of the closures, i.e. $\mathrm{cl} \bigcup \{B : B \in \mathfrak{B}\} = \bigcup \{\mathrm{cl}\, B: B \in \mathfrak{B}\}$. The collection $\mathfrak{A}$ is *σ-closure-preserving* if it is the union of a sequence of closure-preserving subcollections **(8)**.

We shall need the following easily proved lemma **(8**, Lemma **1)**.

LEMMA 3. *If an indexed family $\{U_\alpha, \alpha \in \Gamma\}$ is refined by a locally finite (closure-preserving) family $\mathfrak{B}$, there exists a locally finite (a closure-preserving) indexed family $\{V_\alpha, \alpha \in \Gamma\}$ such that $V_\alpha \subset U_\alpha\}$ for all $\alpha \in \Gamma$ and*

$$\bigcup \{V: V \in \mathfrak{B}\} = \bigcup \{V_\alpha, \alpha \in \Gamma\}.$$

Moreover, if each V in $\mathfrak{B}$ is open (closed) then each V_α may be taken to be open (closed).

THEOREM 4. *A topological space X is countably paracompact if and only if for every countable directed open cover $\mathfrak{U}$ there exists a σ-closure-preserving open cover $\mathfrak{B}$ such that $\{\mathrm{cl}\, V: V \in \mathfrak{B}\}$ refines $\mathfrak{U}$.*

Proof. According to Lemma 1, a countable directed cover $\mathfrak{U}$ admits a subcover $\{U_n: n = 1, 2, \ldots\}$ such that $U_n \subset U_{n+1}$. It follows from **(5)** that if X is countably paracompact, then there exists an open cover $\{V_n\}$ such that $\mathrm{cl}\, V_n \subset U_n$. Clearly $\{V_n\}$ satisfies all the requirements for $\mathfrak{B}$. Conversely, let $\mathfrak{U} = \{U_n: n = 1, 2, \ldots\}$ be an open cover of X with $U_n \subset U_{n+1}$. If $\mathfrak{U}$ has a σ-closure-preserving open refinement whose closures refine $\mathfrak{U}$, then it follows, using Lemma 3, that there exists an open cover $\{V_n\}$ such that $\mathrm{cl}\, V_n \subset U_n$ for each index n. Hence by **(5)**, X is countably paracompact.

Remark. Using the notation of **(9)**, Theorem 4 can be stated as follows: *X is countably paracompact if and only if every countable directed open cover has a σ-cushioned open refinement.*

THEOREM 5. *For any topological space X and any infinite cardinal $\mathfrak{M}$, the following are equivalent:*

(a) *X is $\mathfrak{M}$-paracompact.*

(b) *X is countably paracompact and every open cover of X having power $\leqslant \mathfrak{M}$ has a σ-locally finite open refinement.*

(c) *Every directed open cover of X having power $\leqslant \mathfrak{M}$ has a locally finite open refinement.*

(d) *For every directed open cover $\mathfrak{U}$ of X having power $\leqslant \mathfrak{M}$, there exists a locally finite open cover $\mathfrak{B}$ such that $\{\mathrm{cl}\, V: V \in \mathfrak{B}\}$ refines $\mathfrak{U}$.*

(e) *Every directed open cover of X having power $\leqslant \mathfrak{M}$ has a locally finite closed refinement.*

(f) *Every well-ordered open cover of X having power $\leqslant \mathfrak{M}$ has a locally finite open refinement.*

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(g) *For every well-ordered open cover* $\mathfrak{U}$ *of* X *having power* $\leqslant \mathfrak{M}$, *there exists a* σ*-locally finite open cover* $\mathfrak{V}$ *such that* $\{\mathrm{cl}\ V\colon V \in \mathfrak{V}\}$ *refines* $\mathfrak{U}$.

Proof. Clearly (a) and (b) are equivalent; it is also obvious that (a) implies (c) and that (d) implies (e).

(c) $\Rightarrow$ (d). Let $\mathfrak{U}$ be a directed open cover having power $\leqslant \mathfrak{M}$. We may suppose $\mathfrak{U}$ is indexed by a directed set Γ (with card $\Gamma \leqslant \mathfrak{M}$) such that $\alpha \leqslant \beta$ if and only if $U_\alpha \subset U_\beta$. By (c) and Lemma 3, there exists a locally finite open cover $\mathfrak{G} = \{G_\alpha\colon \alpha \in \Gamma\}$ such that $G_\alpha \subset U_\alpha$ for each $\alpha \in \Gamma$. Set $W_\alpha = X \setminus \bigcup_{\beta \not\leqslant \alpha} \mathrm{cl}\ G_\beta$. Then $\{W_\alpha\colon \alpha \in \Gamma\}$ is a directed open cover for which $\mathrm{cl}\ W_\alpha \subset U_\alpha$ for each $\alpha \in \Gamma$. By (c) the cover $\{W_\alpha\}$ has a locally finite open refinement $\mathfrak{V}$. Clearly $\mathfrak{V}$ has all the properties required in (d). Note that the above argument also shows that (f) implies (g).

(e) $\Rightarrow$ (f). To prove this, we shall modify the proof of Lemma 1 in **(7)**. Let $\{U_\alpha\}$ be a directed open cover having power $\leqslant \mathfrak{M}$. By hypothesis there is a locally finite closed cover $\{F_\alpha\}$ such that $F_\alpha \subset U_\alpha$ for all α. Let Λ denote the collection of all finite sets of indices. For $\lambda \in \Lambda$, set $V_\lambda = X \setminus \bigcup_{\alpha \notin \lambda} F_\alpha$. Then $\{V_\lambda\colon \lambda \in \Lambda\}$ is a directed open cover having power $\leqslant \mathfrak{M}$; thus there is a locally finite closed cover $\{H_\lambda\colon \lambda \in \Lambda\}$ such that $H_\lambda \subset V_\lambda$ for all $\lambda \in \Lambda$. Finally set $W_\alpha = U_\alpha \setminus \bigcup\{H_\lambda\colon H_\lambda \cap F_\alpha = \emptyset\}$. Then $\{W_\alpha\}$ is a locally finite open refinement of $\{U_\alpha\}$.

(g) $\Rightarrow$ (b). It follows from Theorem 4 that (g) implies X is countably paracompact. We shall use transfinite induction to prove the implication for arbitrary infinite cardinal $\mathfrak{M}$. Suppose X is $\mathfrak{k}$-paracompact for every infinite cardinal $\mathfrak{k}$ less than $\mathfrak{M}$ and let $\mathfrak{U}$ be an open cover of X having power $\leqslant \mathfrak{M}$. Index $\mathfrak{U}$ with an initial interval Γ of the set of ordinals less than the first ordinal having power $\mathfrak{M}$. Then $\mathfrak{U}' = \{\bigcup_{\beta < \alpha} U_\beta\colon \alpha \in \Gamma\}$ is a well-ordered open cover having power $\leqslant \mathfrak{M}$. By assumption (g), there exists a σ-locally finite open cover $\mathfrak{V}$ whose closures refine $\mathfrak{U}'$. For $V \in \mathfrak{V}$ there exists an index α such that $\{U_\beta\colon \beta \leqslant \alpha\}$ covers $\mathrm{cl}\ V$. Since by the induction hypothesis X is $\mathfrak{k}$-paracompact for $\mathfrak{k} = \mathrm{card}\ \alpha$, there exists a locally finite collection $\mathfrak{W}_V$ of open subsets of X such that $\mathfrak{W}_V$ refines $\mathfrak{U}$ and covers $\mathrm{cl}\ V$. Then

$$\{W \cap V\colon W \in \mathfrak{W}_V,\ V \in \mathfrak{V}\}$$

is a σ-locally finite open refinement of $\mathfrak{U}$.

COROLLARY 6. *A topological space is paracompact if and only if every well-ordered open cover has a locally finite open refinement.*

COROLLARY 7. *A topological space* X *is paracompact if and only if every directed open cover has a locally finite closed refinement.*

Note that unlike Lemma 1 of **(7)**, X is not assumed to be regular in Corollary 7.

THEOREM 8. *A T_1-space is $\mathfrak{M}$-compact if and only if it is $\mathfrak{M}$-paracompact and countably compact.*

Proof. See **(2)**.

A topological space X is *pseudocompact* if every real-valued continuous function on X is bounded.

COROLLARY 9. *A completely regular Hausdorff space is $\mathfrak{M}$-compact if and only if it is $\mathfrak{M}$-paracompact and pseudocompact.*

Proof. See **(4)**.

Remark. Theorem 8 can be modified to read: *X is $\mathfrak{M}$-compact if and only if X is countably compact and every open cover of X having power $\leqslant \mathfrak{M}$ has a point-finite open refinement.*

A subset A of a topological space X is a *zero-set* if there exists a real-valued continuous function f on X such that $A = f^{-1}(0)$. The complement of a zero-set is a *cozero-set*. A set of A is a *generalized F_σ-set* (*generalized cozero-set*) if every open set containing A contains an F_σ-set (cozero-set) which contains A.

THEOREM 10. *Let Y be a generalized F_σ-subset of a countably paracompact space X. If the subspace is Y normal, then it is countably paracompact.*

Proof. Let $\mathfrak{U} = \{U_k\}$ be a countable open cover of Y. For each k, let V_k be an open set in X such that $V_k \cap Y = U_k$ and then set $G = \bigcup V_k$. Since G is an open subset of X which contains Y, there exists a sequence $\{F_n\}$ of closed subsets of X such that $Y \subset \bigcup F_n \subset G$. Then for each n,

$$\{V_k\colon k = 1, 2, \ldots\} \cup \{X \diagdown F_n\}$$

is a countable open cover of X. Hence, for each n, there exists a locally finite open cover

$$\{W_{nk}, k = 1, 2, \ldots\} \cup \{X \diagdown F_n\}$$

such that $W_{nk} \subset V_k$ for each k. Thus

$$\mathfrak{W} = \{W_{nk} \cap [Y \diagdown \bigcup_{m<n} F_m];\ n = 1, 2, \ldots;\ k = 1, 2, \ldots\}$$

is a countable refinement of $\mathfrak{U}$. Since

$$\{Y \diagdown \bigcup_{m<n} F_m\colon n = 1, 2, \ldots\}$$

is point-finite, $\mathfrak{W}$ is also point-finite. In a normal space each countable point-finite open cover has a locally finite open refinement **(10)**. Thus $\mathfrak{W}$ (and hence $\mathfrak{U}$) has a locally finite open refinement.

THEOREM 11. *Let Y be a generalized F_σ-subspace of an $\mathfrak{M}$-paracompact space X. If Y is either countably paracompact or normal, then Y is $\mathfrak{M}$-paracompact.*

Proof. A standard argument **(7**, Proposition 3**)** can be used to show that each open cover of Y having power $\leqslant \mathfrak{M}$ has a σ-locally finite open refinement. If Y is normal, then it is countably paracompact by Theorem 10 and if it is countably paracompact it is $\mathfrak{M}$-paracompact by Theorem 5.

THEOREM 12. *Let X be an $\mathfrak{M}$-paracompact space and let Y be a subset of X such that for every open set G containing Y, there exists a sequence $\{F_n\}$ of closed sets contained in G for which $\bigcup$ int $F_n \supset Y$. Then Y is $\mathfrak{M}$-paracompact.*

Proof. In view of Theorem 11, it suffices to show that Y is countably paracompact. This may be done as in Theorem 10. Note that under the hypotheses given here, the cover $\mathfrak{W}$ is locally finite as well as being point-finite.

COROLLARY 13. *Every generalized cozero subspace of an $\mathfrak{M}$-paracompact space is $\mathfrak{M}$-paracompact.*

LEMMA 14. *Let f be a mapping from a topological space X onto a second space Y such that $f^{-1}(y)$ is $\mathfrak{M}$-compact for every $y \in Y$. If $\mathfrak{U}$ is a directed open cover of X having power $\leqslant \mathfrak{M}$, then $f^*(\mathfrak{U}) = \{Y \setminus f(X \setminus U) \colon U \in \mathfrak{U}\}$ is a directed cover of Y. Moreover, if f is a closed mapping, then $f^*(\mathfrak{U})$ is an open cover.*

Proof. Let $y \in Y$. Then $\{U \cap f^{-1}(y) \colon U \in \mathfrak{U}\}$ is a directed open cover having power $\leqslant \mathfrak{M}$, of the $\mathfrak{M}$-compact space $f^{-1}(y)$. According to Theorem 2, there exists $U \in \mathfrak{U}$ such that $f^{-1}(y) = f^{-1}(y) \cap U \subset U$. Thus

$$Y \setminus f(X \setminus U) \supset Y \setminus f(X \setminus f^{-1}(y)) = \{y\};$$

whence $f^*(\mathfrak{U})$ covers Y.

The following lemma is implicit in the proof of Corollary 2.3 in **(11)**.

LEMMA 15. *Let f be a closed mapping from X onto Y such that $f^{-1}(y)$ is compact for each $y \in Y$. Then the image under f of a locally finite collection of subsets of X is locally finite in Y.*

THEOREM 16. *Let f be a closed continuous mapping from a topological space X onto a space Y.*

(a) *If $f^{-1}(y)$ is $\mathfrak{M}$-compact for each $y \in Y$ and Y is $\mathfrak{M}$-paracompact, then X is $\mathfrak{M}$-paracompact.*

(b) *If $f^{-1}(y)$ is compact for each $y \in Y$ and X is $\mathfrak{M}$-paracompact, then Y is $\mathfrak{M}$-paracompact.*

Proof. Part (a) follows directly from Lemma 14 while (b) follows from Theorem 5(e) and Lemma 15.

THEOREM 17. *Let X be an $\mathfrak{M}$-paracompact space such that each point has a basic neighbourhood system having power $\leqslant \mathfrak{M}$. Then the product of X with an $\mathfrak{M}$-compact space Y is $\mathfrak{M}$-paracompact.*

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Proof. It is easy to show (**11**, footnote 6) that the projection of $X \times Y$ onto X is a closed mapping. This theorem is then an immediate consequence of Theorem 16.

COROLLARY 18. *The product of a metric space with an $\mathfrak{M}$-compact space is $\mathfrak{M}$-paracompact.*

Remark. The special case of Theorem 17 for which X is paracompact and normal is proved in (**11**, 4.1).

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