BULL. AUSTRAL. MATH. SOC. 46B20, 46A17, 46B03, 58C20
VOL. 54 (1996) [155–166]

ON WEAK HADAMARD DIFFERENTIABILITY OF CONVEX FUNCTIONS ON BANACH SPACES

J.R. GILES AND SCOTT SCIFFER

We study two variants of weak Hadamard differentiability of continuous convex functions on a Banach space, uniform weak Hadamard differentiability and weak Hadamard directional differentiability, and determine their special properties on Banach spaces which do not contain a subspace topologically isomorphic to ℓ_1.

Given a real Banach space X with dual X^*, the *Mackey topology* on X^*, $\tau(X^*,X)$, is the topology of uniform convergence on weakly compact subsets of X. It has recently been shown that a Banach space X does not contain a subspace topologically isomorphic to ℓ_1 if and only if every Mackey convergent sequence in X^* is norm convergent, [**1**, p.1132]. This characterisation has implications for the differentiability of continuous convex functions on X.

A continuous convex function ϕ on an open convex subset A of X is said to be *Gâteaux differentiable* at $x \in A$ if there exists a continuous linear functional $\phi'(x)$ on X and given any $\varepsilon > 0$ and $v \in X$ there exists a $\delta(\varepsilon, x, v) > 0$ such that

$$\left|\frac{\phi(x+tv)-\phi(x)}{t} - \phi'(x)(v)\right| < \varepsilon \quad \text{for all } 0 < t < \delta.$$

Further, ϕ is said to be *weak Hadamard differentiable* at $x \in A$ if given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, x, K) > 0$ such that the inequality holds for all $0 < t < \delta$ and $v \in K$. Moreover, ϕ is said to be *Fréchet differentiable* at $x \in A$ if given $\varepsilon > 0$ there exists a $\delta(\varepsilon, x) > 0$ such that the inequality holds for all $0 < t < \delta$ and $v \in X$, $\|v\| = 1$. In a reflexive space X, ϕ is weak Hadamard differentiable at $x \in A$ if and only if it is Fréchet differentiable at x, and in ℓ_1, ϕ is weak Hadamard differentiable at $x \in A$ if and only if ϕ is Gâteaux differentiable at x. It has recently been shown that on a Banach space X which does not contain a subspace topologically isomorphic to ℓ_1, ϕ is weak Hadamard differentiable at $x \in A$ if and only if it is Fréchet differentiable at x, [**1**, p.1124].

In this paper we study two variants of weak Hadamard differentiability of continuous convex functions, uniform weak Hadamard differentiability and weak Hadamard directional differentiability, and using the recent sequential characterisation for Banach spaces which do not contain a subspace topologically isomorphic to ℓ_1, we determine the special properties of these differentiability conditions on such spaces.

Received 27th September, 1995.

1. Uniform weak Hadamard differentiability

The uniform variant of weak Hadamard differentiability of the norm has implications for the geometry of the Banach space.

The norm of a Banach space X is said to be *uniformly Gâteaux differentiable* if it is Gâteaux differentiable at each $x \in X$, $\|x\| = 1$, and given $\varepsilon > 0$ and $v \in X$ there exists a $\delta(\varepsilon, v) > 0$ such that

$$\left|\frac{\|x+tv\| - \|x\|}{t} - \|.\|'(x)(v)\right| < \varepsilon \quad \text{for all } 0 < t < \delta \text{ and } x \in X, \|x\| = 1.$$

Similarly, we say that the norm is *uniformly weak Hadamard differentiable* if it is Gâteaux differentiable at each $x \in X$, $\|x\| = 1$, and given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that the inequality holds for all $0 < t < \delta$, $v \in K$ and $x \in X$, $\|x\| = 1$. The norm is *uniformly Fréchet differentiable* if it is Gâteaux differentiable at each $x \in X$, $\|x\| = 1$, and given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that the inequality holds for all $0 < t < \delta$ and $x, v \in X$, $\|x\| = \|v\| = 1$.

Just as with uniform Gâteaux and uniform Fréchet differentiability, uniform weak Hadamard differentiability can be characterised by a continuity property of the subdifferential mapping of the norm. For each $x \in X$ the *subdifferential* of the norm is the set

$$\partial\|x\| \equiv \{f \in X^* : f(v) \leqslant \|.\|'_+(x)(v) \text{ for all } v \in X\}$$

where $\|.\|'_+(x)(v)$ is the right-hand derivative of the norm at x in the direction v.

Lemma 1.1. *A Banach space X has uniformly weak Hadamard differentiable norm if and only if there exists a selection $x \mapsto f_x$ where $f_x \in \partial\|x\|$ with the following property:*
given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that

$$|(f_x - f_y)(v)| < \varepsilon \quad \textit{for all } x, y \in X,\ \|x\| = \|y\| = 1,\ v \in K \textit{ and } \|x - y\| < \delta.$$

Proof: Given $x \in X$, $\|x\| = 1$, any $f_x \in \partial\|x\|$ and $f_{x+tv} \in \partial\|x+tv\|$ we have

$$f_x(v) \leqslant \frac{\|x+tv\| - \|x\|}{t} \leqslant f_{x+tv}(v)$$

for $t > 0$ and $v \in X$, with the inequalities reversed for $t < 0$. So it is clear that the continuity property for the selection implies uniform weak Hadamard differentiability of the norm.

Conversely, given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that

$$\left|\frac{\|x+tv\| - \|x\|}{t} - f_x(v)\right| < \varepsilon \quad \text{for all } x \in X,\ \|x\| = 1,\ v \in K \text{ when } \frac{1}{2}\delta < t < \delta.$$

Then

$$\begin{aligned}
&|(f_x - f(y)(v))| \\
&\leqslant \left| f_x(v) - \frac{\|x+tv\|-1}{t} \right| + \left| \frac{\|x+tv\|-1}{t} - \frac{\|y+tv\|-1}{t} \right| + \left| \frac{\|y+tv\|-1}{t} - f_y(v) \right| \\
&< 4\varepsilon
\end{aligned}$$

for all $x, y \in X$, $\|x\|, \|y\| = 1$ and $v \in K$ when $\|x - y\| < \varepsilon\delta/2$. □

There is a significant rotundity property dual to uniform weak Hadamard differentiability of the norm, similar to the rotundity properties dual to uniform Gâteaux and uniform Fréchet differentiability of the norm.

A Banach space X is said to be *weakly uniformly rotund* if given $\varepsilon > 0$ and $f \in X^*$ there exists a $\delta(\varepsilon, f) > 0$ such that

$$|f(x-y)| < \varepsilon \quad \text{when } \|x+y\| > 2-\delta \text{ and } \|x\|, \|y\| \leqslant 1,$$

and the dual X^* is said to be *weak* uniformly rotund* if given $\varepsilon > 0$ and $x \in X$ there exists a $\delta(\varepsilon, x) > 0$ such that

$$|(f-g)(x)| < \varepsilon \quad \text{when } \|f+g\| > 2-\delta \text{ and } \|f\|, \|g\| \leqslant 1.$$

A Banach space is weakly uniformly rotund if and only if its dual X^* has uniformly Gâteaux differentiable norm, and X has uniformly Gâteaux differentiable norm if and only if its dual is weak* uniformly rotund, [**5**, p,63].

A Banach space X is said to be *uniformly rotund* if given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that

$$\|x-y\| < \varepsilon \quad \text{when } \|x+y\| > 2-\delta \text{ and } \|x\|, \|y\| \leqslant 1.$$

A Banach space X has uniformly Fréchet differentiable norm if and only if its dual X^* is uniformly rotund. A Banach space which is uniformly rotund is reflexive.

We say that a Banach space X has *uniformly Mackey rotund* dual X^* if given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that

$$\|(f-g)(v)\| < \varepsilon \quad \text{for all } v \in K \text{ when } \|f+g\| > 2-\delta \text{ and } \|f\|, \|g\| \leqslant 1.$$

LEMMA 1.2. *A Banach space X has uniformly weak Hadamard differentiable norm if and only if its dual X^* is uniformly Mackey rotund.*

PROOF: Given any selection $x \mapsto f_x$ where $f_x \in \partial\|x\|$ on the unit sphere, we have

$$\|f_x + f_y\| + \|x - y\| \geqslant 2.$$

https://doi.org/10.1017/S000497270001515X Published online by Cambridge University Press

When X^* is uniformly Mackey rotund then given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that

$$|(f_x - f_y)(v)| < \varepsilon \quad \text{when } \|f_x + f_y\| > 2 - \delta \text{ for all } v \in K.$$

Therefore, when $\|x - y\| < \delta$ we conclude that

$$|(f_x - f_y)(v)| < \varepsilon \quad \text{for all } v \in K.$$

Conversely, when X has uniformly weak Hadamard differentiable norm, given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, K) > 0$ such that

$$\left|\frac{\|x + tv\| - \|x\|}{t} - f_x(v)\right| < \varepsilon \quad \text{and} \quad \left|\frac{\|-x + tv\| - \|-x\|}{t} - f_{-x}(v)\right| < \varepsilon$$

for all $x \in X$, $\|x\| = 1$ and $v \in K$ for $0 < t < \delta$. Therefore,

$$\|x + tv\| + \|x - tv\| < 2 + 2\varepsilon t \quad \text{for all } x \in X,\ \|x\| = 1,\ v \in K \text{ and } 0 < t < \delta.$$

For any $f, g \in X^*$, $\|f\|, \|g\| \leqslant 1$ such that for some $v_0 \in K$, $(f - g)(v_0) \geqslant 3\varepsilon$ we have

$$\begin{aligned}
\|f + g\| &= \sup\{(f + g)(x) : \|x\| \leqslant 1\} \\
&= \sup\{f(x + tv_0) + g(x - tv_0) - t(f - g)(v_0) : \|x\| \leqslant 1\} \\
&\leqslant \sup\{\|x + tv_0\| + \|x - tv_0\| - 3\varepsilon t : \|x\| \leqslant 1\} \quad \text{for all } t > 0, \\
&< 2 - \varepsilon\delta \quad \text{when } 0 < t < \delta.
\end{aligned}$$

Therefore,

$$|(f - g)(v)| < 3\varepsilon \quad \text{for all } v \in K \text{ when } \|f + g\| > 2 - \varepsilon\delta \text{ and } \|f\|, \|g\| \leqslant 1. \qquad \square$$

We are now in a position to determine the geometrical implications of these properties in Banach spaces which do or do not contain a subspace topologically isomorphic to ℓ_1.

Theorem 1.3. *A Banach space X which does not contain a subspace topologically isomorphic to ℓ_1 is reflexive if its norm is uniformly weak Hadamard differentiable.*

Proof: We show that the dual X^* is uniformly rotund. Suppose that this is not so. Then there exists an $r > 0$ and sequences $\{f_n\}, \{g_n\}$ in X^* where $\|f_n\|, \|g_n\| \leqslant 1$ for all $n \in \mathbb{N}$ such that $\|f_n + g_n\| \to 2$ as $n \to \infty$ but $\|f_n - g_n\| > r$ for all $n \in \mathbb{N}$. Since X does not contain a subspace topologically isomorphic to ℓ_1 we can conclude that $\{f_n - g_n\}$ is not Mackey convergent to 0; that is, there exists a weakly compact

https://doi.org/10.1017/S000497270001515X Published online by Cambridge University Press

set K in X such that $\{(f_n - g_n)(v)\}$ does not converge to 0 uniformly for all $v \in K$. This contradicts X^* being uniformly Mackey rotund. $\square$

However we should note that a non-reflexive Banach space which does not contain a subspace topologically isomorphic to ℓ_1 may have a dual whose norm is uniformly weak Hadamard differentiable. The space c_0 can be equivalently renormed to be weakly uniformly rotund and then ℓ_∞ is weak* uniformly rotund. Since weak compact sets in ℓ_1 are norm compact, the Mackey topology on ℓ_∞ is the weak* topology and so ℓ_∞ is uniformly Mackey rotund. Further, a Banach space whose dual has uniformly weak Hadamard differentiable norm cannot contain a subspace topologically isomorphic to ℓ_1. Weakly convergent sequences in ℓ_1 are norm convergent, so if such a space did contain a subspace topologically isomorphic to ℓ_1 then ℓ_1 could be equivalently renormed to be uniformly rotund, which is impossible.

Given a measure space (Ω, Σ, μ) it has been shown that the Banach space $L_1(\Omega, \Sigma, \mu)$ admits an equivalent weak Hadamard differentiable norm if and only if μ is σ-finite, [**2**, p,409]. However this can be extended to give the following stronger result.

Theorem 1.4. *A Banach space $L_1(\Omega, \Sigma, \mu)$ has an equivalent uniformly weak Hadamard differentiable norm if and only if μ is σ-finite.*

If the measure μ is not σ-finite then $L_1(\Omega, \Sigma, \mu)$ does not admit an equivalent Gâteaux differentiable norm, [**4**, p.161].

The proof of the theorem depends upon the following strengthening of Proposition 2.3, [**2**, p.409].

Lemma 1.5. *Given a Banach space $L_1(\Omega, \Sigma, \mu)$ with μ finite, the dual $L_\infty(\Omega, \Sigma, \mu)$ has an equivalent uniformly Mackey rotund dual norm.*

Proof: On $L_\infty(\Omega, \Sigma, \mu)$ consider the equivalent dual norm

$$\|f\| = \sqrt{\|f\|_\infty^2 + \|f\|_2^2}.$$

We show that this is a uniformly Mackey rotund norm. Suppose not, then there exist sequences $\{f_n\}, \{g_n\}$ in $L_\infty(\Omega, \Sigma, \mu)$ such that $\|f_n\|, \|g_n\| \leqslant 1$ for all $n \in \mathbb{N}$ and $\|f_n + g_n\| \to 2$ as $n \to \infty$ but $\{f_n - g_n\}$ does not converge to 0 in the Mackey topology on $L_\infty(\Omega, \Sigma, \mu)$. Then by [**2**, p.408], $\|f_n - g_n\|_1$ does not converge to 0 as $n \to \infty$. Then $\|f_n - g_n\|_2$ does not converge to 0 as $n \to \infty$, but as $L_2(\Omega, \Sigma, \mu)$ is uniformly rotund we deduce that $\|f_n + g_n\|_2$ does not converge to 2 as $n \to \infty$. However this contradicts the original choice of the sequences $\{f_n\}$ and $\{g_n\}$. $\square$

Again since weakly compact sets in ℓ_1 are norm compact, an equivalent norm on ℓ_1 is uniformly weak Hadamard differentiable if and only if it is uniformly Gâteaux differentiable.

2. Weak Hadamard directional differentiability

Consider a continuous convex function ϕ on an open convex subset A of a Banach space X. Now at each $x \in A$ the right-hand derivative $\phi'_+(x)(v)$ exists for all $v \in X$ and is a continuous sublinear functional in v.

We introduce the following weaker differentiability conditions. We say that ϕ is *weak Hadamard directionally differentiable* at $x \in A$ if given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, x, K) > 0$ such that

$$\left| \frac{\phi(x+tv) - \phi(x)}{t} - \phi'_+(x)(v) \right| < \varepsilon \quad \text{for all } 0 < t < \delta \text{ and } v \in K.$$

We say that ϕ is *Fréchet directionally differentiable* at $x \in A$ if given $\varepsilon > 0$ there exists a $\delta(\varepsilon, x) > 0$ such that the inequality holds for all $0 < t < \delta$ and $v \in X$, $\|v\| = 1$.

The differentiability of a continuous convex function ϕ on an open convex subset A of a Banach space X is studied by means of the *subdifferential mapping* $x \mapsto \partial\phi(x)$ where

$$\partial\phi(x) \equiv \{f \in X^* : f(v) \leqslant \phi'_+(x)(v) \text{ for all } v \in X\}.$$

This mapping is a *weak* cusco*; that is, for $x \in A$, $\partial\phi(x)$ is weak*compact and convex and given a weak*open set W in X^* such that $\partial\phi(x) \subset W$ there exists an open neighbourhood N of x such that $\partial\phi(N) \subset W$. But it is also *minimal*; that is, given any open set U in X and weak*open half-space W in X^* where $\partial\phi(U) \cap W \neq \emptyset$, there exists a non-empty open set $V \subset U$ such that $\partial\phi(V) \subset W$.

We say that the subdifferential mapping $x \mapsto \partial\phi(x)$ is *Hausdorff Mackey upper semi-continuous* at $x \in A$ if given a weakly compact set K in X with polar K^o in X^* there exists an open neighbourhood N of x such that $\partial\phi(N) \subset \partial\phi(x) + K^o$, and we say that it is *Hausdorff norm upper semi-continuous* at $x \in A$ if given $\varepsilon > 0$ there exists an open neighbourhood N of x such that $\partial\phi(N) \subset \partial\phi(x) + \varepsilon B(X^*)$.

Fréchet directional differentiability for continuous convex functions has recently been characterised by Hausdorff norm upper semi-continuity of the subdifferential mapping, [**6**, Theorem 3.2]. A similar characterisation can be given for weak Hadamard directional differentiability.

Lemma 2.1. *A continuous function ϕ on an open convex subset A of a Banach space X is weak Hadamard directionally differentiable at $x \in A$ if and only if the subdifferential mapping $x \mapsto \partial\phi(x)$ is Hausdorff Mackey upper semi-continuous at x.*

Proof: Suppose that ϕ is weak Hadamard directionally differentiable at $x \in A$ but the subdifferential mapping $x \mapsto \partial\phi(x)$ is not Hausdorff Mackey upper semi-continuous at x. Then there exists a weakly compact set K in X and for every open neighbourhood U of x,

$$\partial\phi(U) \not\subset \partial\phi(x) + K^o = \{f \in X^* : (f - f_x)(v) \leqslant 1 \text{ for some } f_x \in \partial\phi(x) \text{ and all } v \in K\}.$$

So given an open neighbourhood U of x there exists an $f_0 \in \partial\phi(U)$ and a $v_0 \in K$ such that

$$(f_0 - f_x)(v_0) > 1 \text{ for all } f_x \in \partial\phi(x).$$

That is, $v_0 \in K$ separates f_0 from $\partial\phi(x) + K^o$. Since the subdifferential mapping $x \mapsto \partial\phi(x)$ is a minimal weak* cusco there exists a non-empty open set $V \subset U$ such that $\partial\phi(V)$ is separated from $\partial\phi(x) + K^o$ by $\widehat{v}_0$. Then

$$(f - f_x)(v_0) > 1 \text{ for all } f \in \partial\phi(V) \text{ and } f_x \in \partial\phi(x).$$

Now given $0 < \varepsilon < 1/2$ there exists a $\delta(\varepsilon, K) > 0$ such that

$$\left|\frac{\phi(x+tv) - \phi(x)}{t} - \phi'_+(x)(v)\right| < \varepsilon \quad \text{for all } 0 < t \leqslant \delta \text{ and } v \in K.$$

Write $y_0 \equiv \delta v_0$. Then since

$$\phi'_+(x)(y_0) = \sup\{f_x(y_0) : f_x \in \partial\phi(x)\}$$

there exists an $f'_x \in \partial\phi(x)$ such that

$$|\phi(x + y_0) - \phi(x) - f'_x(y_0)| < \varepsilon\delta.$$

Now $f((x + y_0) - x') \leqslant \phi(x + y_0) - \phi(x')$ for all $x' \in V$ and $f \in \partial\phi(x')$. So

$$f(y_0) \leqslant \phi(x + y_0) - \phi(x) + f(x' - x) + \phi(x) - \phi(x').$$

Then

$$\begin{aligned}\delta &< (f - f'_x)(y_0) \\ &\leqslant \phi(x + y_0) - \phi(x) - f'_x(y_0) + f(x' - x) + \phi(x) - \phi(x') \\ &< \varepsilon\delta + M\,\|x' - x\| + \phi(x) - \phi(x') \quad \text{for some } M > 0 \text{ and all } x' \in V \subset U.\end{aligned}$$

But this implies ϕ is not continuous at x.

Conversely, suppose that the subdifferential mapping $x \mapsto \partial\phi(x)$ is Hausdorff Mackey upper semi-continuous at $x \in A$. Then given $\varepsilon > 0$ and a weakly compact set K in X there exists a $\delta(\varepsilon, x, K) > 0$ such that

$$\partial\phi(x') \subset \partial\phi(x) + \varepsilon K^o \quad \text{for all } x' \in B(x; \delta);$$

that is, $(f - f_x)(v) \leqslant \varepsilon$ for some $f_x \in \partial\phi(x)$, all $f \in \partial\phi(x')$, $x' \in B(x;\delta)$ and all $v \in K$. Now K is bounded so there exists a $\delta' > 0$ such that $\|tv\| < \delta$ for all $0 < t < \delta'$

https://doi.org/10.1017/S000497270001515X Published online by Cambridge University Press

and $v \in K$. So $(f_{x+tv} - f_x)(v) \leqslant \varepsilon$ for some $f_x \in \partial\phi(x)$, all $f_{x+tv} \in \partial\phi(x+tv)$, $0 < t < \delta'$ and all $v \in K$. As $\phi'_+(x)(v) = \sup\{f_x(v) : f_x \in \partial\phi(x)\}$ and $\phi'_+(x)(v)$ is positively homogeneous in v,

$$t\phi'_+(x)(v) \leqslant \phi(x+tv) - \phi(x) \leqslant tf_{x+tv}(v) \leqslant tf_x(v) + \varepsilon t \leqslant t\phi'_+(x)(v) + \varepsilon t \quad \text{for } 0 < t < \delta'.$$

We conclude that

$$\left| \frac{\phi(x+tv) - \phi(x)}{t} - \phi'_+(x)(v) \right| < \varepsilon \quad \text{for all } 0 < t < \delta' \text{ and } v \in K.$$

∎

We now determine the special properties of weak Hadamard directionally differentiable convex functions on a Banach space which does not contain a subspace topologically isomorphic to ℓ_1.

Theorem 2.2. *Consider a continuous convex function ϕ on an open convex subset A of a Banach space X which does not contain a subspace topologically isomorphic to ℓ_1. If ϕ is weak Hadamard directionally differentiable on A then ϕ is Fréchet differentiable on a dense G_δ subset of A.*

Proof: If X is separable it is a weak Asplund space and so ϕ is weak Hadamard differentiable on a dense G_δ subset of A. But since X does not contain a subspace topologically isomorphic to ℓ_1, then ϕ is Fréchet differentiable on a dense G_δ subset of A. If X is not separable, it follows from [**7**, p.162] that it is sufficient to prove that for every separable closed subspace Y of X where $A \cap Y \neq \emptyset$, $\phi|_Y$ is weak Hadamard differentiable on $A \cap Y$. This follows since the injection is continuous, mapping Y with its weak topology into X with its weak topology and so K weakly compact in Y is weakly compact in X. ∎

We cannot expect such a result to extend to Banach spaces in general. Phelps has shown that ℓ_1 can be given an equivalent norm which is Gâteaux differentiable except at the origin and is nowhere Fréchet differentiable, [**8**, p.86]. Again since weakly compact sets in ℓ_1 are norm compact, such a norm is weak Hadamard differentiable except at the origin.

3. Counterexamples

Although it has been shown that for a Banach space X which does not contain a subspace topologically isomorphic to ℓ_1, every Mackey convergent sequence in X^* is norm convergent, it is instructive to see that this does not necessarily imply that sequences which are Hausdorff Mackey convergent to a set are Hausdorff norm convergent to the set.

EXAMPLE 3.1. We exhibit a sequence $\{f_n\}$ in $\ell_1 \cong c_0^*$ which is Hausdorff Mackey convergent to a weak* compact convex set in ℓ_1 but not Hausdorff norm convergent to that set.

Consider $\{e_n\}$ the canonical basis for c_0 and $\{e_n^*\}$ the dual basis for ℓ_1. Consider the set

$$C \equiv \overline{co}^{w^*}\{e_m^* - e_n^* : m \leqslant n, \quad m, n \in \mathbf{N}\}.$$

For each $n \in \mathbf{N}$ consider $f_n = -e_n^*$. Then $d(f_n, C) = 1$ for all $n \in \mathbf{N}$ and so the sequence $\{f_n\}$ is not Hausdorff norm convergent to the set C.

Suppose that $\{f_n\}$ is not Hausdorff Mackey convergent to C. Then there exists a weakly compact convex set $K \subset c_0$ and a subsequence $\{f_{n_k}\}$ such that

$$\left(f_{n_k} + K^o\right) \cap C = \emptyset \quad \text{for all } k \in \mathbf{N}.$$

Then there exists a sequence $\{v_{n_k}\}$ in K such that

$$\left(f_{n_k} + K^o\right) \cap C = \emptyset;$$

that is,

$$\left(f - f_{n_k}\right)\left(v_{n_k}\right) > 1 \quad \text{for all } f \in C.$$

Now consider that we have passed to the subsequence and write $v_n = \sum_{i=1}^{\infty} \alpha_{ni} e_i$. Then since $0 \in C$, we have

$$-f_n(v_n) = e_n^*\left(\sum_{i=1}^{\infty} \alpha_{ni} e_i\right) = \alpha_{nn} > 1.$$

Also, since $(e_m^* - e_n^*) \in C$ for all $m \leqslant n$, we have

$$((e_m^* - e_n^*) - f_n)(v_n) = e_m^*\left(\sum_{i=1}^{\infty} \alpha_{ni} e_i\right) = \alpha_{nm} > 1 \quad \text{for all } m \leqslant n.$$

But then the sequence $\{v_n\}$ cannot converge weakly in c_0 and this contradicts the weak compactness of K.

However, it is interesting to note that Hausdorff Mackey convergence to the unit ball of ℓ_1 does imply Hausdorff norm convergence.

EXAMPLE 3.2. Consider a bounded sequence $\{f_n\}$ in $\ell_1 \cong c_0^*$ which is Hausdorff Mackey convergent to the unit ball in ℓ_1. We show it is Hausdorff norm convergent to the ball.

Suppose $\{f_n\}$ is not Hausdorff norm convergent to the ball. Then $\|f_n\| > 1 + r$ for some $r > 0$ and all $n \in \mathbf{N}$. We may assume that $\{f_n\}$ is weak* convergent to $f \in B(\ell_1)$. Given $\varepsilon > 0$ there exists a $v \in c_0$, $\|v\| = 1$, such that $f(v) > \|f\| - \varepsilon$. Writing $f_n \equiv \{\lambda_{n1}, \lambda_{n2}, \dots, \lambda_{nk}, \dots\}$ and $f \equiv \{\lambda_1, \lambda_2, \dots, \lambda_k, \dots\} \in \ell_1$, since $\{f_n - f\}$ is weak* convergent to 0, given $\varepsilon > 0$ and $p \in \mathbf{N}$ there exists an $n_p \in \mathbf{N}$ such that

$$\sum_{k=p+1}^{\infty} |\lambda_{nk} - \lambda_k| > \|f_n - f\| - \varepsilon \quad \text{for all } n \geqslant n_p.$$

So for each $p \in \mathbf{N}$ we can choose $v_p \in c_0$, $\|v_p\| = 1$, with zero entries for the first p terms such that

$$(f_{n_p} - f)(v_p) > \|f_n - f\| - \varepsilon.$$

Then the sequence $\{v_p\}$ is weakly convergent to 0 and

$$f_{n_p}(v + v_p) = \big(f_{n_p} - f\big)(v_p) + f_{n_p}(v) + f(v_p).$$

So for sufficiently large $p \in \mathbf{N}$ we have

$$\begin{aligned} f_{n_p}(v + v_p) &> \big(\|f_{n_p} - f\| - \varepsilon\big) + \big(\|f\| - \varepsilon\big) \\ &\geqslant \|f_{n_p}\| - 2\varepsilon > 1 + r - 2\varepsilon. \end{aligned}$$

Now the set $K \equiv \{v, v + v_p : p \in \mathbf{N}\}$ is weakly compact, and choosing $\varepsilon = r/3$ we see that

$$\big(f_{n_p} + \varepsilon K^o\big) \cap B(\ell_1) = \emptyset$$

for sufficiently large p. But this contradicts $\{f_n\}$ being Hausdorff Mackey convergent to the ball.

Although it has been shown that for a Banach space which does not contain a subspace topologically isomorphic to ℓ_1, every continuous convex function which is weak Hadamard differentiable at a point is Fréchet differentiable there, it is also instructive to see that a comparable result does not extend to directional differentiability.

EXAMPLE 3.3. We exhibit a continuous convex function ϕ on c_0 which is weak Hadamard directionally differentiable at 0 but not Fréchet directionally differentiable there.

Given $x = \sum_{i=1}^{\infty} \lambda_i e_i \in c_0$, we define

$$\phi(x) = \sup\{\lambda_m - \lambda_n, -\lambda_n - \frac{1}{n} : m \leqslant n, \quad m, n \in \mathbf{N}\}.$$

https://doi.org/10.1017/S000497270001515X Published online by Cambridge University Press

Clearly ϕ is continuous and convex. For $\phi(x) > 0$, since $\lambda_m - \lambda_n \to 0$ as $m,n \to \infty$ and $-\lambda_n - 1/n \to 0$ as $n \to \infty$, there are only finitely many choices of m and n for which $\phi(x)$ is (strongly) attained. If $\phi(x) = 0$ then of course $\phi(x)$ is attained for $m = n$. There may be a sequence $\{n_k\}$ such that $-e_{n_k}^*(x) - 1/n_k \to \phi(x) = 0$ but in any case $-e_{n_k}^*$ is weak* convergent to 0, and so we conclude that, given $x \in c_0$,

$$\partial\phi(x) = \overline{co}^{w^*}\{e_m^* - e_n^* \text{ where } \phi(x) = \lambda_m - \lambda_n \text{ for } m \leqslant n \text{ and } \\ -e_n^* \text{ where } \phi(x) = -\lambda_n - 1/n \text{ for } m,n \in \mathbb{N}\}.$$

We now use the characterisation of directional differentiability given in Lemma 2.1 and show that the subdifferential mapping $x \mapsto \partial\phi(x)$ is Hausdorff Mackey upper semi-continuous at 0 but not Hausdorff norm upper semi-continuous at 0.

Consider the sequence $\{x_n\}$ in c_0 where $x_n = \sum_{i=1}^{n} -(1/n)e_i$ for all $n \in \mathbb{N}$. Then $x_n \to 0$ as $n \to \infty$ and $\partial\phi(0) = \overline{co}^{w^*}\{e_m^* - e_n^* : m \leqslant n, \quad m,n \in \mathbb{N}\}$. However $-e_n^* \in \partial\phi(x_n)$ and $d(-e_n^*, \partial\phi(0)) = 1$ for all $n \in \mathbb{N}$, so the subdifferential mapping $x \mapsto \partial\phi(x)$ is not Hausdorff norm upper semi-continuous.

Now for each $n \in \mathbb{N}$ there exists a neighbourhood U of 0 such that $-e_n^* \notin \partial\phi(U)$, so to show that the subdifferential mapping $x \mapsto \partial\phi(x)$ is Hausdorff Mackey upper semi-continuous at 0 it is sufficient to prove that the sequence $\{-e_n^*\}$ is Hausdorff Mackey convergent to $\partial\phi(0)$. But this follows from Example 3.1.

4. Remarks

We note that the characterisation given in Lemma 1.1 holds for any continuous convex function and Lemmas 1.1, 1.2 and 2.1 hold for a Banach space with any of the recognised bornologies considered for differentiability questions, [**2**, p.410]. However we have confined ourselves to Banach spaces with the weak Hadamard bornology as our interest is in determining the special properties of uniform weak Hadamard differentiability of the norm and weak Hadamard directional differentiability of continuous convex functions on Banach spaces which do not contain a subspace topologically isomorphic to ℓ_1.

In [**3**, p.453] it was shown that a Banach space X is an Asplund space if the subdifferential mapping for the norm $x \mapsto \partial\|x\|$ is Hausdorff weak upper semi-continuous on the unit sphere. Lemma 2.1 and Theorem 2.2 prompt us to pose the associated problem.

PROBLEM. Is a Banach space X which does not contain a subspace topologically isomorphic to ℓ_1 an Asplund space if the norm on X is weak Hadamard directionally differentiable on the unit sphere?

References

[1] J.M. Borwein and M. Fabian, 'On convex functions having points of Gâteaux differentiability which are not points of Fréchet differentiability', *Canad. J. Math.* **45** (1993), 1121–1134.

[2] J.M. Borwein and S. Fitzpatrick, 'A weak Hadamard smooth renorming of $L_1(\Omega, \mu)$', *Canad. Math. Bull.* **36** (1993), 407–413.

[3] M.D. Contreras and R. Paya, 'On upper semi-continuity of duality mappings', *Proc. Amer. Math. Soc.* **121** (1994), 451–459.

[4] M.M. Day, *Normed linear spaces*, (3rd edition) (Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[5] R.Deville, G.Godefroy and V.Zizler, *Smoothness and renormings in Banach spaces*, Pitman monographs **64** (Longman, New York, 1993).

[6] J.R. Giles and W.B. Moors, *Generic continuity of restricted weak upper semi-continuous set-valued mappings*, Set-Valued Analysis (to appear).

[7] J.R. Giles and Scott Sciffer, 'Separable determination of Fréchet differentiability of convex functions', *Bull. Austral. Math. Soc.* **52** (1995), 161–167.

[8] R.R. Phelps, *Convex functions, monotone operators and differentiability*, Lecture notes in mathematics **1364**, (2nd edition) (Springer-Verlag, Berlin, Heidelberg, New York, 1993).

Department of Mathematics
The University of Newcastle
New South Wales 2308
Australia